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EDEXCEL A LEVEL FURTHER MATHEMATICS: QUESTIONS BY TOPIC

Complex Numbers (2)

Core Pure Mathematics 1 · Topic 2

11 questions · 95 marks · Mark schemes included

Pearson Edexcel A Level Further Mathematics (9FM0), June 2019 - June 2025

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HOW TO USE THIS BOOKLET

Questions are grouped by specification sub-topic and run from the lowest mark tariff to the highest within each one.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme.

Diagrams, graphs, tables and data are printed exactly as they appeared in the original papers.

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9.8 DAMPED OSCILLATIONS		1 Qs · 12 marks	45
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QUESTIONS

11 questions · 95 marks · 6 sub-topics



SUB-TOPIC 2.2

ARITHMETIC OF COMPLEX NUMBERS

Sub-topic 2.2 · 2 questions · 12 marks · lowest tariff first



Question 7 continued



Question 7 continued

(Total for Question 7 is 7 marks)



3. The complex number $z = a + bi$ where a and b are real constants.

Given that $\frac{z}{z^*}$ is purely imaginary,

- (a) show that $a^2 = b^2$

(2)

Given also that $zz^* = 50$

- (b) determine the possible complex numbers z

(3)

SUB-TOPIC 2.3

COMPLEX CONJUGATES AND ROOTS OF POLYNOMIALS

Sub-topic 2.3 · 4 questions · 30 marks · lowest tariff first



1. $f(z) = z^3 + az + 52$ where a is a real constant

Given that $2 - 3i$ is a root of the equation $f(z) = 0$

(a) write down the other complex root.

(1)

(b) Hence

(i) solve completely $f(z) = 0$

(ii) determine the value of a

(4)

(c) Show all the roots of the equation $f(z) = 0$ on a single Argand diagram.

(1)

Question 1 continued

(Total for Question 1 is 6 marks)



1. $f(z) = z^4 - 6z^3 + az^2 + bz + 145$

where a and b are real constants.

Given that $2 + 5i$ is a root of the equation $f(z) = 0$

(a) determine the other roots of the equation $f(z) = 0$

(7)

(b) Show all the roots of $f(z) = 0$ on a single Argand diagram.

(2)



Question 1 continued



Question 1 continued



Question 1 continued

(Total for Question 1 is 9 marks)



1.

$$f(z) = 3z^3 + pz^2 + 57z + q$$

where p and q are real constants.

Given that $3 - 2\sqrt{2}i$ is a root of the equation $f(z) = 0$

(a) show all the roots of $f(z) = 0$ on a single Argand diagram,

(7)

(b) find the value of p and the value of q .

(3)





Question 1 continued



Question 1 continued

(Total for Question 1 is 10 marks)







Question 6 continued

SUB-TOPIC 2.5

MODULUS-ARGUMENT FORM

Sub-topic 2.5 · 1 question · 10 marks · lowest tariff first



3.

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

$$z_1 = -4 + 4i$$

(a) Express z_1 in the form $r(\cos \theta + i \sin \theta)$, where $r \in \mathbb{R}$, $r > 0$ and $0 \leq \theta < 2\pi$

(2)

$$z_2 = 3\left(\cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12}\right)$$

(b) Determine in the form $a + ib$, where a and b are exact real numbers,

$$(i) \quad \frac{z_1}{z_2}$$

(2)

$$(ii) \quad (z_2)^4$$

(2)

(c) Show on a single Argand diagram

(i) the complex numbers z_1, z_2 and $\frac{z_1}{z_2}$

(ii) the region defined by $\{z \in \mathbb{C} : |z - z_1| < |z - z_2|\}$

(4)







(Total for Question 3 is 10 marks)



SUB-TOPIC 2.8

DE MOIVRE'S THEOREM

Sub-topic 2.8 · 2 questions · 22 marks · lowest tariff first



4. The complex number $z = e^{i\theta}$, where θ is real.

(a) Show that

$$z^n + \frac{1}{z^n} \equiv 2 \cos n\theta$$

where n is a positive integer.

(2)

(b) Show that

$$\cos^5 \theta = \frac{1}{16}(\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta)$$

(5)

(c) Hence, making your reasoning clear, determine all the solutions of

$$\cos 5\theta + 5 \cos 3\theta + 12 \cos \theta = 0$$

in the interval $0 \leq \theta < 2\pi$

(3)







10. (a) Given that, for $n \in \mathbb{N}$

$$z^n + \frac{1}{z^n} = 2 \cos n\theta$$

$$z^n - \frac{1}{z^n} = 2i \sin n\theta$$

show that

$$8 \sin^4 \theta \equiv \cos 4\theta - 4 \cos 2\theta + 3 \quad (5)$$



Figure 1

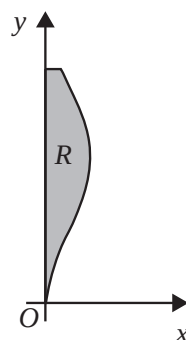


Figure 2

Figure 1 shows the central vertical cross-section of a solid wooden ornament.

Figure 2 shows the curve with equation

$$x = \sin^2\left(\frac{1}{2}y\right) \quad 0 \leq y \leq \frac{8\pi}{5}$$

The region R , shown shaded in Figure 2, is bounded by the curve, the line with

equation $y = \frac{8\pi}{5}$ and the y -axis.

The ornament is modelled by the solid of revolution formed when R is rotated 360° about the y -axis. The units are centimetres.

- (b) Using algebraic integration and the result in part (a), determine, in cm^3 , the volume of wood needed to make the ornament, according to the model. Give your answer to 2 significant figures.

[Solutions based entirely on calculator technology are not acceptable.]

(5)

Given that

- the density of the wood is 0.85 g/cm^3
- the mass of the ornament is 6 grams

- (c) comment on the suitability of the model.

(2)





SUB-TOPIC 9.2

GENERAL AND PARTICULAR SOLUTIONS IN CONTEXT

Sub-topic 9.2 · 1 question · 9 marks · lowest tariff first





Question 5 continued



Question 5 continued

(Total for Question 5 is 9 marks)



SUB-TOPIC 9.8

DAMPED OSCILLATIONS

Sub-topic 9.8 · 1 question · 12 marks · lowest tariff first



Question 6 continued



(Total for Question 6 is 12 marks)

