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EDEXCEL A LEVEL FURTHER MATHEMATICS: QUESTIONS BY TOPIC

Proof (1)

Core Pure Mathematics 2 · Topic 1

11 questions · 95 marks · Mark schemes included

Pearson Edexcel A Level Further Mathematics (9FM0), June 2019 - June 2025

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HOW TO USE THIS BOOKLET

Questions are grouped by specification sub-topic and run from the lowest mark tariff to the highest within each one.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme.

Diagrams, graphs, tables and data are printed exactly as they appeared in the original papers.

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QUESTIONS

11 questions · 95 marks · 3 sub-topics



SUB-TOPIC 1.3

INDUCTION: MATRICES

Sub-topic 1.3 · 2 questions · 23 marks · lowest tariff first



3.

$$\mathbf{M} = \begin{pmatrix} 5 & a \\ 0 & 1 \end{pmatrix} \quad \text{where } a \text{ is a constant}$$

(a) Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\mathbf{M}^n = \begin{pmatrix} 3^n & \frac{a}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix} \quad (6)$$

Triangle T has vertices A , B and C .

Triangle T is transformed to triangle T' by the transformation represented by $\mathbf{M}^n$ where $n \in \mathbb{N}$

Given that

- triangle T has an area of 5 cm^2
- triangle T' has an area of 1215 cm^2
- vertex $A(2, -2)$ is transformed to vertex $A'(123, -2)$

(b) determine

- (i) the value of n
- (ii) the value of a

(5)







(Total for Question 3 is 11 marks)



3. Given

$$\mathbf{A} = \begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}$$

(a) prove by mathematical induction that, for $n \in \mathbb{N}$

$$\mathbf{A}^n = \begin{pmatrix} 1 & 5(2^n - 1) \\ 0 & 2^n \end{pmatrix} \quad (6)$$

Given

$$\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) describe fully the single geometrical transformation P represented by the matrix $\mathbf{B}$ (2)

The transformation Q is represented by the matrix $\mathbf{A}^n$

The transformation P followed by the transformation Q is the transformation R , which is represented by the matrix $\mathbf{C}$

(c) Determine $\mathbf{C}$ in terms of n (1)

Given that, for a particular value of n , the transformation R maps the point with coordinates $(27, 1)$ to the point with coordinates (a, a) , where a is a constant,

(d) determine the matrix that represents the transformation Q (3)







SUB-TOPIC 1.4

INDUCTION: DERIVATIVES

Sub-topic 1.4 · 1 question · 6 marks · lowest tariff first



6. Given that

$$y = e^{2x} \sinh x$$

prove by induction that for $n \in \mathbb{N}$

$$\frac{d^n y}{dx^n} = e^{2x} \left(\frac{3^n + 1}{2} \sinh x + \frac{3^n - 1}{2} \cosh x \right)$$

(6)



SUB-TOPIC 1.5

OTHER PROOFS

Sub-topic 1.5 · 8 questions · 66 marks · lowest tariff first



$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix}$$

Prove that, for the linear transformation M , there are no invariant lines.

(5)



(Total for Question 2 is 5 marks)



(Total for Question 5 is 6 marks)



1. (a) Using the definition of $\sinh x$ in terms of exponentials, prove that

$$4\sinh^3 x + 3\sinh x \equiv \sinh 3x \quad (2)$$

- (b) Hence solve the equation

$$\sinh 3x = 19 \sinh x$$

giving your answers as simplified natural logarithms where appropriate. (5)











(Total for Question 9 is 8 marks)





(Total for Question 4 is 10 marks)







(Total for Question 5 is 10 marks)

7.

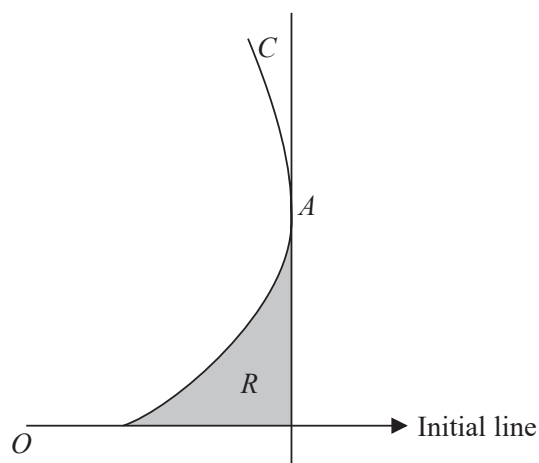


Figure 1

Figure 1 shows a sketch of the curve C with equation

$$r = 1 + \tan \theta \qquad 0 \leq \theta < \frac{\pi}{3}$$

Figure 1 also shows the tangent to C at the point A . This tangent is perpendicular to the initial line.

- (a) Use differentiation to prove that the polar coordinates of A are $\left(2, \frac{\pi}{4}\right)$

The finite region R , shown shaded in Figure 1, is bounded by C , the tangent at A and the initial line.

- (b) Use calculus to show that the exact area of R is $\frac{1}{2}(1 - \ln 2)$ (6)





(Total for Question 7 is 10 marks)



MARK SCHEMES

official mark schemes



Question	Scheme	Marks	AOs
3(a)	$n = 1 \Rightarrow \mathbf{M}^1 = \begin{pmatrix} 3^1 & \frac{a}{2}(3^1 - 1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$ {So the result is true for $n = 1$ }	B1	2.2a
	Assume true for $n = k$ Or assume $\mathbf{M}^n$ or $\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & \frac{a}{2}(3^k - 1) \\ 0 & 1 \end{pmatrix}$	M1	2.4
	A correct method to find an expression for $n = k + 1$ $\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3^k & \frac{a}{2}(3^k - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3^k & \frac{a}{2}(3^k - 1) \\ 0 & 1 \end{pmatrix}$	M1	1.1b
	$\begin{pmatrix} 3(3^k) & a(3^k) + \frac{a}{2}(3^k - 1) \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 3(3^k) & 3 \times \frac{a}{2}(3^k - 1) + a \\ 0 & 1 \end{pmatrix}$	A1	1.1b
	$\begin{pmatrix} 3^{k+1} & \frac{a}{2}[2(3^k) + (3^k - 1)] \\ 0 & 1 \end{pmatrix} =$ $\begin{pmatrix} 3^{k+1} & \frac{a}{2}[3(3^k) - 1] \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3^{k+1} & \frac{a}{2}[3^{k+1} - 1] \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 3(3^k) & 3 \times \frac{a}{2}(3^k - 1) + a \\ 0 & 1 \end{pmatrix} =$ $\begin{pmatrix} 3^{k+1} & \frac{a}{2}(3(3^k - 1) + 2) \\ 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 3^{k+1} & \frac{a}{2}(3^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$	A1	2.1
	If true for $n = k$ then true for $n = k + 1$ and as it is true for $n = 1$ the statement is true for all (positive integers) n	A1	2.4
		(6)	
(b)(i)	$\det(\mathbf{M}^n) = 3^n$ or $\det(\mathbf{M}) = 3$	B1	1.1b
	Uses $5 \times \det(\mathbf{M}^n) = 1215 \Rightarrow p^n = q \Rightarrow n = \dots$ $5 \times 3^n = 1215 \Rightarrow 3^n = 243 \Rightarrow n = \dots$	M1	3.1a
	$n = 5$	A1	1.1b
	$\begin{pmatrix} 3^n & \frac{a}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 123 \\ -2 \end{pmatrix} \Rightarrow 2(3^n) - 2 \frac{a}{2}(3^n - 1) = 123$ $\Rightarrow a = \dots$	M1	1.1b
(ii)			

