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EDEXCEL A LEVEL FURTHER MATHEMATICS: QUESTIONS BY TOPIC

Complex Numbers (2)

Core Pure Mathematics 2 · Topic 2

11 questions · 93 marks · Mark schemes included

Pearson Edexcel A Level Further Mathematics (9FM0), June 2019 - June 2025

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HOW TO USE THIS BOOKLET

Questions are grouped by specification sub-topic and run from the lowest mark tariff to the highest within each one.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme.

Diagrams, graphs, tables and data are printed exactly as they appeared in the original papers.

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QUESTIONS

11 questions · 93 marks · 5 sub-topics



SUB-TOPIC 2.6

MULTIPLYING AND DIVIDING IN MODULUS-ARGUMENT FORM

Sub-topic 2.6 · 2 questions · 9 marks · lowest tariff first



1. A student was asked to answer the following:

For the complex numbers $z_1 = 3 - 3i$ and $z_2 = \sqrt{3} + i$, find the value of $\arg\left(\frac{z_1}{z_2}\right)$

The student's attempt is shown below.

Line 1 $\longrightarrow$ $\arg(z_1) = \tan^{-1}\left(\frac{3}{3}\right) = \frac{\pi}{4}$

Line 2 $\longrightarrow$ $\arg(z_2) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Line 3 $\longrightarrow$ $\arg\left(\frac{z_1}{z_2}\right) = \frac{\arg(z_1)}{\arg(z_2)}$

Line 4 $\longrightarrow$ $= \frac{\left(\frac{\pi}{4}\right)}{\left(\frac{\pi}{6}\right)} = \frac{3}{2}$

The student made errors in line 1 and line 3

Correct the error that the student made in

- (a) (i) line 1

- (ii) line 3

(2)

- (b) Write down the correct value of $\arg\left(\frac{z_1}{z_2}\right)$

(1)



(Total for Question 1 is 3 marks)



In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.

$$z = 2 - 2\sqrt{3}i \text{ and } w = -1 + \sqrt{3}i$$
$$(a) \quad \left| \frac{z}{w} \right| = \frac{|z|}{|w|} \quad (3)$$

$$(b) \quad \arg(zw) = \arg(z) + \arg(w) \tag{3}$$



SUB-TOPIC 2.7

LOCI IN THE ARGAND DIAGRAM

Sub-topic 2.7 · 3 questions · 28 marks · lowest tariff first







(Total for Question 2 is 7 marks)









Question 4 continued



Question 4 continued



SUB-TOPIC 2.8

DE MOIVRE'S THEOREM

Sub-topic 2.8 · 2 questions · 18 marks · lowest tariff first















(Total for Question 4 is 10 marks)

SUB-TOPIC 2.10

NTH ROOTS OF COMPLEX NUMBERS

Sub-topic 2.10 · 1 question · 9 marks · lowest tariff first



Question 7 continued



Question 7 continued

Question 7 continued

(Total for Question 7 is 9 marks)



SUB-TOPIC 2.11

ROOTS OF UNITY AND GEOMETRIC PROBLEMS

Sub-topic 2.11 · 3 questions · 29 marks · lowest tariff first



Question 5 continued

Question 5 continued

(Total for Question 5 is 9 marks)



6. In an Argand diagram, the points A , B and C are the vertices of an equilateral triangle with its centre at the origin. The point A represents the complex number $6 + 2i$.
- (a) Find the complex numbers represented by the points B and C , giving your answers in the form $x + iy$, where x and y are real and exact.

(6)

The points D, E and F are the midpoints of the sides of triangle ABC .

- (b) Find the exact area of triangle DEF .