MARK SCHEMES

official mark schemes



Question	Scheme	Marks	AOs
7(a)	$z^* = a - bi$ then $zz^* = (a + bi)(a - bi) = \dots$	M1	1.1b
	$zz^* = a^2 + b^2$ therefore, a real number	A1	2.4
		(2)	
(b)	$\frac{z}{z^*} = \frac{a+bi}{a-bi} = \frac{(a+bi)(a+bi)}{(a-bi)(a+bi)} = \frac{(a^2-b^2)+2abi}{a^2+b^2} = \frac{7}{9} + \frac{4\sqrt{2}i}{9}$ or $\frac{z}{z^*} = \frac{z^2}{zz^*} = \frac{z^2}{18} \Rightarrow$ $z^2 = 14 + 8\sqrt{2}i$ or $a + bi = \left(\frac{7}{9} + \frac{4\sqrt{2}i}{9}\right)(a - bi) = \dots + \dots i$	M1	1.1b
	Forms two equations from $a^2 + b^2 = 18$ or $\frac{a^2-b^2}{18} = \frac{7}{9}$ or $\frac{a^2-b^2}{a^2+b^2} = \frac{7}{9}$ or $\frac{2ab}{18} = \frac{4\sqrt{2}}{9}$ or $\frac{2ab}{a^2+b^2} = \frac{4\sqrt{2}}{9}$ or $a = \frac{7}{9}a + \frac{4\sqrt{2}}{9}b$ oe	M1 A1	3.1a 1.1b
	Solves the equations simultaneously e.g. $a^2 + b^2 = 18$ and $a^2 - b^2 = 14$ leading to a value for a or b	dM1	1.1b
	$z = \pm(4 + \sqrt{2}i)$	A1	2.2a
		(5)	
	(7 marks)		
Notes:			
(a)(i)			
M1: States or implies $z^* = a - bi$ and finds an expression for zz^*			
A1: Achieves $zz^* = a^2 + b^2$ and draws the conclusion that zz^* is a real number. Accept $\in \mathbb{R}$ as conclusion, but not just “no imaginary part”.			
(b)			
M1: Starts the process of solving by using the conjugate to form an equation with real denominators, and without z^* or i^2 in the equation. Accept as shown in scheme, or may multiply through by $a - bi$ and expand and gather terms. May be implied by correct extraction of equation(s).			
M1: Uses the given information to form two equations involving a and b at least one of which includes both. It must involve equating real or imaginary parts of $\frac{z}{z^*} = \frac{7}{9} + \frac{4\sqrt{2}i}{9}$			
A1: Any two correct equations arising from use of both given facts. (Note: if multiplying through by $a - bi$ then equating real and imaginary terms gives the same equation.)			
dM1: Dependent on previous method mark, solves the equations to find a value for either a or b .			
A1: Deduces the correct complex numbers and no extras. Do not accept $\pm 4 \pm \sqrt{2}i$			
Note: it is possible to solve via polar coordinates, but unlikely to succeed. If you see responses you think are worthy of credit but are unsure how to mark, use review. Example solutions shown below.			



(b) Alt	$\frac{z}{z^*} = \frac{z^2}{zz^*} = \frac{z^2}{18} \Rightarrow z^2 = 14 + 8\sqrt{2}i$ or let $\arg z = \theta$. then $\frac{z}{z^*} = \frac{re^{i\theta}}{re^{-i\theta}} = e^{2i\theta} = \cos 2\theta + i\sin 2\theta$	M1	1.1b
	$z^2 = 18(\cos \alpha + i\sin \alpha)$ where $\tan \alpha = \frac{4\sqrt{2}}{7} \Rightarrow z = \pm\sqrt{18}\left(\cos \frac{1}{2}\alpha + i\sin \frac{1}{2}\alpha\right)$ Or $\cos 2\theta + i\sin 2\theta = \frac{7}{9} + \frac{4\sqrt{2}i}{9} \Rightarrow 2\cos^2 \theta - 1 = \frac{7}{9}, 2\sin \theta \cos \theta = \frac{4\sqrt{2}}{9}$	M1 A1	1.1b 1.1b
	$\cos \frac{1}{2}\alpha = \sqrt{\frac{1}{2}(1 + \cos \alpha)} = \sqrt{\frac{1}{2}\left(1 + \frac{7}{9}\right)} = \dots$ and $\sin \frac{1}{2}\alpha = \sqrt{\frac{1}{2}(1 - \cos \alpha)} = \sqrt{\frac{1}{2}\left(1 - \frac{7}{9}\right)} = \dots$ or $\Rightarrow \cos \theta = \frac{2\sqrt{2}}{3}, \sin \theta = \frac{1}{3}, r = z = \sqrt{zz^*} = \sqrt{18}$	dM1	3.1a
	$z = \pm(4 + \sqrt{2}i)$	A1	2.2a
		(5)	

Question	Scheme	Marks	AOs
3(a)	$\frac{z}{z^*} = \frac{a+bi}{a-bi} = \frac{(a+bi)(a+bi)}{(a-bi)(a+bi)} = \frac{(a^2-b^2)+2abi}{a^2+b^2}$	M1	2.1
	$a^2 - b^2 = 0 \Rightarrow a^2 = b^2$	A1*	1.1b
		(2)	
	Alternative		
	$\frac{z}{z^*} = \frac{a+bi}{a-bi} = ki \Rightarrow a+bi = ki(a-bi)$ $\Rightarrow a+bi = kai + kb$	M1	2.1
	$\Rightarrow a = kb, b = ka$ e.g. $a = \frac{b}{a} \times b$ or $(k =) \frac{a}{b} = \frac{b}{a}$ $\Rightarrow a^2 = b^2$	A1*	1.1b
(b)	$zz^* = (a+bi)(a-bi) = a^2 + b^2 = 50$ Solves simultaneously with $a^2 = b^2$ to find a value for a or b $2a^2 = 50 \Rightarrow a = \dots$ or $2b^2 = 50 \Rightarrow b = \dots$	M1	3.1a
	$(z) = 5-5i, 5+5i, -5-5i, -5+5i$	A1 A1	1.1b 2.2a
		(3)	
(5 marks)			

Notes:

(a)

M1: Finds an expression for $\frac{z}{z^*}$ by rationalising the denominator and collects real parts, eliminating i^2 terms.

e.g. Accept $\frac{a^2 - b^2 + 2abi}{a^2 + b^2}$ or $\frac{a^2 - b^2}{a^2 + b^2} + \frac{2abi}{a^2 + b^2}$

but $\frac{a^2 + 2abi - b^2}{a^2 + b^2}$ is insufficient unless they then go on to identify the real parts

Allow sign slips but must be a correct use of rationalising the denominator.

A1*: Sets real part equal to zero and then achieves the correct equation, with no errors seen.

$\frac{z}{z^*} = \frac{a^2 - b^2 + 2abi}{a^2 + b^2} \Rightarrow a^2 = b^2$ is insufficient, as they have not set or identified their real part as zero.

Alternative:

M1: Substitutes $z = a + bi$ and $z^* = a - bi$, sets equal to an imaginary number such as ki but not i .

Then expands brackets, eliminating i^2 terms, achieving an equation equivalent to $a + bi = kai + kb$

A1*: Equates real parts **and** imaginary parts and uses their pair of equations to eliminate k and then achieve the correct equation, with no errors seen.

If you see a method that involves using exponentials, or modulus argument form, or working backwards, or any other method which may be worthy of credit, please send to review.

(b)

M1: Uses the information $zz^* = 50$ to form another equation for a and b , with no imaginary terms present. Then solves simultaneously to find a value for a or b

If they obtain $a^2 - b^2 = 50$ this will be M0

A1: At least two correct complex numbers. Accept e.g. $\pm(5 + 5i)$

A1: Deduces all 4 correct complex numbers. Accept e.g. $\pm 5 \pm 5i$ or $\pm(5 \pm 5i)$

Note: Correct answers with no workings can achieve M1A1A1



Question	Scheme	Marks	AOs
1(a)	$2 + 3i$	B1	1.1b
		(1)	
(b) (i)	$z^* = 2 + 3i$ so $z + z^* = 4$, $zz^* = 13$ $z + z^* + \alpha = 0 \Rightarrow \alpha = \dots$ or $azz^* = -52 \Rightarrow \alpha = -\frac{52}{13} = \dots$ or $z^2 - (\text{sum roots})z + (\text{product roots}) = 0$ or $(z - (2 + 3i))(z - (2 - 3i)) = \dots$ $\Rightarrow (z^2 - 4z + 13)(z + 4) \Rightarrow z = \dots$	M1	3.1a
	$z = 2 \pm 3i, -4$	A1	1.1b
	$(z^2 - 4z + 13)(z + 4)$ expands the brackets to find value for a Or $a = \text{pair sum} = -4(2 + 3i + 2 - 3i) + 13 = \dots$ Or $f(-4)/f(2 \pm 3i) = 0 \Rightarrow \dots \Rightarrow a = \dots$	M1	1.1b
	$a = -3$	A1	2.2a
		(4)	
(c)		B1ft	1.1b
		(1)	
(6 marks)			

Notes:

(a)

B1: $2 + 3i$

(b)

(i)

M1: A complete method to find the third root. E.g. forms the quadratic factor and uses this to find the linear factor leading to roots. Alternatively uses sum of roots = 0 or product of roots = ± 52 (condone sign error) with their complex roots to find the third. Note they may have used the factor theorem to find a first, which is fine. If they have found a first, then the correct third root seen implies this mark. The method may be implied by the third root seen on the diagram.