	$\begin{pmatrix} 243 & \frac{a}{2}(243-1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 123 \\ -2 \end{pmatrix} \Rightarrow 2(243) - 2\frac{a}{2}(243-1) = 123 \Rightarrow a = \dots$ $\frac{1}{243} \begin{pmatrix} 1 & -\frac{a}{2}(243-1) \\ 0 & a \end{pmatrix} \begin{pmatrix} 123 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \Rightarrow \frac{123 - 2\frac{a}{2}(243-1)}{243} = -2 \Rightarrow a = \dots$		
	$a = 1.5$	A1	1.1b
		(5)	
(11 marks)			
Notes:			
(a) B1: Shows that the result holds for $n = 1$. Must see substitution in the RHS minimum required is $\begin{pmatrix} 3 & \frac{a}{2}(3-1) \\ 0 & 1 \end{pmatrix}$ and reaches $\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$ M1: Assumes the result is true for some value of $n = k$. Assume (true for) $n = k$ is sufficient. Alternatively states assume $\mathbf{M}^n$ or $\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}$ M1: Sets up a matrix multiplication of their assumed result multiplied by the original matrix, either way round. Allow a slip as long as the intention is clear. A1: Achieves a correct un-simplified matrix A1: Reaches a correct simplified matrix with no errors, the correct un-simplified matrix seen previously and at least one intermediate line which must be correct. A1: Correct conclusion. This mark is dependent on all previous marks except B mark but $n = 1$ must have been attempted. It is gained by conveying the ideas of all four bold points either at the end of their solution or as a narrative in their solution. Condone $n \in \mathbb{Z}$			
(b)(i) B1: States correct determinant. This can be implied by a correct equation M1: Correct method to find a value of n using $5 \times \text{'their } \det(\mathbf{M}^n) = 1215$ which involves solving an index equation of the form $p^n = q$ where $n > 1$ A1: $n = 5$ (ii) M1: Sets up an equation by multiplying the matrix $\mathbf{M}^n$ by $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ setting equal to $\begin{pmatrix} 123 \\ -2 \end{pmatrix}$ and reaches a value for a. You may just see $2(3^n) - 2\frac{a}{2}(3^n-1) = 123 \Rightarrow a = \dots$ Follow through on their value for n. A1: $a = 1.5$			

Question	Scheme	Marks	AOs
3(a)	$n=1, \text{LHS} = \begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}^1 = \begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}, \text{RHS} = \begin{pmatrix} 1 & 5(2^1-1) \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}$ So the result is true for $n = 1$	B1	2.2a
	Assume true for $n = k, \begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 1 & 5(2^k-1) \\ 0 & 2^k \end{pmatrix}$ then for $n = k+1$ $\begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 5(2^k-1) \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 5(2^k-1) \\ 0 & 2^k \end{pmatrix}$	M1	2.4
	$= \begin{pmatrix} 1 & 5 + 2 \times 5(2^k-1) \\ 0 & 2(2^k) \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 5(2^k-1) + 5(2^k) \\ 0 & 2(2^k) \end{pmatrix}$	M1 A1	1.1b 1.1b
	$\begin{pmatrix} 1 & 10(2^k) - 5 \\ 0 & 2^{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 5[2(2^k) - 1] \\ 0 & 2^{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 5(2^{k+1} - 1) \\ 0 & 2^{k+1} \end{pmatrix}$	A1	2.1
	“If true for $n = k$ then true for $n = k + 1$” and as it is “true for $n = 1$” the statement is “true for all (positive integers) n”	A1	2.4
		(6)	
(b)	Reflection	B1	1.1b
	Reflection in the y-axis or line $x = 0$	B1	1.1b
		(2)	
(c)	$\begin{pmatrix} 1 & 5(2^n-1) \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 5(2^n-1) \\ 0 & 2^n \end{pmatrix}$	B1	1.1b
		(1)	
(d)	$\begin{pmatrix} -1 & 5(2^n-1) \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 27 \\ 1 \end{pmatrix} = \begin{pmatrix} -27 + 5(2^n-1) \\ 2^n \end{pmatrix}$ Sets $-27 + 5(2^n-1) = 2^n \Rightarrow 2^n = \dots$	M1	3.1a
	$(\mathbf{A}^n =) \begin{pmatrix} 1 & 5('8'-1) \\ 0 & '8' \end{pmatrix} = \begin{pmatrix} 1 & 35 \\ 0 & 8 \end{pmatrix}$	M1 A1	1.1b 1.1b
		(3)	
(12 marks)			

Notes:**(a)****B1:** Shows that the result holds for $n = 1$. Must see substitution in the RHS minimum required is
$$\begin{pmatrix} 1 & 5(2-1) \\ 0 & 2 \end{pmatrix} \text{ and reaches } \begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}$$
 No need to state “true for $n = 1$ ” for this mark.

M1: Assumes the result is true for some value of $n = k$, and sets up a matrix multiplication of the assumed result multiplied by the original matrix, either way round (no need to carry out for this mark, it is for essentially explaining the procedure). “Assume (true for) $n = k$ ” (oe) is sufficient for the assumption and may even be part of the conclusion and accept any alternative wording that indicates the assumption has been made. Allow for setting up the multiplication in reverse (ie working from $n = k + 1$ towards $n = k$, $\mathbf{A}^k = \mathbf{A}^{k+1} \mathbf{A}^{-1}$ oe)

M1: Carries out the multiplication. (Allow the reverse case.)**A1:** Achieves a correct un-simplified matrix. Accept in working in reverse.

A1: Reaches a correct simplified matrix with **no errors, the correct un-simplified matrix seen previously and at least one intermediate line which must be correct**. If working from both sides all steps in showing the two sides are equal must be seen. If working in reverse they must return to $\mathbf{A}^{k+1} = \dots$ for this mark.

A1: Correct formal conclusion. This mark is dependent on the M and previous A marks having been scored and an attempt at the check for $n = 1$ (if e.g. they didn’t show sufficient detail). It is gained by conveying the ideas of **all three bold points at the end of their solution**.

Note: Some cases may use $n = 0$ as the base case. These can score full marks if dealt with correctly, but the conclusion must be consistent with their initial check to score the final A.

For the B mark minimum $\begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (or **I**) and $\begin{pmatrix} 1 & 5(1-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ should be seen. (No need to state true for $n = 0$ for this mark – but must have consistent conclusion for the final A as noted above). If unsure send to review.

(b)**B1:** Identifies the transformation as a reflection. Accept stretch with scale factor -1 .

B1: Identifies reflection and the correct line of reflection. Condone phrasing such as “across the y-axis”. Must be a single transformation – B0 if they state something else also happens.

NB Allow B1B0 for answers such as “flip in the y-axis” that convey the correct transformation in imprecise language.

(c)**B1:** Correct matrix**(d)**

M1: A complete method to find a value for 2^n (or n). Multiplies the coordinates $(27, 1)$ by their matrix **C** and sets the x and y coordinates equal to reach a value for 2^n or n (condone negative values for 2^n). Note you may allow this for reach a value of a if $a = 2^n$ is stated or clearly implied by their working.

M1: Uses their value of 2^n or n , to find the matrix $\mathbf{A}^n$. Note substituting into **C** is M0 without further work to find $\mathbf{A}^n$.

A1: Correct matrix.