(3)

Question 6 continued



Question 6 continued

(Total for Question 6 is 9 marks)



Question 8 continued



Question 8 continued



(Total for Question 8 is 11 marks)



MARK SCHEMES

official mark schemes



Question	Scheme	Marks	AOs
1(a) (i) (a) (ii)	$\{arg(z_1) = \} \tan^{-1}\left(\frac{-3}{3}\right)$ or $\{arg(z_1) = \} \tan^{-1}(-1)$ or $\{arg(z_1) = \} - \tan^{-1}\left(\frac{3}{3}\right)$ or $\{arg(z_1) = \} - \frac{\pi}{4}$ or $\{arg(z_1) = \} 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ or states should be -3 not 3 on top	B1	2.3
	States that $\{arg\left(\frac{z_1}{z_2}\right) = \} arg(z_1) - arg(z_2)$ Or states that the arguments should be subtracted	B1	2.3
		(2)	
(b)	$\left\{arg\left(\frac{z_1}{z_2}\right) = \left(\text{their } -\frac{\pi}{4}\right) - \frac{\pi}{6} = \right\} -\frac{5\pi}{12}$ Or $\left\{arg\left(\frac{z_1}{z_2}\right) = \left(\text{their } \frac{7\pi}{4}\right) - \frac{\pi}{6} = \right\} \frac{19\pi}{12}$	B1ft	2.2a
		(1)	
(3 marks)			
Notes:			
(a) (i) B1: See scheme, Condone – 45 Any incorrect arguments seen is B0. $arg(z_1) = \tan^{-1}\left(\frac{3}{-3}\right)$ is B0 Note: They used 3 instead of -3 is B0, there are two 3's in line 1 do they mean both should -3 It should be negative is B0 (a) (ii) B1: See scheme (b) B1ft: States a correct value for $arg\left(\frac{z_1}{z_2}\right)$ Follow through on their answer to part (a) (i), do not ISW			

Question	Scheme	Marks	AOs
1(a)	$ z = \left(\sqrt{2^2 + (-2\sqrt{3})^2} \right) = 4$ and $ w = \left(\sqrt{(-1)^2 + (\sqrt{3})^2} \right) = 2$ Alt: $z = -2(-1 + \sqrt{3}i) = -2w$	B1	1.1b
	$\frac{z}{w} = \frac{2 - 2\sqrt{3}i}{-1 + \sqrt{3}i} = -2 \Rightarrow \left \frac{z}{w} \right = 2$	M1	2.1
	$\frac{ z }{ w } = \frac{4}{2} = 2$ therefore $\left \frac{z}{w} \right = \frac{ z }{ w }$	A1cso	2.4
		(3)	
(b)	$\arg(z) = -\frac{\pi}{3}$ or $\frac{5\pi}{3}$ and $\arg(w) = \frac{2\pi}{3}$ Alt: $\arg(w) = \frac{2\pi}{3}$ and $\arg(w^2) = \frac{4\pi}{3}$	B1	1.1b
	$zw = (2 - 2\sqrt{3}i)(-1 + \sqrt{3}i) = 4 + 4\sqrt{3}i \Rightarrow \arg(zw) = \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right) = \frac{\pi}{3}$ (or $zw = -2w^2 = -2(-2 - 2\sqrt{3}i) \Rightarrow \arg(zw) = \dots$) Alt: $\arg z = \arg(-2w) = \arg w \pm \pi$, $\arg(zw) = \arg(-2w^2) = \arg w^2 \pm \pi$	M1	2.1
	$\arg(z) + \arg(w) = -\frac{\pi}{3} + \frac{2\pi}{3} = \frac{\pi}{3}$ therefore $\arg(zw) = \arg(z) + \arg(w)$	A1cso	2.4
		(3)	
(6 marks)			
Notes:			
(a) B1: Correct values for z and w. No need for method, may see modulus-argument forms (polar or exponential) used. Alternatively, correctly identifies $z = -2w$ M1: Finds $\frac{z}{w}$ and then $\left \frac{z}{w} \right$. Look for multiplication of numerator and denominator by the conjugate of the denominator for finding $\frac{z}{w}$. In the look for alternative factoring out the -2 and cancelling z oe. Allow this mark if modulus-argument forms (polar or exponential) are used to deduce the modulus.			