A1: Correct roots, all three must be clearly stated somewhere in (b), not just seen on a diagram in part (c).

(ii)

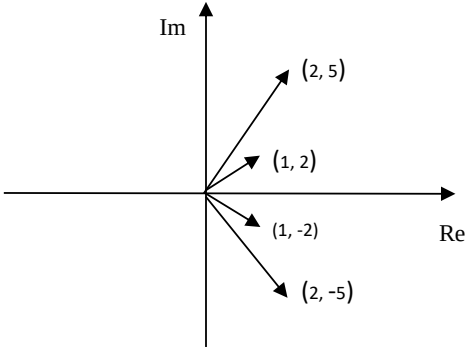
M1: Complete method to find a value for a e.g. multiplies out their quadratic and linear factors to find the coefficient of z , or uses pair sum, or uses factor theorem with one of the roots (may be done before finding the third root) but must reach a value for a .

A1: Deduces the correct value of a . May be seen as the z coefficient in the cubic (need not be extracted, but if it is it must be correct).

(c)

B1ft: Correctly plots all three roots following through their third root in part (b). Must be labelled with the “ -4 ” further from O than 2, but don’t be concerned about x and y scale. If correct look for one root on the negative real axis, with the other two symmetric about real axis in quadrants 1 and 4, but follow through their real root if positive. Accept $(0, -4)$ labelled on the real axis in correct place as a label.



Question	Scheme	Marks	AOs
1(a)	$z = 2 - 5i$	B1	1.2
	$(z - (2 + 5i))(z - (2 - 5i)) = \dots$ or Sum of roots = 4, Product of roots = 29 $\rightarrow z^2 + \dots$	M1	3.1a
	$= z^2 - 4z + 29$	A1	1.1b
	$z^4 - 6z^3 + az^2 + bz + 145 = (z^2 - 4z + 29)(z^2 + cz + 5)$	M1	3.1a
	$z^2 - 2z + 5 = 0$	A1	2.2a
	$z^2 - 2z + 5 = 0 \Rightarrow z = \dots$	M1	1.1b
	$z = 1 \pm 2i$	A1	1.1b
		(7)	
(b)		B1ft	1.1b
		B1	1.1b
		(2)	

(9 marks)

Notes

(a)

Note: if there are multiple attempts or attempts which start in main scheme before switching to an Alt, then mark the most complete attempt.

B1: Identifies the correct complex conjugate as another root.

M1: Formulates a correct strategy – main scheme. Uses the conjugate pair and a correct method to find a quadratic factor.

A1: Correct quadratic.

M1: Uses the given quartic and their quadratic to establish the other quadratic factor – by inspection or long division etc. resulting in a 3TQ

A1: Deduces the correct second quadratic.

M1: Solves their second quadratic.

A1: Correct second conjugate pair (isw if they try to write in a factorised equation).

(b)

B1ft: A pair of complex roots plotted correctly, either both $2 \pm 5i$ or follow through their second pair of *complex* roots. Just the points are needed. Allow if there is no labelling for this mark, as long as there is a pair of roots symmetric in the real axis.

B1: Fully correct and labelled sketch with both sets of roots shown in approximately the correct locations. Just the points are needed. Labelling may be by coordinates or labels on axes, and accept tick marks as unit increments for labels. The $(1, \pm 2)$ roots must be within the sector spanned by $(2, \pm 5)$ (as in the diagram).



(a) Alt 1	$z = 2 - 5i$	B1	1.2
	$\alpha + \beta = 4, \alpha\beta = 2^2 + 5^2 = \dots$ and $\alpha + \beta + \gamma + \delta = \pm 6, \alpha\beta\gamma\delta = \pm 145$	M1	3.1a
	$\alpha + \beta = 4, \alpha\beta = 29$ and $\alpha + \beta + \gamma + \delta = 6, \alpha\beta\gamma\delta = 145$	A1	1.1b
	$\Rightarrow \gamma + \delta = 2, \gamma\delta = \frac{145}{29} \Rightarrow (2 - \delta)\delta = \frac{145}{29}$	M1	3.1a
	$\delta^2 - 2\delta + 5 = 0$	A1	2.2a
	$\delta^2 - 2\delta + 5 = 0 \Rightarrow \delta = \dots$	M1	1.1b
	$= 1 \pm 2i$	A1	1.1b
		(7)	

Notes

(a)

B1: Identifies the correct complex conjugate as another root

M1: Formulates a correct strategy – sum and product approach. Attempts formulae for sum and product of the two known roots and all four roots – allow sign errors, $\alpha + \beta = 4, \alpha\beta = 2^2 + 5^2 = \dots$
 $\alpha + \beta + \gamma + \delta = \pm 6, \alpha\beta\gamma\delta = \pm 145$

A1: Correct equations $\alpha + \beta = 4, \alpha\beta = 29$ $\alpha + \beta + \gamma + \delta = 6, \alpha\beta\gamma\delta = 145$ (seen or implied).

M1: Uses the sum and product of known roots to reduce to equations in just the unknown roots and attempts to solve simultaneously. Note allow if they assume complex roots and use $\lambda \pm \mu i$ for the two unknown roots.

A1: Deduces the correct quadratic for remaining roots.

M1: Solves their quadratic

A1: Correct second conjugate pair

(a) Alt 2	$z = 2 - 5i$	B1	1.2
	$f(2 \pm 5i) = 0 \Rightarrow -21a + 2b + 1038 \pm (-20a - 5b + 450)i = 0$	M1	3.1a
	$\Rightarrow 21a - 2b - 1038 = 0, 20a + 5b - 450 = 0$	A1	1.1b
	$\Rightarrow a = \dots, b = \dots$	M1	3.1a
	$a = 42$ and $b = -78$	A1	2.2a
	$z^4 - 6z^3 + 42z^2 - 78z + 145 = 0 \Rightarrow z = \dots$	M1	1.1b
	$z = 1 \pm 2i, (2 \pm 5i)$	A1	1.1b
		(7)	

Notes

(a)

B1: Identifies the correct complex conjugate as another root- seen anywhere

M1: Formulates a correct strategy – factor theorem approach. Applies the factor theorem with either complex root and equates real and imaginary terms to form simultaneous equations in a and b

A1: Correct equations need not be simplified.

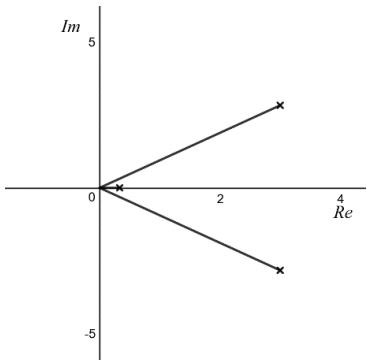
M1: Solves the equations to find values for a and b . Do not be concerned with the algebra.