Question	Scheme	Marks	AOs
6	$\frac{dy}{dx} = 2e^{2x} \sinh x + e^{2x} \cosh x = ae^{2x} \sinh x + be^{2x} \cosh x$ or $e^{2x} (a \sinh x + b \cosh x)$	M1	2.2a
	$\frac{dy}{dx} = e^{2x} (2 \sinh x + \cosh x)$ $n = 1$ then $\frac{dy}{dx} = e^{2x} \left(\frac{3+1}{2} \sinh x + \frac{3-1}{2} \cosh x \right)$ {so the result is true for $n = 1$ }	A1	2.4
	(Assume the result is true for $n = k$, then) Must be an attempt at the product rule, with k 's in all terms $\frac{d^{k+1}y}{dx^{k+1}} = Ae^{2x} (f(k) \sinh x + g(k) \cosh x) + e^{2x} (f(k) \cosh x + g(k) \sinh x)$ $\frac{d^{k+1}y}{dx^{k+1}} = 2e^{2x} \left(\frac{3^k+1}{2} \sinh x + \frac{3^k-1}{2} \cosh x \right) + e^{2x} \left(\frac{3^k+1}{2} \cosh x + \frac{3^k-1}{2} \sinh x \right)$	M1	2.1
	$= e^{2x} \left(\left(3^k + 1 + \frac{3^k-1}{2} \right) \sinh x + \left(3^k - 1 + \frac{3^k+1}{2} \right) \cosh x \right)$ or $= e^{2x} \left(\frac{3 \times 3^k + 1}{2} \sinh x + \frac{3 \times 3^k - 1}{2} \cosh x \right)$	dM1	1.1b
	$= e^{2x} \left(\frac{3^{k+1}+1}{2} \sinh x + \frac{3^{k+1}-1}{2} \cosh x \right)$	A1	2.1
	If true for $n = k$ then true for $n = k + 1$, and as also true for $n = 1$, so the result is true for all positive integers or true $n \in \mathbb{N}$	A1	2.4
		(6)	
	Alternative using exponential definitions $y = e^{2x} \sinh x \Rightarrow y = e^{2x} \left(\frac{e^x - e^{-x}}{2} \right) = \frac{1}{2} (e^{3x} - e^x)$ $\frac{dy}{dx} = \frac{1}{2} (3e^{3x} - e^x)$	M1	2.2a
	$n = 1$ then	A1	2.4

	(Assume the result is true for $n = k$, then) $\frac{d^k y}{dx^k} = e^{2x} \left(\frac{3^k + 1}{2} \left(\frac{e^x - e^{-x}}{2} \right) + \frac{3^k - 1}{2} \left(\frac{e^x + e^{-x}}{2} \right) \right)$ $= e^{2x} \left(\frac{3^k}{2} e^x - \frac{1}{2} e^{-x} \right) = \frac{3^k}{2} e^{3x} - \frac{1}{2} e^x$ Then differentiates $\frac{d^{k+1} y}{dx^{k+1}} = e^{2x} \left(\frac{3^{k+1} + 1}{2} \left(\frac{e^x - e^{-x}}{2} \right) + \frac{3^{k+1} - 1}{2} \left(\frac{e^x + e^{-x}}{2} \right) \right)$	M1	2.1
	Simplifies $\frac{d^{k+1} y}{dx^{k+1}} = e^{2x} \left(\frac{3^{k+1}}{2} e^x - \frac{1}{2} e^{-x} \right) = \frac{3^{k+1}}{2} e^{3x} - \frac{1}{2} e^x$	dM1	1.1b
	Using the given result $\frac{d^{k+1} y}{dx^{k+1}} = e^{2x} \left(\frac{3^{k+1} + 1}{2} \left(\frac{e^x - e^{-x}}{2} \right) + \frac{3^{k+1} - 1}{2} \left(\frac{e^x + e^{-x}}{2} \right) \right)$ $= 3 \times \frac{3^k}{2} e^{3x} - \frac{1}{2} e^x = \frac{3^{k+1}}{2} e^{3x} - \frac{1}{2} e^x$	A1	2.1
	If true for $n = k$ then true for $n = k + 1$, and as also true for $n = 1$, so the result is true for all positive integers or true $n \in \mathbb{N}$	A1	2.4
		(6)	
(6 marks)			
Notes:			
M1: Differentiates to a correct form A1: Correct derivative and reaches appropriate form to deduce the result is true for $n = 1$, minimum $\frac{dy}{dx} = e^{2x} \left(\frac{4}{2} \sinh x + \frac{2}{2} \cosh x \right)$ M1: (Makes the inductive assumption and) attempts the $(k + 1)$ th derivative from the k th derivative. Allow slips in coefficients but must be evidence of the use of the product rule. Must be of the form $\frac{d^{k+1} y}{dx^{k+1}} = Ae^{2x} (f(k) \sinh x + g(k) \cosh x) + e^{2x} (f(k) \cosh x + g(k) \sinh x)$ dM1: Dependent on the previous method mark. Factors out the exponential and gathers the $\sinh x$ and $\cosh x$ terms. Accept either form shown or equivalent.			

A1: Reaches the correct form from correct work. Must have the “ $k+1$ ” showing. Depends on the previous two method marks.

A1: Makes appropriate concluding sentence covering the points indicated in scheme. Depends on all method marks having been scored. Must have reached at least the second line shown in the dM mark, and made an attempt at checking $n=1$ (though the first A mark need not have been scored if insufficient detail shown).

Maybe seen as a narrative throughout their solution.

This mark requires all necessary brackets throughout.

Alternative: using exponentials

M1: Uses the exponential definition of $\sinh x$ and differentiates to a correct form

A1: Correct derivative and uses the exponential definitions of $\sinh x$ and $\cosh x$ to deduce the result is true for $n=1$

M1: (Makes the inductive assumption and) attempts the $(k+1)$ th derivative from the k th derivative. Uses the exponential definition of $\sinh x$ first and then differentiates to a correct form

dM1: Dependent on the previous method mark. Collects exponential terms and simplifies

A1: Reaches the $k+1$ th derivative. Uses the exponential definitions of $\sinh x$ and $\cosh x$ in the result, simplifies and achieves the correct result

A1: Makes appropriate concluding sentence covering the points indicated in scheme. Depends on all method marks having been scored. Must have reached at least the second line shown in the dM mark, and made an attempt at checking $n=1$ (though the first A mark need not have been scored if insufficient detail shown).

Maybe seen as a narrative throughout their solution.

This mark requires all necessary brackets throughout.