A1: Correctly shows that $\frac{|z|}{|w|} = 2$ and $\left|\frac{z}{w}\right| = 2$ hence state that $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ cso. If the result is not stated at the end, we require both $\frac{|z|}{|w|}$ and $\left|\frac{z}{w}\right|$ to have explicitly been seen at some stage in their work and a minimal conclusion (e.g. “LHS=RHS”) to be given. Note $\frac{z}{w} = \dots = -2$ followed by $|-2| = 2$ without ever seeing $\left|\frac{z}{w}\right|$ will be A0.

If modulus-argument forms (polar or exponential) were used all working must have been correct.

(b)

B1: Correct values for $\arg(z)$ and $\arg(w)$. Accept degrees equivalents here and throughout.

Alternatively, correct values for $\arg(w)$ and $\arg(w^2)$.

M1: Finds zw and then $\arg(zw)$. Do not allow attempts via modulus-argument forms (polar or exponential) that use the sum of arguments to prove the sum of arguments of the product.

Alternatively, finds both $\arg(z)$ and $\arg(zw)$ in terms of $\arg(w)$ using $\arg(-p) = \arg(p) \pm \pi$.

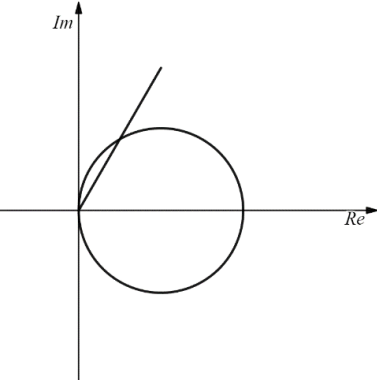
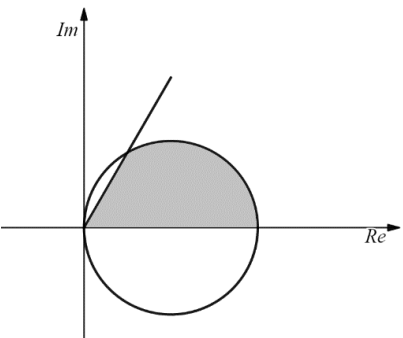
A1: Shows that $\arg(z) + \arg(w) = \frac{\pi}{3}$ and following $\arg(zw) = \frac{\pi}{3}$ hence that

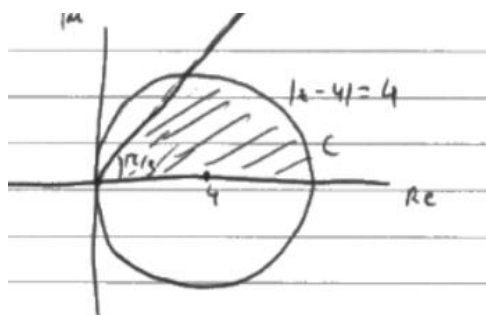
$\arg(zw) = \arg(z) + \arg(w)$ cso If the result is not stated at the end, we require both sides of the result to have been seen at some stage in their work and a minimal conclusion (e.g. “LHS=RHS”) to be given.

Alternatively, correctly establishes the result using $\arg(zw) = \arg(w^2) \pm \pi = 2\arg(w) \pm \pi$ and

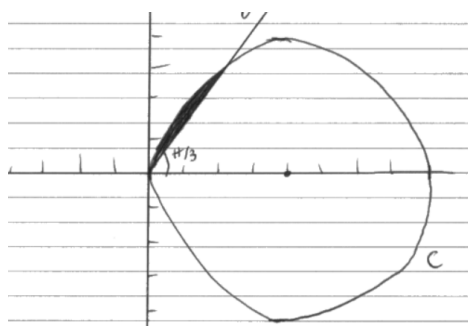
$\arg z + \arg w = \arg w \pm \pi + \arg w = 2\arg w \pm \pi$ with $\arg w^2 = 2\arg w$ verified.