A1: Correct values

M1: Solves the resulting quartic – may be by calculator. Note that if roots are stated by calculator they must correspond to their found a and b , but allow for rounding to nearest integer.

A1: Correct second conjugate pair from fully correct work.



Question	Scheme	Marks	AOs
1(a)	$\beta = 3 + 2\sqrt{2}i$ is also a root	B1	1.2
	$\alpha\beta = 17, \alpha + \beta = 6$	B1	1.1b
	$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{57}{3}$	M1	1.1b
	$\alpha\gamma + \beta\gamma = \frac{57}{3} - 17 = \gamma(\alpha + \beta) = 6\gamma \Rightarrow \gamma = \dots$	M1	3.1a
	$\gamma = \frac{1}{3}$	A1	2.2a
		B1	1.1b
		B1ft	1.1b
	(7)		
(a) Alternative:	(a) Alternative:		
	$\beta = 3 + 2\sqrt{2}i$ is also a root	B1	1.2
	$(z - (3 + 2\sqrt{2}i))(z - (3 - 2\sqrt{2}i)) = z^2 - 6z + 17$	B1	1.1b
	$f(z) = (z^2 - 6z + 17)(3z + a) = 3z^3 + az^2 - 18z^2 - 6az + 51z + 17a$	M1	1.1b
	$\Rightarrow 51 - 6a = 57 \Rightarrow a = -1 \Rightarrow \gamma = \dots$	M1	3.1a
	$\gamma = \frac{1}{3}$	A1	2.2a
	Then B1 B1ft as above		
	(7)		
(b)	$3 - 2\sqrt{2}i + 3 + 2\sqrt{2}i + \frac{1}{3} = -\frac{p}{3} \Rightarrow p = \dots$ or $(3 - 2\sqrt{2}i)(3 + 2\sqrt{2}i) \times \frac{1}{3} = -\frac{q}{3} \Rightarrow q = \dots$	M1	3.1a
	$p = -19$ or $q = -17$	A1	1.1b
	$p = -19$ and $q = -17$	A1	1.1b
	(3)		
	(b) Alternative:		
	$f(z) = (z^2 - 6z + 17)(3z - 1) = 3z^3 + pz^2 + 57z + q$	M1	3.1a

	$\Rightarrow p = \dots, q = \dots$		
	$p = -19$ or $q = -17$	A1	1.1b
	$p = -19$ and $q = -17$	A1	1.1b
		(3)	
(10 marks)			
Notes			
(a) B1: Identifies the correct complex conjugate as another root B1: Correct values for the sum and product for the conjugate pair M1: Correct application of the pair sum M1: Identifies a complete and correct strategy for identifying the third root A1: Deduces the correct third root B1: $3 \pm 2\sqrt{2}i$ plotted correctly, in quadrants 1 and 4 which are reflections in the real axis. Do not be concerned about labelling or scaling. B1ft: Their real root plotted correctly, in correct relative position to the two complex roots. Scales are not needed but if correct, the real root must be close to the origin compared to the complex roots. Alternative: B1: Identifies the correct complex conjugate as another root B1: Correct quadratic factor obtained M1: Expands their quadratic $\times (3z + "a")$ or attempts to factor out the quadratic, or use long division, leading to a factor $(3z + "a")$. Implied by seeing $(z^2 - 6z + 17)(3z + a)$ with any value of a (or with their quadratic). M1: Proceeds to extract the root from their third factor of from $(3z + "a")$. A1: Deduces the correct third root. If not explicitly stated, look for it on their diagram. B1: $3 \pm 2\sqrt{2}i$ plotted correctly, as above B1ft: Their real root plotted correctly as above. (b) M1: Correct strategy used for identifying at least one of p or q A1: At least one value correct A1: Both values correct Alternative: M1: Correct strategy by expanding their quadratic and linear factors to identifying at least one of p or q A1: At least one value correct A1: Both values correct Note: some may attempt to use the factor theorem with the complex root. $f(3 - 2i\sqrt{2}) = 36 + p + q + i(-228\sqrt{2} - 12\sqrt{2}p) = 0$ 2nd B1: equate real and imaginary components to 0 to get correct equations $36 + p + q = 0, -228\sqrt{2} - 12\sqrt{2}p = 0$ 1st M1: solves their equations $\Rightarrow p = -19, q = -17$ 2nd M1: Solves the cubic (may be from calculator). The 1st B1 may then be implied for the second complex root, and the rest as main scheme.			

Question	Scheme	Marks	AOs
6	Method 1: using roots		
	$z_2 = 2 - 4i$	B1	1.1b
	$12 = \frac{1}{2}(4 - -4) \times h \Rightarrow h = \dots \Rightarrow z_3 = 2 + h \text{ or } 2 - h$	M1	3.1a
	$(z - (2 - 4i))(z - (2 + 4i))(z - 5)$ or $(z - (2 - 4i))(z - (2 + 4i))(z - (-1))$	dM1	2.1
	$(f(z)) = z^3 - 9z^2 + 40z - 100$ $(f(z)) = z^3 - 3z^2 + 16z + 20$	ddM1 A1	1.1b 2.2a
		(5)	
	Method 2: using sum, pair sum and product of roots		
	$z_2 = 2 - 4i$	B1	1.1b
	$12 = \frac{1}{2}(4 - -4) \times h \Rightarrow h = \dots \Rightarrow z_3 = 2 + h \text{ or } 2 - h$	M1	3.1a
	Sum = $2 + 4i + 2 - 4i + "5" = 9 = -a$ Pair sum = $(2 + 4i)(2 - 4i) + "5"(2 + 4i) + "5"(2 - 4i) = 40 = b$ Product = $"5"(2 + 4i)(2 - 4i) = 100 = -c$ OR Sum = $2 + 4i + 2 - 4i + "(-1)" = 3 = -a$ Pair sum = $(2 + 4i)(2 - 4i) + "(-1)(2 + 4i) + "(-1)(2 - 4i) = 16 = b$ Product = $"(-1)(2 + 4i)(2 - 4i) = -20 = -c$	dM1	2.1
	$(f(z)) = z^3 - 9z^2 + 40z - 100$ $(f(z)) = z^3 - 3z^2 + 16z + 20$	ddM1 A1	1.1b 2.2a
		(5)	
(5 marks)			

Notes:**Method 1:**

B1: States the other complex root. May be implied by later working.

M1: Uses the area of the triangle to determine at least one value for the real root. You can award for sight of one correct value of z_3

(**Note:** $h = 3 \Rightarrow z_3 = 5$ or -1)

dm1: Dependent on the previous method mark. Deduces a correct expression for one of their real roots. Award for $(z - (2 - 4i))(z - (2 + 4i))(z - z_3)$ or equivalent e.g. $(z^2 - 4z + 20)(z - z_3)$

z_3 must be real.

ddm1: Dependent on previous method marks. Multiplies out to find values for the constants a , b and c . Award for the sight of at least 2 correct values of the constants a , b and c in any one of their attempts at the function.

A1: Deduces the correct two functions.

Method 2:

B1: States the other complex root. May be implied by later working.

M1: Uses the area of the triangle to determine at least one value for the real root. You can award for sight of one correct value of z_3 (**Note:** $h = 3 \Rightarrow z_3 = 5$ or -1)

dm1: Dependent on the previous method mark. Attempts sum, pair sum and product for at least one of their real roots.

ddm1: Dependent on previous method marks. Multiplies out to find values for the constants a , b and c

Award for the sight of at least 2 correct values of the constants a , b and c in any one of their attempts at the function.