Question	Scheme	Marks	AOs
2	$\begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} X \\ mX+c \end{pmatrix}$ leading to an equation in x , m , c and X	M1	3.1a
	$4x - 2(mx + c) = X$ and $5x + 3(mx + c) = mX + c$	A1	1.1b
	$5x + 3(mx + c) = m(4x - 2(mx + c)) + c$ leading to $5 + 3m = 4m - 2m^2$ $(3c = -2mc + c)$	M1	2.1
	$2m^2 - m + 5 = 0 \Rightarrow b^2 - 4ac =$ $(-1)^2 - 4(2)(5) = \dots$	Solves $3c = -2mc + c \Rightarrow m = \dots$ dM1	1.1b
	Correct expression for the discriminant = $\{-39\} < 0$ therefore there are no invariant lines.	$m = -1$ and shows a contradiction in $5 + 3m = 4m - 2m^2$ therefore there are no invariant lines. A1	2.4
Alternative			
	$\begin{pmatrix} 4 & -2 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} X \\ mX \end{pmatrix}$ leading to an equation in x , m and X	M1	3.1a
	$4x - 2(mx) = X$ and $5x + 3(mx) = mX$	A1	1.1b
	$5x + 3(mx) = m(4x - 2(mx))$ leading to $5 + 3m = 4m - 2m^2$	M1	2.1
	$2m^2 - m + 5 = 0 \Rightarrow b^2 - 4ac = (-1)^2 - 4(2)(5) = \dots$	dM1	1.1b
	Correct expression for the discriminant = $\{-39\} < 0$ therefore there are no invariant lines that pass through the origin no invariant lines.	A1	2.4
		(5)	
(5 marks)			
Notes: M1: Sets up a matrix equation in an attempt to find a fixed line and extract at least one equation. A1: Correct equations. M1: Eliminates X from the simultaneous equations and equates the coefficients of x leading to a quadratic equation in terms of m. dM1: Dependent on the previous method, finds the value of the discriminant, this can be seen in an attempt to solve the quadratic using the formula. Alternatively solves $3c = -2mc + c$ and finds a value for m Note: If the quadratic equation in m is solved on a calculator and complex roots given this is M0 as they are not showing why there are no real roots. A1: Correct expression for the discriminant, states < 0 and draws the required conclusion. Alternatively, correct value for m, shows a contradiction in $5 + 3m = 4m - 2m^2$ and draws the required conclusion.			
Alternative M1: Sets up a matrix equation in an attempt to find a fixed line and extract at least one equation. A1: Correct equations.			

M1: Eliminates X from the simultaneous equations and equates the coefficients of x leading to a quadratic equation in terms of m .

dM1: Dependent on the previous method, finds the value of the discriminant.

A1: Correct expression for the discriminant, states < 0 and draws the required conclusion.



Question	Scheme		Marks	AOs
5(a)	$\sin y = x \Rightarrow \cos y \frac{dy}{dx} = 1$	$\sin y = x \Rightarrow \frac{dx}{dy} = \cos y$	M1	1.1b
	Use $\sin^2 y + \cos^2 y = 1 \Rightarrow \cos y = \sqrt{1 - \sin^2 y} \Rightarrow \sqrt{1 - x^2}$		M1	2.1
	$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} * \text{CSO}$		A1*	1.1b
			(3)	
(b)	Using the answer to (a) $f'(x) = \frac{1}{\sqrt{1-e^{2x}}} \times \dots$	Restart $\sin y = e^x \Rightarrow \cos y \frac{dy}{dx} = e^x$	M1	3.1a
	$f'(x) = \frac{1}{\sqrt{1-e^{2x}}} \times e^x$	$f'(x) = \frac{e^x}{\cos y}$	A1	1.1b
	$e^x \neq 0$ (or $e^x > 0$) therefore, there are no stationary points Alternatively, $e^x = 0$ leading to $x = \ln 0$ which is impossible/undefined therefore there are no stationary points.		A1	2.4
			(3)	
(6 marks)				
Notes:				
(a) M1: Finds x in terms of y and differentiates M1: Uses the trig identity $\sin^2 y + \cos^2 y = 1$ to express $\cos y$ in terms of x. This may be seen in their derivative or stated on the side A1*: Correctly achieves the printed answer $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \cdot \text{CSO}$				
(b) M1: Differentiates using the chain rule to achieve the correct form, condone $f'(x) = \frac{1}{\sqrt{1-e^{2x}}}$ Note $f'(x) = \frac{1}{\sqrt{1-e^x}}$ is B0 for incorrect form Alternatively restart, finds x in terms of y and differentiates A1: Correct differentiation A1: Follows correct differentiation. States that as $e^x \neq 0$ (or $e^x > 0$) or no solutions to $e^x = 0$ therefore there are no stationary points. Alternatively, $e^x = 0$ leading to $x = \ln 0$ which is impossible/undefined/error therefore there are no stationary points. Ignore any reference to the denominator = 0				



Question	Scheme	Marks	AOs
1(a)	$4\sinh^3 x + 3\sinh x \equiv 4\left(\frac{e^x - e^{-x}}{2}\right)^3 + 3\left(\frac{e^x - e^{-x}}{2}\right)$ $\equiv 4\left(\frac{e^{3x} - 3e^x + 3e^{-x} - e^{-3x}}{8}\right) + 3\left(\frac{e^x - e^{-x}}{2}\right)$	M1	2.1
	$\equiv \frac{e^{3x} - e^{-3x}}{2} \equiv \sinh 3x^*$	A1*	1.1b
		(2)	
(b)	$\sinh 3x = 19\sinh x \Rightarrow 4\sinh^3 x + 3\sinh x = 19\sinh x$ $\sinh 3x = 19\sinh x \Rightarrow 4\sinh^3 x - 16\sinh x = 0$ $4\sinh x(\sinh^2 x - 4) = 0$	M1	3.1a
	$\sinh x = 0 \Rightarrow x = 0$	B1	2.2a
	$\sinh^2 x = 4 \Rightarrow \sinh x = \pm 2$ $\Rightarrow x = \ln\left(\pm 2 + \sqrt{(\pm 2)^2 + 1}\right)$	M1	1.1b
	$x = \ln(2 + \sqrt{5}) \text{ or } x = \ln(-2 + \sqrt{5}) \text{ oe e.g. } x = -\ln(2 + \sqrt{5})$	A1	1.1b
	$x = \ln(2 + \sqrt{5}) \text{ and } x = \ln(-2 + \sqrt{5})$ $\text{Alternatively, } x = \ln(\sqrt{5} \pm 2) \text{ oe e.g. } x = \pm \ln(2 + \sqrt{5})$ $\text{or } \frac{1}{2} \ln(9 \pm 4\sqrt{5})$	A1	1.1b
		(5)	
(7 marks)			
Notes			
(a) M1: Begins the proof by expressing $\sinh x$ correctly in terms of exponentials, substitutes and makes progress in cubing the bracket. Award for obtaining an expression of the form $Ae^{3x} + Be^x + Ce^{-x} + De^{-3x}$ but terms do not need to be collected. Note that $(e^{2x} - 2 + e^{-2x})(e^x - e^{-x}) = e^{3x} - e^x - 2e^x + 2e^{-x} + e^{-x} - e^{-3x}$ A1*: Fully correct proof with no errors. Must see $= \sinh 3x$ or e.g. LHS = RHS (b) M1: Uses part (a), collects terms and attempts to factorise or cancel $\sinh x$. This can be implied if they go straight from their cubic to writing all correct answers for $\sinh x$ including zero from their calculator. B1: Deduces the root $x = 0$, allow $\ln 1$ but not $\ln[0 + \ln \sqrt{0+1}]$ M1: Proceeds to $\sinh x = \alpha$ and uses the correct logarithmic form of arsinh to obtain at least one exact value for x Alternatively, candidates proceed from $\sinh x = \alpha$ to substitute the exponential form and then solve a 3TQ in e^x to obtain at least one exact value for x A1: One correct non-zero solution A1: Both correct non-zero solutions and no incorrect other solutions, but isw if they go on to evaluate these answers incorrectly. Allow $x = \ln(\sqrt{5} \pm 2)$ for listing both solutions			

Alternative if candidates substitute the exponential form at the start:

M1: Candidates substitutes the exponential form for each term and proceeds to find a four term cubic equation in $e^{2x} = 0$, which may not be correct.