Question	Scheme	Marks	AOs
2(a)	Centre of circle C is $(1, -1)$	B1	1.1b
	$r = \sqrt{(5-1)^2 + (-4+1)^2} = 5$ or $r = \sqrt{(-3-1)^2 + (2+1)^2} = 5$ or $r = \frac{1}{2}\sqrt{(-3-5)^2 + (2+4)^2} = 5$	M1	3.1a
	$ z - 1 + i = 5$ or $ z - (1 - i) = 5$	A1	2.5
		(3)	
(b)	$(x-1)^2 + (y+1)^2 = 25, \quad (x-2)^2 + (y-3)^2 = 4$ $x^2 - 2x + 1 + y^2 + 2y + 1 = 25$ $x^2 - 4x + 4 + y^2 - 6y + 9 = 4$ $\Rightarrow 2x + 8y = 32$	M1	3.1a
	$(16-4y)^2 - 4(16-4y) + 4 + y^2 - 6y + 9 = 4$ or $x^2 - 4x + 4 + \left(\frac{16-x}{4}\right)^2 - 6\left(\frac{16-x}{4}\right) + 9 = 4$	M1	1.1b
	$17y^2 - 118y + 201 = 0$ or $17x^2 - 72x + 16 = 0$	A1	1.1b
	$17y^2 - 118y + 201 = 0 \Rightarrow (17y - 67)(y - 3) = 0 \Rightarrow y = \frac{67}{17}, 3$ or $17x^2 - 72x + 16 = 0 \Rightarrow (17x - 4)(x - 4) = 0 \Rightarrow x = \frac{4}{17}, 4$	M1	1.1b
	$y = \frac{67}{17}, 3 \Rightarrow x = \frac{4}{17}, 4$ or $x = \frac{4}{17}, 4 \Rightarrow y = \frac{67}{17}, 3$	M1	2.1
	$4 + 3i, \frac{4}{17} + \frac{67}{17}i$	A1	2.2a
		(6)	
(9 marks)			
Notes			
(a) B1: Correct coordinates of centre M1: Fully correct strategy for identifying the radius. If the diameter is calculated this must be halved to achieve this mark. A1: Correct equation using the required notation (b) M1: Begins the process of finding z_1 and z_2 by using the Cartesian equations to obtain the equation of the line of intersection M1: Substitutes back into the equation of one of the circles to obtain an equation in one variable A1: Correct 3 term quadratic M1: Solves their 3TQ M1: Substitutes to find values of the other variable to complete the process of finding z_1 and z_2 A1: Correct complex numbers			

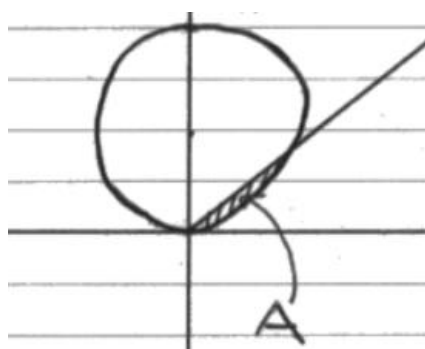
Question	Scheme	Marks	AOs
5(a)		M1 A1 M1 A1	1.1b 1.1b 1.1b 1.1b
		(4)	
(b)		B1ft	2.3
		(1)	
Notes			
(a) M1: Circle drawn with centre on the real axis Look for real axis acting as a line of symmetry of the circle. A1: Circle in the correct position with the imaginary axis as a tangent Centre need not be labelled for either mark. M1: Half line starting at the origin, must be in the first quadrant. Do not award if their line continues into the third quadrant A1: Fully correct diagram that requires • A circle in the correct position • A half line intersecting the circle at the origin and in the first quadrant • The x coordinate of the intersection in the first quadrant must be to the left of the centre of the circle (b) B1ft: Shades the region in their circle above the real axis and below the half line For this mark to be awarded their line must intersect the circle.			