A1: Deduces the correct two functions.

Note: There are several methods that can be used to obtain the first method mark:

- Uses vertices $(2, 4)$, $(2, -4)$ and $(x_3, 0)$ to determine at least one value for the real root.

Alternatively uses (x_3, y_3) . This may come from a diagram. Do not be concerned about the variables they use.

$$\text{e.g. } 12 = \frac{1}{2} \times 8 \times |x_3 - 2| \Rightarrow z_3 = \text{ or } x_3 =$$

- Uses the formula for the area of a triangle with vertices (x_1, y_1) , (x_2, y_2) , (x_3, y_3) to determine at least one value for the real root. Do not be concerned about the variables they use.

$$\left[\frac{1}{2} |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| \right]$$

$$12 = \frac{1}{2} |2(-4 - y_3) + 2(y_3 - 4) + x_3(4 - -4)|$$

$$24 = |-16 + 8x_3|$$

$$\Rightarrow z_3 = \text{ or } x_3 =$$

- Uses the shoelace method with no sign errors.



$$\text{e.g. Area} = \frac{1}{2} \begin{vmatrix} 2 & 2 & x_3 & 2 \\ 4 & -4 & 0 & 4 \end{vmatrix}$$

$$12 = \frac{1}{2} |(2 \times -4 + 4x_3) - (2 \times 4 - 4x_3)|$$

$$24 = |-16 + 8x_3|$$

$$\Rightarrow z_3 = \text{ or } x_3 =$$

If you are uncertain in how to apply a method, then please send to review.



Question	Scheme	Marks	AOs
3(a)	e.g. $ z_1 = \sqrt{(-4)^2 + 4^2}$ or $\arg z_1 = \pi - \frac{\pi}{4}$ oe	M1	1.1b
	$(z_1 =) 4\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$ or e.g. $(z_1 =) \sqrt{32} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$	A1	1.1b
		(2)	
(b)(i)	$\frac{z_1}{z_2} = \frac{4\sqrt{2}}{3} \left(\cos \left(\frac{3\pi}{4} - \frac{17\pi}{12} \right) + i \sin \left(\frac{3\pi}{4} - \frac{17\pi}{12} \right) \right) = \dots$ or $\frac{z_1}{z_2} = \frac{4\sqrt{2}}{3e^{\frac{17\pi}{12}i}} e^{\frac{3\pi}{4}i} = \frac{4\sqrt{2}}{3} e^{\left(\frac{3\pi}{4} - \frac{17\pi}{12} \right)i}$ or $\frac{z_1}{z_2} = \frac{-4 + 4i}{3 \left(\left(\frac{\sqrt{2} - \sqrt{6}}{4} \right) - i \left(\frac{\sqrt{2} + \sqrt{6}}{4} \right) \right)} \times \frac{\left(\frac{\sqrt{2} - \sqrt{6}}{4} \right) + i \left(\frac{\sqrt{2} + \sqrt{6}}{4} \right)}{\left(\frac{\sqrt{2} - \sqrt{6}}{4} \right) + i \left(\frac{\sqrt{2} + \sqrt{6}}{4} \right)} = \dots$	M1	3.1a
	$= -\frac{2\sqrt{2}}{3} - \frac{2\sqrt{6}}{3}i \text{ or } -\frac{2\sqrt{2}}{3} - i\frac{2\sqrt{6}}{3} \text{ or } -\frac{2\sqrt{2}}{3} + i\left(-\frac{2\sqrt{6}}{3}\right)$	A1	1.1b
		(2)	

Notes

(a) **Correct answer with no working scores both marks in (a)**

M1: Any correct expression for $|z_1|$ or $\arg z_1$ e.g. $|z_1| = \sqrt{(-4)^2 + 4^2}$ or $\arg z_1 = \pi - \frac{\pi}{4}$

A1: Correct expression. The " $z_1 =$ " is not required.

This mark is not for correct modulus and correct argument it is for the complex number written in the required form. Condone the missing closing bracket e.g. $(z_1 =) \sqrt{32} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$

(b)(i) **Correct answer with no working scores no marks in (b)(i)**

M1: Employs a correct method to find the quotient. E.g.

- uses modulus argument form and divides moduli and subtracts arguments the right way round
- uses exponential form and divides moduli and subtracts arguments the right way round
- converts z_2 to Cartesian form and multiplies numerator and denominator by the complex conjugate of the denominator. Allow if the "3" is missing for this method. Allow with decimals for this method e.g. $\frac{z_1}{z_2} = \frac{-4 + 4i}{-0.258... - 0.965...i} \times \frac{-0.258... + 0.965...i}{-0.258... + 0.965...i} = \dots$

If they convert z_2 to Cartesian form it must be correct as shown or correct decimals.

A1: Correct exact answer in the required form.

Do not allow e.g. $-\frac{2}{3}(\sqrt{2} + \sqrt{6}i)$ or $\frac{-2\sqrt{2} - 2\sqrt{6}i}{3}$ unless a correct form is seen previously then apply isw.

Provided a correct method is shown as above, allow to go from the forms in the main scheme to the correct exact answer with no intermediate step.



(ii)	$z_2^4 = 3^4 \left(\cos \left(4 \times \frac{17\pi}{12} \right) + i \sin \left(4 \times \frac{17\pi}{12} \right) \right)$ or $(z_2)^4 = \left(3e^{\frac{17\pi}{12}i} \right)^4 = 3^4 e^{\frac{17\pi}{12} \times 4i}$ or $z_2^4 = \left\{ 3 \left(\left(\frac{\sqrt{2}-\sqrt{6}}{4} \right) - i \left(\frac{\sqrt{2}+\sqrt{6}}{4} \right) \right) \right\}^4 = \dots$	M1	1.1b
	$= \frac{81}{2} - \frac{81\sqrt{3}}{2}i \text{ or } \frac{81}{2} - i \frac{81\sqrt{3}}{2} \text{ or } \frac{81}{2} + i \left(-\frac{81\sqrt{3}}{2} \right)$		
		A1	1.1b
		(2)	

(b)(ii) Correct answer with no working scores no marks in (b)(ii)

M1: Applies De Moivre's theorem correctly to z_2 . E.g. uses polar form or exponential form and

calculates the modulus as 3^4 and the argument as $4 \times \frac{17\pi}{12}$

For attempts at $z_2^4 = \left\{ 3 \left(\left(\frac{\sqrt{2}-\sqrt{6}}{4} \right) - i \left(\frac{\sqrt{2}+\sqrt{6}}{4} \right) \right) \right\}^4$ you would need to see:

- the correct exact form used
- a clear and convincing attempt to expand the brackets e.g. by using a full binomial expansion or a complete attempt to multiply all 4 brackets together but you are not expected to check every detail
- a final answer in the required form with no obvious errors seen

$$\text{So } z_2^4 = \left\{ 3 \left(\left(\frac{\sqrt{2}-\sqrt{6}}{4} \right) - i \left(\frac{\sqrt{2}+\sqrt{6}}{4} \right) \right) \right\}^4 = \frac{81}{2} - \frac{81\sqrt{3}}{2}i \text{ scores no marks.}$$

Similar guidance applies if they attempt to expand $\left\{ 3 \left(\cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12} \right) \right\}^4$

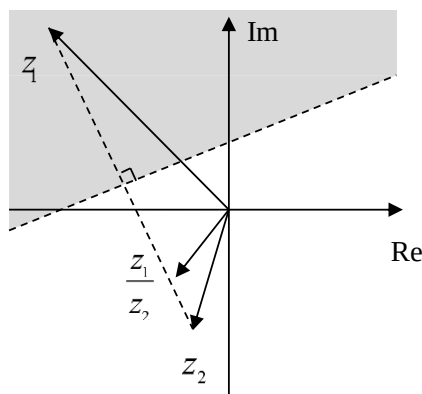
A1: Correct exact answer in the required form.