B1: Deduces the root $x = 0$, allow $\ln 1$ but not $\ln[0 + \ln \sqrt{0+1}]$

M1: They factorise their cubic, proceed to obtain exact values for e^{2x} , then take logs to obtain at least one exact value for x

If they go directly to decimal answers this will usually score M0 unless they recover to exact form.

A1: One correct non-zero solution

A1: Both correct non-zero solutions and no incorrect other solutions, but isw if they go on to evaluate these answers incorrectly.

This is how this would be awarded:

$$\frac{(e^{3x} - e^{-3x})}{2} = 19 \left(\frac{e^x - e^{-x}}{2} \right)$$

$$e^{3x} - e^{-3x} - 19e^x + 19e^{-x} = 0$$

$$e^{6x} - 1 - 19e^{4x} + 19e^{2x} = 0$$

M1 (for cubic in $e^{2x} = 0$)

$$e^{6x} + 19e^{4x} + 19e^{2x} - 1 = 0$$

$$\Rightarrow (e^{2x} - 1)(e^{4x} - 18e^{2x} + 1) = 0$$

$$e^{2x} = 1, 9 \pm 4\sqrt{5}$$

$$2x = \ln 1, 2x = \ln(9 \pm 4\sqrt{5})$$

$$x = 0, x = \frac{1}{2} \ln(9 \pm 4\sqrt{5})$$

B1, M1, A1, A1

Question	Scheme	Marks	AOs
9(a)	$\frac{1}{1-z}$	B1	2.2a
		(1)	
(b)(i)	$1+z+z^2+z^3+\dots$ $=1+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)^3+\dots$ $=1+\frac{1}{2}(\cos\theta+i\sin\theta)+\frac{1}{4}(\cos 2\theta+i\sin 2\theta)+\frac{1}{8}(\cos 3\theta+i\sin 3\theta)+\dots$	M1	3.1a
	$\frac{1}{1-z} = \frac{1}{1-\frac{1}{2}(\cos\theta+i\sin\theta)} \times \frac{1-\frac{1}{2}\cos\theta+\frac{1}{2}i\sin\theta}{1-\frac{1}{2}\cos\theta+\frac{1}{2}i\sin\theta}$ or $\frac{1}{1-z} = \frac{2}{2-(\cos\theta+i\sin\theta)} \times \frac{2-(\cos\theta-i\sin\theta)}{2-(\cos\theta-i\sin\theta)}$	M1	3.1a
	$\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin 2\theta)+\frac{1}{8}(\sin 3\theta)+\dots\right\} = \frac{\frac{1}{2}\sin\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2}$ or $\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin 2\theta)+\frac{1}{8}(\sin 3\theta)+\dots\right\} = \frac{2\sin\theta}{(2-\cos\theta)^2+(\sin\theta)^2}$	M1	2.1
	$\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2 = 1-\cos\theta+\frac{1}{4}\cos^2\theta+\frac{1}{4}\sin^2\theta$ $=\frac{5}{4}-\cos\theta$ or $(2-\cos\theta)^2+(\sin\theta)^2 = 4-4\cos\theta+\cos^2\theta+\sin^2\theta$ $=5-4\cos\theta$	M1	1.1b
	$\frac{1}{2}\sin\theta+\frac{1}{4}\sin 2\theta+\frac{1}{8}\sin 3\theta+\dots = \frac{\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta} = \frac{2\sin\theta}{5-4\cos\theta}^*$	A1*	1.1b
	Alternative $1+z+z^2+z^3+\dots$ $=1+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)^3+\dots$ $=1+\frac{1}{2}(\cos\theta+i\sin\theta)+\frac{1}{4}(\cos 2\theta+i\sin 2\theta)+\frac{1}{8}(\cos 3\theta+i\sin 3\theta)+\dots$	M1	3.1a

	$\frac{1}{1-z} = \frac{1}{1-\frac{1}{2}e^{i\theta}} \times \frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{2}e^{-i\theta}}$	M1	3.1a
	$\frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{4}e^{i\theta}-\frac{1}{4}e^{-i\theta}+\frac{1}{4}} = \frac{4-2e^{-i\theta}}{5-2(e^{i\theta}+e^{-i\theta})} = \frac{4-2(\cos\theta-i\sin\theta)}{5-2(2\cos\theta)}$	M1	2.1
	Select the imaginary part $\frac{2\sin\theta}{5-4\cos\theta}$	M1	1.1b
	$\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \dots = \frac{2\sin\theta}{5-4\cos\theta}^*$	A1*	1.1b
		(5)	
(b)(ii)	$\frac{1-\frac{1}{2}\cos\theta}{\frac{5}{4}-\cos\theta} = 0 \Rightarrow \cos\theta = 2$	M1	3.1a
	As $(-1 \leq) \cos\theta \leq 1$ therefore there is no solution to $\cos\theta = 2$ so there will also be a real part, hence the sum cannot be purely imaginary.	A1	2.4
	Alternative 1 Imaginary part is $\frac{4-2\cos\theta}{5-4\cos\theta} = \frac{1}{2} + \frac{3}{2(5-4\cos\theta)}$	M1	3.1a
	$-1 \leq \cos\theta \leq 1$ therefore $\frac{1}{6} < \frac{3}{2(5-4\cos\theta)} < \frac{3}{2}$ so sum must contain real part	A1	2.4
	Alternative 2 $\frac{1}{1-z} = ki \Rightarrow z = 1 + \frac{i}{k}$	M1	3.1a
	mod $ z > 1$ contradiction hence cannot be purely imaginary	A1	2.4
		(2)	
(8 marks)			
Notes:			
(a) B1: See scheme			
(b)(i) M1: Substitutes $z = \frac{1}{2}(\cos\theta + i\sin\theta)$ into at least 3 terms of the series and applies de Moivre's theorem. M1: Substitutes $z = \frac{1}{2}(\cos\theta + i\sin\theta)$ into their answer to part (a) and rationalises the denominator. M1: Equates the imaginary terms. M1: Multiplies out the denominator and simplifies by using the identity $\cos^2\theta + \sin^2\theta = 1$			

A1*: cso. Achieves the printed answer having substituted $z = \frac{1}{2}(\cos \theta + i \sin \theta)$ into 4 terms of the series.

Alternative

M1: Substitutes $z = \frac{1}{2}(\cos \theta + i \sin \theta)$ into at least 3 terms of the series and applies de Moivre's theorem.

M1: Substitutes $z = \frac{1}{2}e^{i\theta}$ into their answer to part (a) and rationalises the denominator.

M1: Uses $e^{-i\theta} = \cos \theta - i \sin \theta$ and $e^{i\theta} + e^{-i\theta} = 2 \cos \theta$ to express in terms of $\sin \theta$ and $\cos \theta$

M1: Select the imaginary terms.

A1*: cso Achieves the printed answer having substituted $z = \frac{1}{2}(\cos \theta + i \sin \theta)$ into 4 terms of the series.

(b)(ii)

M1: Setting the real part of the series = 0 and rearranges to find $\cos \theta = \dots$

A1: See scheme

Alternative 1

M1: Rearranges imaginary part so that $\cos \theta$ only appears once

A1: Uses $-1 \leq \cos \theta \leq 1$ to show that the sum must always be positive so must contain a real part

Alternative 2

M1: Sets sum as purely imaginary and rearranges to make z the subject

A1: Shows a contradiction and draws an appropriate conclusion

Question	Scheme	Marks	AOs
1(a)	$y = \tanh^{-1}(x) \Rightarrow \tanh y = x \Rightarrow x = \frac{\sinh y}{\cosh y} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$	M1 A1	2.1 1.1b
	Note that some candidates only have one variable and reach e.g. $x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ or } \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ Allow this to score M1A1		
	$x(e^{2y} + 1) = e^{2y} - 1 \Rightarrow e^{2y}(1 - x) = 1 + x \Rightarrow e^{2y} = \frac{1+x}{1-x}$	M1	1.1b
	$e^{2y} = \frac{1+x}{1-x} \Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right) \Rightarrow y = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)^*$	A1*	2.1
	Note that $e^{2y}(x-1) + x + 1 = 0$ can be solved as a quadratic in e^y : $e^y = \frac{-\sqrt{0-4(x-1)(x+1)}}{2(x-1)} = \frac{-\sqrt{4(1-x)(x+1)}}{2(x-1)} = \frac{2\sqrt{(1-x)(x+1)}}{2(1-x)}$ $= \frac{\sqrt{(x+1)}}{\sqrt{(1-x)}} \Rightarrow y = \frac{1}{2} \ln\left(\frac{x+1}{1-x}\right)^*$ Score M1 for an attempt at the quadratic formula to make e^y the subject (condone $\pm \sqrt{\dots}$) and A1* for a correct solution that rejects the positive root at some point and deals with the $(x-1)$ bracket correctly		
	$k = 1 \text{ or } -1 < x < 1$	B1	1.1b
		(5)	
(a) Way 2	$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \Rightarrow x = \tanh\left(\frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)\right) = \frac{e^{\frac{\ln(1+x)}{1-x}} - 1}{e^{\frac{\ln(1+x)}{1-x}} + 1}$	M1 A1	2.1 1.1b
	$x = \frac{e^{\frac{\ln(1+x)}{1-x}} - 1}{e^{\frac{\ln(1+x)}{1-x}} + 1} = \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} = x$ Hence true, QED, tick etc.	M1 A1	1.1b 2.1
(b)	$2x = \tanh(\ln \sqrt{2-3x}) \Rightarrow \tanh^{-1}(2x) = \ln \sqrt{2-3x}$	M1	3.1a
	$\frac{1}{2} \ln\left(\frac{1+2x}{1-2x}\right) = \frac{1}{2} \ln(2-3x) \Rightarrow \frac{1+2x}{1-2x} = 2-3x$	M1	2.1
	$6x^2 - 9x + 1 = 0$	A1	1.1b
	$6x^2 - 9x + 1 = 0 \Rightarrow x = \dots$	M1	1.1b
	$x = \frac{9 - \sqrt{57}}{12}$	A1	3.2a
		(5)	

Alternative for first 2 marks of (b)			
	$2x = \tanh\left(\ln\sqrt{2-3x}\right) \Rightarrow 2x = \frac{e^{2\ln\sqrt{2-3x}} - 1}{e^{2\ln\sqrt{2-3x}} + 1}$	M1	3.1a
	$\Rightarrow \frac{2-3x-1}{2-3x+1} = 2x$	M1	2.1
(10 marks)			
Notes			
(a) If you come across any attempts to use calculus to prove the result – send to review M1: Begins the proof by expressing tanh in terms of exponentials and forms an equation in exponentials. The exponential form can be any of $\frac{(e^y - e^{-y})/2}{(e^y + e^{-y})/2}, \frac{e^y - e^{-y}}{e^y + e^{-y}}, \frac{e^{2y} - 1}{e^{2y} + 1}$ Allow any variables to be used but the final answer must be in terms of x. Allow alternative notation for $\tanh^{-1}x$ e.g. artanh, arctanh. A1: Correct expression for “x” in terms of exponentials M1: Full method to make e^{2y} the subject of the formula. This must be correct algebra so allow sign errors only. A1*: Completes the proof by using logs correctly and reaches the printed answer with no errors. Allow e.g. $\frac{1}{2}\ln\left(\frac{x+1}{1-x}\right), \frac{1}{2}\ln\frac{x+1}{1-x}, \frac{1}{2}\ln\left \frac{x+1}{1-x}\right$. Need to see $\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ as a conclusion but allow if the proof concludes that $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ with y defined as $\tanh^{-1}x$ earlier. B1: Correct value for k or writes $-1 < x < 1$ Way 2 M1: Starts with result, takes tanh of both sides and expresses in terms of exponentials A1: Correct expression M1: Eliminates exponentials and logs and simplifies A1: Correct result (i.e. $x = x$) with conclusion B1: Correct value for k or writes $-1 < x < 1$ (b) M1: Adopts a correct strategy by taking $\tanh^{-1}$ of both sides M1: Makes the link with part (a) by replacing artanh(2x) with $\frac{1}{2}\ln\left(\frac{1+2x}{1-2x}\right)$ and demonstrates the use of the power law of logs to obtain an equation with logs removed correctly. A1: Obtains the correct 3TQ M1: Solves their 3TQ using a correct method (see General Guidance – if no working is shown (calculator) and the roots are correct for their quadratic, allow M1) A1: Correct value with the other solution rejected (accept rejection by omission) so $x = \frac{9 \pm \sqrt{57}}{12}$ scores A0 unless the positive root is rejected Alternative for first 2 marks of (b) M1: Adopts a correct strategy by expressing tanh in terms of exponentials M1: Demonstrates the use of the power law of logs to obtain an equation with logs removed correctly			

Question	Scheme	Marks	AOs
4(a)	$(\cos \theta + i \sin \theta)^7 = \cos^7 \theta + \binom{7}{1} \cos^6 \theta (i \sin \theta) + \binom{7}{2} \cos^5 \theta (i \sin \theta)^2 + \dots$ Some simplification may be done at this stage e.g. $c^7 + 7c^6 is - 21c^5 s^2 - 35c^4 is^3 + 35c^3 s^4 + 21c^2 is^5 - 7cs^6 - is^7$	M1	1.1b
	$i \sin 7\theta = {}^7C_1 c^6 is + {}^7C_3 c^4 i^3 s^3 + {}^7C_5 c^2 i^5 s^5 + i^7 s^7$ or $= 7c^6 is + 35c^4 i^3 s^3 + 21c^2 i^5 s^5 + i^7 s^7$	M1	2.1
	$\sin 7\theta = 7c^6 s - 35c^4 s^3 + 21c^2 s^5 - s^7$	A1	1.1b
	$= 7(1-s^2)^3 s - 35(1-s^2)^2 s^3 + 21(1-s^2)s^5 - s^7$ $= 7(1-3s^2+3s^4-s^6)s - 35(1-2s^2+s^4)s^3 + 21(1-s^2)s^5 - s^7$	M1	2.1
	$\{7s - 21s^3 + 21s^5 - 7s^7 - 35s^3 + 70s^5 - 35s^7 + 21s^5 - 21s^7 - s^7\}$ leading to $\sin 7\theta = 7 \sin \theta - 56 \sin^3 \theta + 112 \sin^5 \theta - 64 \sin^7 \theta *$	A1*	1.1b
		(5)	
(b)	$1 + \sin 7\theta = 0 \Rightarrow \sin 7\theta = -1$	M1	3.1a
	$7\theta = -450, -90, 270, 630, \dots$ or $7\theta = -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \dots$	A1	1.1b
	$\theta = -\frac{450}{7}, -\frac{90}{7}, \frac{270}{7}, \frac{630}{7}, \dots \Rightarrow \sin \theta = \dots$ or $\theta = -\frac{5\pi}{14}, -\frac{\pi}{14}, \frac{3\pi}{14}, \frac{7\pi}{14}, \dots \Rightarrow \sin \theta = \dots$	M1	2.2a
	$x = \sin \theta = -0.901, -0.223, 0.623, 1$	A1	1.1b
		A1	2.3
		(5)	
(10 marks)			
Notes			
(a) M1: Attempts to expand $(\cos \theta + i \sin \theta)^7$ including a recognisable attempt at binomial coefficients Some simplification may be done at this stage. (May only see imaginary terms) M1: Identifies imaginary terms with $\sin 7\theta$ A1: Correct expression with coefficients evaluated and i's dealt with correctly M1: Replaces $\cos^2 \theta$ with $1 - \sin^2 \theta$ and applies the expansions of $(1 - \sin^2 \theta)^2$ and $(1 - \sin^2 \theta)^3$ to their expression A1*: Reaches the printed answer with no errors and expansion of brackets seen. (b) M1: Makes the connection with part (a) and realises the need to solve $\sin 7\theta = -1$ A1: At least one correct value for 7θ M1: Divides by 7 and deduces that x values are found by finding at least one value for $\sin \theta$ A1: Awrt 2 correct values for x A1: Awrt all 4 x values correct and no extras			