Do not allow e.g. $\frac{81}{2} \left(1 - \frac{81\sqrt{3}}{2}i \right)$ or $\frac{81-81\sqrt{3}i}{2}$ unless a correct form is seen previously then apply isw.

Provided a correct method is shown as above, allow to go from the forms in the main scheme to the correct exact answer with no intermediate step.



(c)(i) and (ii)


Notes:
(c)(i)

B1: z_1 and z_2 correctly positioned. Look for correct quadrants with z_1 approximately on $y = -x$ and z_2 below $y = x$ closer to the origin than z_1 . Note that the points are usually labelled but mark positively if it is clear which points are which if there is no labelling.

B1

1.1b

B1ft: $\frac{z_1}{z_2}$ in the correct quadrant. Follow through their answer to (b)(i).

Note that the point is usually labelled but mark positively if it is clear which point it is. It is sometimes labelled as z_3 which is fine.

B1ft

1.1b

(ii)

M1: Draws a line (solid or dashed) that is the perpendicular bisector of z_1z_2 **or** draws a line that crosses z_1z_2 and shades one of the sides of this line.

M1

3.1a

A1: A line drawn (solid or dashed) that is the perpendicular bisector of z_1z_2 with either side shaded as long as it is clear they are not discounting the upper region. The B1 in part (i) may not have been scored but z_1 must be in quadrant 2 and z_2 in quadrant 3.

A1

1.1b

Note that some candidates are drawing the region on a separate diagram and this is acceptable. You do not need to see a line joining z_1 to z_2 .

(4)
(10 marks)


Question	Scheme	Marks	AOs
4(a)	$z^n + \frac{1}{z^n} \equiv e^{in\theta} + \frac{1}{e^{in\theta}} \equiv e^{in\theta} + e^{-in\theta}$	M1	1.1b
	$\equiv \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta \equiv 2 \cos n\theta^*$	A1*	2.1
		(2)	
(b)	$(z + z^{-1})^5 = 32 \cos^5 \theta$	B1	2.2a
	$(z + z^{-1})^5 = z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5}$	M1 A1	1.1b 1.1b
	$32 \cos^5 \theta = (z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1})$ $= 2 \cos 5\theta + 10 \cos 3\theta + 20 \cos \theta$	M1	2.1
	$\cos^5 \theta = \frac{1}{16} (\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta)^*$	A1*	1.1b
		(5)	
(c)	$\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta = -2 \cos \theta \Rightarrow 16 \cos^5 \theta = -2 \cos \theta$	B1	3.1a
	$2 \cos \theta (8 \cos^4 \theta + 1) = 0 \Rightarrow \theta = \dots$	M1	1.1b
	$8 \cos^4 \theta + 1 = 0$ has no solution so $\cos \theta = 0$ $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$	A1	2.2a
		(3)	
(10 marks)			
Notes			
(a) M1: Substitutes z into the LHS and simplifies the powers as shown. Allow if they go direct to trigonometric expressions without exponentials. The mark is for sorting out the negative index. A1*: Converts the exponential form to trigonometric form correctly and correctly completes the proof with no errors seen. The trigonometric expansion must be clearly seen. Condone missing brackets in e.g. $\cos -n\theta$ terms if intent is clear. Note the LHS of the identity may be implied. (b) B1: Deduces that $(z + z^{-1})^5 = 32 \cos^5 \theta$ Do not accept 2^5 for 32. May be implied. M1: Attempts to expand $(z + z^{-1})^5$. Correct binomial coefficients must be used, terms need not be simplified. Condone at most one slip in powers. A1: Correct expansion, terms need not be gathered but powers must have been simplified. M1: Sets their expressions equal and applies the result from (a) – grouping must be shown. A1*: Reaches the printed answer with no errors and relevant steps all shown. (c) B1: Uses the result from (b) to deduce the correct equation. M1: Must have attempted to use part (b) to obtain $\alpha \cos^5 \theta = \beta \cos \theta$ or equivalent. Collects to one side and attempts to factorise and solve. Note dividing through by $\cos \theta$ is M0. A1: Rejects the inappropriate solution and selects $\cos \theta = 0$ and obtains the correct values only. The equation must have been correct. There must have been some consideration of the $8 \cos^4 \theta + 1$ e.g. stating $8 \cos^4 \theta > 0$ so no solutions, or attempting to find complex roots and deducing no answers. May be minimal, but some consideration that no roots arise from this part must have been given. Note: The correct answer will appear from incorrect attempts – the M must be gained in order to award the A. E.g. assuming the equation reduces to $\cos^5 \theta = 0$ will score B0M0A0. Likewise, answers only scores B0M0A0 (questions says hence so use of (b) must be seen).			



Alt (a)	$z^n + \frac{1}{z^n} = \cos n\theta + i \sin n\theta + \frac{1}{\cos n\theta + i \sin n\theta} = \frac{\cos^2 n\theta + 2i \cos n\theta \sin n\theta - \sin^2 n\theta + 1}{\cos n\theta + i \sin n\theta}$	M1	1.1b
	$= \frac{2 \cos^2 n\theta + 2i \cos n\theta \sin n\theta}{\cos n\theta + i \sin n\theta} = \frac{2 \cos n\theta (\cos n\theta + i \sin n\theta)}{\cos n\theta + i \sin n\theta} = 2 \cos n\theta *$	A1*	2.1
		(2)	
Notes			
Alt: by De Moivre M1: Applies De Moivre on both terms and puts over a common denominator. A1*: Complete correctly, using $1 - \sin^2 n\theta = \cos^2 n\theta$ and cancelling $\cos n\theta + i \sin n\theta$. No errors seen.			
Alt (b)	$\cos 5\theta = \operatorname{Re}(\cos 5\theta + i \sin 5\theta) = \operatorname{Re}(\cos \theta + i \sin \theta)^5$	B1	2.2a
	$(\cos \theta + i \sin \theta)^5 = c^5 + 5ic^4s + 10i^2c^3s^2 + 10i^3c^2s^3 + 5i^4cs^4 + i^5s^5$	M1	1.1b
	$\operatorname{Re}(\cos \theta + i \sin \theta)^5 = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta$	A1	1.1b
	$\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta (1 - \cos^2 \theta) + 5 \cos \theta (1 - \cos^2 \theta)^2$ $= 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$ $= 16 \cos^5 \theta - \frac{20}{4} (\cos 3\theta + 3 \cos \theta) + 5 \cos \theta$	M1	2.1
	$\Rightarrow \cos^5 \theta = \frac{1}{16} (\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta) *$	A1*	1.1b
		(5)	
Notes			
Alt: by De Moivre B1: Correctly stated or clearly implied De Moivre statement for $\cos 5\theta$ M1: Attempts to expand $(\cos \theta + i \sin \theta)^5$ Correct coefficients but allow one slip per main scheme. The powers of i need not be simplified for the attempt at expansion, accept if only the real terms are shown. Allow c and s notation. A1: Correct real terms extracted with the i's removed. M1: Applies $\sin^2 \theta = 1 - \cos^2 \theta$ to reduce to an equation in $\cos \theta$ and applies $\cos^3 \theta = \frac{1}{4} (\cos 3\theta + 3 \cos \theta)$ (quoted or derived – allow a slip if derived) to get to an equation without powers of cos terms. A1*: Reaches the printed answer with no errors and relevant steps all shown.			