Question	Scheme	Marks	AOs
5(a)	$y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow \frac{dx}{dy} = \sec^2 y$ $y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow \frac{dy}{dx} \sec^2 y = 1$	M1	3.1a
	$\frac{dx}{dy} = 1 + \tan^2 y \text{ or } \frac{dy}{dx} (1 + \tan^2 y) = 1$	M1	1.1b
	$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}^*$	A1*	2.1
		(3)	
(b)	$\frac{d(\tan^{-1} 4x)}{dx} = \frac{4}{1 + 16x^2}$	B1	1.1b
	$\int x \tan^{-1} 4x \, dx = \alpha x^2 \tan^{-1} 4x - \int \alpha x^2 \times \frac{4}{1 + 16x^2} \, dx$	M1	2.1
	$\int x \tan^{-1} 4x \, dx = \frac{x^2}{2} \tan^{-1} 4x - \int \frac{x^2}{2} \times \frac{4}{1 + 16x^2} \, dx$	A1	1.1b
	$= \dots - \frac{1}{8} \int \frac{16x^2 + 1 - 1}{1 + 16x^2} \, dx = \dots - \frac{1}{8} \int \left(1 - \frac{1}{1 + 16x^2} \right) \, dx$ or $\text{let } 4x = \tan u \Rightarrow \frac{1}{8} \int \frac{\tan^2 u}{1 + \tan^2 u} \cdot \frac{1}{4} \sec^2 u \, du$ $\Rightarrow \frac{1}{32} \int \tan^2 u \, du = \frac{1}{32} \int \sec^2 u - u \, du$	M1	3.1a
	$= \frac{x^2}{2} \tan^{-1} 4x - \frac{1}{8} x + \frac{1}{32} \tan^{-1} 4x + k$	A1	2.1
		(5)	
(c)	$\text{Mean value} = \left(\frac{1}{\frac{\sqrt{3}}{4} - 0} \right) \left[\frac{x^2}{2} \tan^{-1} 4x - \frac{1}{8} x + \frac{1}{32} \tan^{-1} 4x \right]_0^{\frac{\sqrt{3}}{4}}$ $= \frac{4}{\sqrt{3}} \left(\left(\frac{3}{32} \times \frac{\pi}{3} - \frac{1}{8} \times \frac{\sqrt{3}}{4} + \frac{1}{32} \times \frac{\pi}{3} \right) - 0 \right)$	M1	2.1
	$= \frac{\sqrt{3}}{72} (4\pi - 3\sqrt{3}) \text{ or } \frac{\sqrt{3}}{18} \pi - \frac{1}{8} \text{ oe}$	A1	1.1b
		(2)	
(10 marks)			
Notes			
(a) M1: Makes progress in establishing the derivative by taking the tan of both sides and differentiating with respect to y or implicitly with respect to x M1: Use of the correct identity A1*: Fully correct proof			

(b)

B1: Correct derivative

M1: Uses integration by parts in the correct direction

A1: Correct expression

M1: Adopts a correct strategy for the integration by splitting into two fractions or using a substitution of $4x = \tan u$ to get to an integrable form

A1: Correct answer

(c)

M1: Correctly applies the method for the mean value for their integration. The limit of zero can be implied if it comes to 0.

A1: Correct exact answer. Allow exact equivalents e.g. $\frac{4\pi\sqrt{3}-9}{72}$, $\frac{\pi\sqrt{3}}{18} - \frac{1}{8}$



Question	Scheme	Marks	AOs
7(a)	$x = r \cos \theta = (1 + \tan \theta) \cos \theta = \cos \theta + \sin \theta$ $= \cos \theta + \tan \theta \cos \theta$ $\frac{dx}{d\theta} = \alpha(1 + \tan \theta) \sin \theta + \beta \sec^2 \theta \cos \theta \quad \text{or} \quad \frac{dx}{d\theta} = \alpha \sin \theta + \beta \cos \theta$ $\frac{dx}{d\theta} = \alpha \sin \theta + \beta \sec^2 \theta \cos \theta + \delta \tan \theta \sin \theta$	M1	3.1a
	$\frac{dx}{d\theta} = -(1 + \tan \theta) \sin \theta + \sec^2 \theta \cos \theta \quad \text{or} \quad \frac{dx}{d\theta} = -\sin \theta + \cos \theta$ $\frac{dx}{d\theta} = -\sin \theta + \sec^2 \theta \cos \theta - \tan \theta \sin \theta \quad \text{or} \quad \frac{dx}{d\theta} = -\sin \theta + \sec \theta - \tan \theta \sin \theta$	A1	1.1b
	For example $\left\{ \frac{dx}{d\theta} = \right\} -\sin \theta + \cos \theta = 0 \Rightarrow \tan \theta = 1 \Rightarrow \theta = \dots$ $\left\{ \frac{dx}{d\theta} = \right\} -\sin \theta + \cos \theta = 0 \Rightarrow \sin \theta = \cos \theta \Rightarrow \theta = \dots$ $\left\{ \frac{dx}{d\theta} = \right\} -\sin \theta + \cos \theta = \sqrt{2} \cos \left(\theta + \frac{\pi}{4} \right) = \theta = \dots$ or $\left\{ \frac{dx}{d\theta} = \right\} - (1 + \tan \theta) \sin \theta + \sec^2 \theta \cos \theta = 0$ $\Rightarrow -\sin \theta - \frac{\sin^2 \theta}{\cos \theta} + \frac{1}{\cos \theta} = 0 \Rightarrow -\sin \theta + \frac{1 - \sin^2 \theta}{\cos \theta} = 0$ $\Rightarrow -\sin \theta + \cos \theta = 0 \Rightarrow \tan \theta = 1 \Rightarrow \theta = \dots$ or $\left\{ \frac{dx}{d\theta} = \right\} -\sin \theta - \tan \theta \sin \theta + \sec \theta = 0$ $\Rightarrow -\frac{1}{2} \sin 2\theta - \sin^2 \theta + 1 = 0 \Rightarrow \sin 2\theta + 2 \sin^2 \theta - 1 = 1$ $\Rightarrow \sin 2\theta - \cos 2\theta = 1 \Rightarrow \sqrt{2} \sin \left(2\theta - \frac{\pi}{4} \right) = 1 \Rightarrow \theta = \dots$ or $\left\{ \frac{dx}{d\theta} = \right\} -\sin \left(\frac{\pi}{4} \right) + \cos \left(\frac{\pi}{4} \right) = 0$ $\left\{ \frac{dx}{d\theta} = \right\} - \left(1 + \tan \left(\frac{\pi}{4} \right) \right) \sin \left(\frac{\pi}{4} \right) + \sec^2 \left(\frac{\pi}{4} \right) \cos \left(\frac{\pi}{4} \right) = 0$ $\left\{ \frac{dx}{d\theta} = \right\} -\sin \left(\frac{\pi}{4} \right) + \sec^2 \left(\frac{\pi}{4} \right) \cos \left(\frac{\pi}{4} \right) - \tan \left(\frac{\pi}{4} \right) \sin \left(\frac{\pi}{4} \right) = 0$	dM1	3.1a
	$r = 1 + \tan \left(\frac{\pi}{4} \right) = 2 \quad \text{therefore } A \left(2, \frac{\pi}{4} \right)^*$	A1*	2.1
		(4)	
	Area bounded by the curve $= \frac{1}{2} \int (1 + \tan \theta)^2 \{d\theta\}$	M1	3.1a