Question	Scheme	Marks	AOs
10(a)	$\left(z - \frac{1}{z}\right)^4 = (2i\sin\theta)^4 = 16\sin^4\theta$	B1	1.1b
	$\left(z - \frac{1}{z}\right)^4 = z^4 + 4\left(z^3\right)\left(-\frac{1}{z}\right) + 6\left(z^2\right)\left(-\frac{1}{z}\right)^2 + 4\left(z\right)\left(-\frac{1}{z}\right)^3 + \left(-\frac{1}{z}\right)^4$	M1	2.1
	$= \left[z^4 + \frac{1}{z^4}\right] - 4\left[z^2 + \frac{1}{z^2}\right] + 6$	A1	1.1b
	Uses $z^n + \frac{1}{z^n} = 2\cos n\theta$ $\{16\sin^4\theta\} \equiv 2\cos 4\theta - 8\cos 2\theta + 6$	M1	2.1
	$8\sin^4\theta \equiv \cos 4\theta - 4\cos 2\theta + 3$ * cso	A1 *	1.1b
		(5)	
(b)	$\text{vol} = \pi \int \left(\sin^2\left(\frac{1}{2}y\right)\right)^2 dy$	B1	3.4
	$\text{vol} = \{\pi\} \int \sin^4\left(\frac{1}{2}y\right) (dy)$	M1	1.1b
	$= \{\pi\} \int \frac{1}{8}(\cos(2y) - 4\cos(y) + 3) (dy) = \dots$		
	$= \{\pi\} \left[\frac{1}{8} \left(\frac{1}{2} \sin(2y) - 4\sin(y) + 3y \right) \right]$	A1	1.1b
	$= \pi \left[\frac{1}{8} \left(\frac{1}{2} \sin\left(2 \times \frac{8\pi}{5}\right) - 4\sin\left(\frac{8\pi}{5}\right) + \left(3 \times \frac{8\pi}{5}\right) \right) - 0 \right]$ $= \dots$	dM1	3.4
	awrt 7.3 (cm ³)	A1	1.1b
		(5)	
(c)	Mass = “7.3” x 0.85 $= \dots$	M1	2.2b
	Mass = 6.2 (grams) therefore a good model	A1ft	3.5a
		(2)	
	Alternative 1		
	Volume = $6 \div 0.85 = \dots$	M1	2.2b
	Volume = 7.1 (cm ³) therefore a good model	A1ft	3.5a
		(2)	

	Alternative 2		
	Density = $6 \div 7.3$	M1	2.2b
	Density = $0.82 \text{ (g/cm}^3\text{)}$ therefore a good model	A1ft	3.5a
		(2)	
(12 marks)			



Notes:

(a)

B1: See scheme. This can appear anywhere in the proof.

Accept $2^4 \sin^4 \theta$ for $16 \sin^4 \theta$, but not $(2 \sin \theta)^4$

Alternatively, they may instead substitute $\sin \theta = \frac{1}{2i} \left(z - \frac{1}{z} \right)$ into the expression $8 \sin^4 \theta$. This can be implied but must come from correct work. This can appear anywhere in their proof.

e.g. $8 \sin^4 \theta = 8 \left[\frac{1}{2i} \left(z - \frac{1}{z} \right) \right]^4$ or $8 \sin^4 \theta = 8 \left[\left(z - \frac{1}{z} \right) \right]^4 \frac{1}{2^4}$

and allow $8 \sin^4 \theta = \frac{1}{2} \left[\left(z - \frac{1}{z} \right) \right]^4$

M1: Finds the expansion of $\left(z - \frac{1}{z} \right)^4$ which may be unsimplified. All five terms must be present.

Condone sign slips only

A1: Correct expansion, with terms grouped.

M1: Uses $z^n + \frac{1}{z^n} = 2 \cos n\theta$ to write in terms of $\cos 4\theta$ and $\cos 2\theta$

A1*: Achieves the printed answer with no errors or omissions. Cso Follows A0.

(b)

B1: Correct formula $\pi \int \left(\sin^2 \left(\frac{1}{2} y \right) \right)^2 dy$ (but not $\text{vol} = \pi \int x^2 dy$) used to find a volume, stated or

implied, ignore limits. If there is a missing π or dy in their integral, then withhold this mark only. However, this mark may be awarded if π and dy are seen together later in their integral.

Do not award the following marks if algebraic integration is not used. For example finding an answer of 7.3 without algebraic integration will obtain M0A0dM0A0

M1: Uses the result in part (a) to express the volume in an integrable form and attempts to integrate.

Award for an integral of the form $\int \frac{1}{8} (A \cos(2y) + B \cos(y) + C) (dy)$ with at least one term integrated correctly. Do not be concerned if they use a different variable such as θ for y .

Special Case:

If they have not used part (a) and instead use the double angle formulae:

$$\sin^2 \alpha = \frac{1}{2} - \frac{1}{2} \cos 2\alpha \quad \text{and} \quad \sin^4 \alpha = \left(\frac{1}{2} - \frac{1}{2} \cos 2\alpha \right)^2 \quad \text{with} \quad \alpha = \frac{1}{2} y$$

In this case they must obtain an exact integral equivalent to $\int \frac{1}{8} (\cos(2y) - 4 \cos(y) + 3) (dy)$ and proceed to integrate at least one term correctly.

A1: Correct integration. May be in terms of another variable such as θ . Ignore π



dm1: Dependent on previous method mark. Finds the required volume using $\pi \int_0^{\frac{8\pi}{5}} x^2 dy$ and applies

their limits to their integral and subtracts the correct way round.

If there are no limits seen substituted, then a correct final answer implies the correct use of limits and the inclusion of π . This is provided they have already achieved an integrated expression of

$\frac{1}{8} \left(\frac{1}{2} \sin(2y) - 4 \sin(y) + 3y \right)$ or with correct limits seen, possibly on their integral.

If their integration is incorrect, then there must be evidence of substituting both limits in each of their terms and subtracting. Allow the omission of subtracting zero provided their integration would produce zero for the lower limit.

A1: awrt 7.3

(c)

M1: Finds the mass of the ornament by multiplying their volume by 0.85

A1ft: Draws an appropriate conclusion about the suitability of the model, comparing their two masses.

If the masses differ by 10% then they must conclude it is a good model.

If the masses differ between 10% and 20% then they may conclude it is either a good model or poor model.

If the masses differ by greater than 20% then they must conclude it is a poor model.

Alternative 1

M1: Finds the volume of the ornament by dividing the mass of 6 grams by 0.85 g/cm^3

A1ft: Draws an appropriate conclusion about the suitability of the model, comparing their two volumes.

If the volumes differ by 10% then they must conclude it is a good model.

If the volumes differ between 10% and 20% then they may conclude it is either a good model or poor model.

If the volumes differ by greater than 20% then they must conclude it is a poor model.

Alternative 2

M1: Finds the density of the ornament by dividing the mass of 6 grams by 7.3 g/cm^3

A1ft: Draws an appropriate conclusion about the suitability of the model, comparing their two densities.

If the densities differ by 10% then they must conclude it is a good model.

If the densities differ between 10% and 20% then they may conclude it is either a good model or poor model.

If the densities differ by greater than 20% then they must conclude it is a poor model.