(b)	$= \frac{1}{2} \int (1 + 2 \tan \theta + \tan^2 \theta) \{d\theta\}$ $= \frac{1}{2} \int (1 + 2 \tan \theta + [\sec^2 \theta - 1]) \{d\theta\} = \dots$		
	$= \frac{1}{2} [2 \ln \sec \theta + \tan \theta] \text{ or } \ln \sec \theta + \frac{1}{2} \tan \theta \text{ or } -\ln \cos \theta + \frac{1}{2} \tan \theta \text{ or } = \frac{1}{2} [-2 \ln \cos \theta + \tan \theta]$	A1	1.1b
	$= \frac{1}{2} \left[2 \ln \left \sec \left(\frac{\pi}{4} \right) \right + \tan \left(\frac{\pi}{4} \right) \right] - \frac{1}{2} [2 \ln \sec(0) + \tan(0)]$ $= \left(\ln \left \sec \left(\frac{\pi}{4} \right) \right + \frac{1}{2} \tan \left(\frac{\pi}{4} \right) \right) - \left(\ln \sec 0 + \frac{1}{2} \tan 0 \right)$ $\left\{ = \ln \sqrt{2} + \frac{1}{2} \right\}$	dM1	1.1b
	Area of triangle $= \frac{1}{2} xy = \frac{1}{2} \left(2 \cos \frac{\pi}{4} \right) \left(2 \sin \frac{\pi}{4} \right) = \dots \left\{ \frac{1}{2} \times \sqrt{2} \times \sqrt{2} = 1 \right\}$ The equation of the tangent is $r = \sqrt{2} \sec \theta$ then applies Area bounded of triangle $= \frac{1}{2} \int_0^{\frac{\pi}{4}} (\sqrt{2} \sec \theta)^2 \{d\theta\}$	M1	1.1b
	Finds the required area = area of triangle – area bounded by the curve $= 1 - \left[\ln \sqrt{2} + \frac{1}{2} \right]$ May be seen within an integral $= \frac{1}{2} \int (\sqrt{2} \sec \theta)^2 \{d\theta\} - \frac{1}{2} \int (1 + \tan \theta)^2 \{d\theta\}$	M1	3.1a
	$= \frac{1}{2} (1 - \ln 2) \text{ * cso}$	A1*	2.1
		(6)	
	Alternative Area bounded by the curve $= \frac{1}{2} \int (1 + \tan \theta)^2 \{d\theta\}$ $= \frac{1}{2} \int (1 + 2 \tan \theta + \tan^2 \theta) \{d\theta\} \text{ let } u = \tan \theta \Rightarrow \frac{du}{d\theta} = \sec^2 \theta$ Leading to $= \frac{1}{2} \int \frac{(1 + 2u + u^2)}{1 + u^2} \{du\} = \frac{1}{2} \int 1 + \frac{2u}{1 + u^2} \{du\} = \dots$	M1	3.1a
	$\frac{1}{2} [u + \ln(1 + u^2)]$	A1	1.1b
	$\frac{1}{2} [(1 + \ln(1 + (1)^2)) - (0 + \ln 1)] \text{ or } \frac{1}{2} \left[\left(\tan \left(\frac{\pi}{4} \right) + \ln \left(1 + \tan^2 \left(\frac{\pi}{4} \right) \right) \right) - (\tan(0) + \ln(1 + \tan^2(0))) \right]$ $\left\{ = \frac{1}{2} \ln 2 + \frac{1}{2} \right\}$	dM1	1.1b
	Area of triangle $= \frac{1}{2} xy = \frac{1}{2} \left(2 \cos \frac{\pi}{4} \right) \left(2 \sin \frac{\pi}{4} \right) = \dots \left\{ \frac{1}{2} \times \sqrt{2} \times \sqrt{2} = 1 \right\}$	M1	1.1b

	Finds the required area = area of triangle – area bounded by the curve $= 1 - \left[\ln \sqrt{2} + \frac{1}{2} \right]$	M1	3.1a
	$= \frac{1}{2}(1 - \ln 2) *$	A1*	2.1
		(6)	
(10 marks)			
Notes:			
(a) M1: Substitutes the equation of C into $x = r \cos \theta$ and differentiates to the required form A1: Fully correct differentiation dm1: Dependent on previous method mark. Sets their $\frac{dx}{d\theta} = 0$ and uses correct trig identities to find a value for θ. Alternatively substitutes $\theta = \frac{\pi}{4}$ into their $\frac{dx}{d\theta}$ and shows equals 0. A1*: Shows that $r = 2$ and hence the polar coordinates $\left(2, \frac{\pi}{4}\right)$ from correct working			
(b) M1: Applies area $= \frac{1}{2} \int r^2 \theta \, d\theta$, multiplies out, uses the identity $\pm 1 \pm \tan^2 \theta = \sec^2 \theta$ to get into an integrable form and integrates. Condone missing $d\theta$, limits are not required for this mark A1: Correct integration. Note may include $\theta - \theta$ if the one's were not cancelled earlier. dm1: Dependent on the first method mark. Applies the limits of $\theta = 0$ and $\theta = \frac{\pi}{4}$ and subtracts the correct way round. Since substitution of the limit $\theta = 0$ is 0 so may be implied M1: Correct method to find the area of triangle seen. This may be minimal but area = 1 only is M0, they need to show some method. M1: Finds the required area = area of triangle – area bounded by the curve A1*: Correct answer, with no errors or omissions. cso Alternative M1: Applies area $= \frac{1}{2} \int r^2 \theta \, d\theta$, multiplies out, uses the substitution $u = \tan \theta$ to get into an integrable form and integrates. Limits are not required for this mark A1: Correct integration dm1: Dependent on the first method mark. Applies the limits of $u = 0$ and $u = 1$ or substitutes back using $u = \tan \theta$ and uses the limits $\theta = 0$ and $\theta = \frac{\pi}{4}$ and subtracts the correct way round. Since substitution of the limit $\theta = 0$ is 0 so may be implied M1: Correct method to find the area of triangle M1: Finds the required area = area of triangle – area bounded by the curve A1*: Correct answer, with no errors or omissions. cso			