Question	Scheme	Marks	AOs
5(a)	$(t+2)\frac{dv}{dt} + 3v = k(t+2) - 3 \Rightarrow \frac{dv}{dt} + \frac{3v}{t+2} = k - \frac{3}{t+2}$ $I = e^{\int \frac{3}{t+2} dt} = (t+2)^3$	M1	3.1b
	$\Rightarrow v(t+2)^3 = \int \left(k(t+2)^3 - 3(t+2)^2 \right) dt = \dots$	M1	1.1b
	$v(t+2)^3 = \frac{k}{4}(t+2)^4 - (t+2)^3 (+c)$ $\left[\text{Alt } (t+2)^3 v = \frac{kt^4}{4} + 2kt^3 - t^3 + 6kt^2 - 6t^2 + 8kt - 12t(+c) \right]$	A1	1.1b
	$t = 0, v = 0 \Rightarrow c = 8 - 4k$	M1	3.4
	$v(t+2)^3 = \frac{k}{4}(t+2)^4 - (t+2)^3 + 8 - 4k$ $\Rightarrow v = \frac{k}{4}(t+2) - 1 + \frac{4(2-k)}{(t+2)^3} *$	A1*	2.1
		(5)	
(b)	$v = 4, t = 2 \Rightarrow 4 = k - 1 + \frac{4(2-k)}{64} \Rightarrow k = \dots (5.2)$	M1	3.1b
	$\Rightarrow v = \frac{"5.2"}{4}(5+2) - 1 + \frac{4(2-"5.2")}{(5+2)^3} = \dots$	dM1	3.4
	Velocity is 8.06 ms^{-1} (awrt)	A1	3.2a
		(3)	
(c)	E.g. - The model suggests the velocity increases indefinitely which is unlikely - The raindrop will reach the ground/doesn't fall forever so not valid - The raindrop will reach a terminal velocity after a finite time so not good model	B1	3.5a
		(1)	
(9 marks)			

Notes

(a)

M1: Begins to solve the problem with a correct process to find the integrating factor for the model. Look for an attempt at $e^{\int \frac{A}{t+2} dt} = \dots$ where A is a constant (positive or negative). For attempts at IF by recognition they must achieve form $A(t+2)^3$

M1: Attempts the solution of the differential equation using a correct method with their integrating factor. The dt may be implied. They must attempt the integration as part of the method, though it need not be correct – any changed function will do. So score for

$v \times$ their IF = their attempt at integrating their IF $\times \left(k - \frac{3}{t+2} \right)$ allowing for errors expanding this bracket.

A1: Correct solution. The $+ c$ may be missing for this mark.

M1: Interprets the initial conditions to find the constant of integration – must have a constant of integration for this mark and the correct initial conditions must be clear.

A1*: Obtains the printed answer with no errors and at least one intermediate line between

$v(t+2)^3 = \frac{k}{4}(t+2)^4 - (t+2)^3 + c$ and the stated answer. Allow terms in different order, but the $8 - 4k$ must be factorised.

(b)

M1: Uses the given conditions to establish the value of k in the model

dM1: Uses their value of k and the equation of the model with $t = 5$ to find a value for v . If a value of k has been found accept “ $t = 5 \quad v = \dots$ ” with any value for an attempt.

A1: Correct value **including units** (allow awrt 8.06 ms^{-1}). Accept exact answer $5531/686$

(c)

B1: Evaluates the model by making a suitable comment – see scheme for examples. Accept answers relating to the size of velocity increasing beyond what is reasonable or answers that allude to the situation not being able to continue for ever. Accept if symbols are used, e.g. “as $t \rightarrow \infty, v \rightarrow \infty$ which cannot happen” but must include a comment on validity, not just a statement about what happens (e.g. $t \rightarrow \infty, v \rightarrow \infty$ with no comment).

Do not accept e.g. “ v does not tend to a value” as this is just a statement about what happens. If contrary information is given then award B0.

Some examples of borderline answers seen.

“Not valid as rain is limited to how fast its velocity is therefore can’t be big.” Scores B1

“Due to gravitational force and not huge height between cloud and ground, the model would be inaccurate for large period of time, because the velocity of raindrops would be a big number, which is unlikely.” B1

“Not valid for very large value of t as raindrop will not move that fast.” B0

“Raindrop has very little mass so effect due to gravity is lower. Unlikely to have a 1000 ms^{-1} raindrop. Not very valid model for large values of t ” B1

“The water droplet wouldn’t always follow the model in cases where it hits a surface” B0

“No valid as raindrop will hit ground soon. Speed can be infinitely big which is not plausible.” B1

“Would not be valid as the raindrop likely has already hit the ground.” B1

You may need to use your judgement on whether their comments are acceptable.



Question	Scheme	Marks	AOs
6(a)	$5k(13.6) + 2k(0) + 17(-20) = 0 \Rightarrow k = \dots$	M1	3.3
	$k = 5$	A1	1.1b
		(2)	
(b)	Solves their $25m^2 + 10m + 17 = 0 \Rightarrow m = \dots$	M1	3.1b
	$m = -0.2 \pm 0.8i$	A1	1.1b
	$x = e^{-0.2t} (A \cos 0.8t + B \sin 0.8t)$	A1ft	1.1b
	$t = 0, x = -20 \Rightarrow A = \dots (= -20)$	M1	3.4
	$\frac{dx}{dt} = -0.2e^{-0.2t} (A \cos 0.8t + B \sin 0.8t) + e^{-0.2t} (-0.8A \sin 0.8t + 0.8B \cos 0.8t)$	M1	1.1b
	$t = 0 \frac{dx}{dt} = 0 \Rightarrow -0.2A + 0.8B = 0 \Rightarrow B = \dots (= -5)$	dM1	3.4
	$x = e^{-0.2t} (-20 \cos 0.8t - 5 \sin 0.8t)$ o.e.	A1	1.1b
		(7)	
(c)	Vertical height $= 30 + [e^{-0.2 \times 15} (-20 \cos(0.8 \times 15) - 5 \sin(0.8 \times 15))]$	M1	3.4
	Vertical height = awrt 29.3 m	A1	2.2b
		(2)	
(d)	For example It is unlikely that the rope will remain taut The model predicts the tourist will continue to move up and down, (but in fact they will lose momentum) The tourist is modelled as a particle	B1	3.5b
		(1)	
(12 marks)			
Notes:			
(a) M1: Substitutes $\frac{d^2x}{dt^2} = 13.6$ $\frac{dx}{dt} = 0$ and $x = -20$ into the differential equation to find a value for k . Allow if there are sign slips but must be attempting the values in the correct places. A1: Correct value $k = 5$			
(b) M1: Forms and solves the auxiliary equation. A1: Correct solution to the auxiliary equation (not follow through). A1ft: Correct complementary function for their solutions to their auxiliary equation. (Follow through on distinct real, repeated or complex roots.)			

M1: Uses the information from the model $t = 0 \quad x = -20$ to find a constant or equation linking two constants in their equation.

M1: Differentiates an expression of the form $e^{kt} (A \cos \lambda_1 t + B \sin \lambda_2 t)$ using the product rule to find an expression for the velocity.

dM1: Uses the information from the model, $t = 0 \quad \frac{dx}{dt} = 0$ to find and solve another equation for the constants.

A1: Correct equation for displacement.

(c)

M1: Finds the height above the river by finding the displacement after 15 seconds and adding 30

A1: Vertical height = awrt 29.3 m

(d)

B1: Any suitable comment relating to the given model or the outcomes of it. See scheme for examples. Do not accept just “air resistance has not been considered” as the question does not say this was ignored. However, if a valid consequence of what including air resistance would mean to the model, then the mark may be awarded.