Q5 (a) and (b) examples

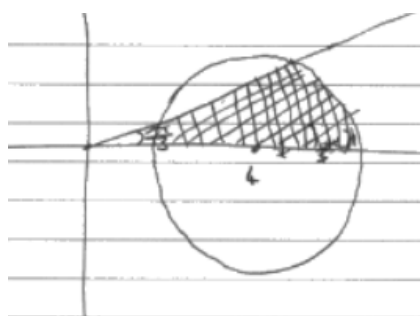
a) M1A1M1A1 b) B1



a) M1A1M1A1 b) B0



a) M0A0M1A0 b) B1



a) M1A0M1A0 b) B1

(c)	$(x-4)^2 + y^2 = 16, \quad y = \sqrt{3}x \Rightarrow x^2 - 8x + 16 + 3x^2 = 16 \Rightarrow x = \dots$ $x = 2, \quad y = 2\sqrt{3}$	M1 A1	3.1a 1.1b
	$\frac{1}{2}\pi \times 4^2 - \left(\frac{1}{2} \times 4^2 \times \frac{\pi}{3} - \frac{1}{2} \times 4^2 \times \sin \frac{\pi}{3} \right)$ or $\frac{1}{2} \times 4^2 \times \frac{2\pi}{3} + \frac{1}{2} \times 4 \times 2\sqrt{3}$ or $\frac{1}{2} \times 4^2 \times \frac{2\pi}{3} + \frac{1}{2} \times 4 \times 4 \times \frac{\sqrt{3}}{2}$	dM1	3.1a
	$= \frac{16}{3}\pi + 4\sqrt{3}$	A1	1.1b
		(4)	
	Alternative 1 for part (c)		
	$x = 4\cos \frac{\pi}{3}, \quad y = 4\sin \frac{\pi}{3}$	M1 A1	3.1a 1.1b
	$\frac{1}{2}\pi \times 4^2 - \left(\frac{1}{2} \times 4^2 \times \frac{\pi}{3} - \frac{1}{2} \times 4^2 \times \sin \frac{\pi}{3} \right)$ or $\frac{1}{2} \times 4^2 \times \frac{2\pi}{3} + \frac{1}{2} \times 4 \times 2\sqrt{3}$ or $\frac{1}{2} \times 4^2 \times \frac{2\pi}{3} + \frac{1}{2} \times 4 \times 4 \times \frac{\sqrt{3}}{2}$	dM1	3.1a
	$= \frac{16}{3}\pi + 4\sqrt{3}$	A1	1.1b
		(4)	
	Alternative 2 for part (c)		
	Deduces $\frac{\pi}{3}$ means there is an equilateral triangle of length 4 4 and $\sin \frac{\pi}{3}$ are seen or used in their workings	M1 A1	3.1a 1.1b
	$\frac{1}{2}\pi \times 4^2 - \left(\frac{1}{2} \times 4^2 \times \frac{\pi}{3} - \frac{1}{2} \times 4^2 \times \sin \frac{\pi}{3} \right)$ or $\frac{1}{2} \times 4^2 \times \frac{2\pi}{3} + \frac{1}{2} \times 4 \times 2\sqrt{3}$ or	dM1	3.1a

	$\frac{1}{2} \times 4^2 \times \frac{2\pi}{3} + \frac{1}{2} \times 4 \times 4 \times \frac{\sqrt{3}}{2}$		
	$= \frac{16}{3} \pi + 4\sqrt{3}$	A1	1.1b
		(4)	
	Alternative 3 for part (c) Polar coordinates		
	$(x-4)^2 + y^2 = 16, \Rightarrow r^2 = 8r \cos \theta \Rightarrow r = 8 \cos \theta$	M1	3.1a
	$\text{Area} = \int \frac{1}{2} r^2 d\theta = \int \frac{1}{2} \cdot 64 \cos^2 \theta d\theta = \int (16 + 16 \cos 2\theta) d\theta$	A1	1.1b
	$= [16\theta + 8 \sin 2\theta]_0^{\frac{\pi}{3}} = \dots$	dM1	3.1a
	$= \frac{16}{3} \pi + 4\sqrt{3}$	A1	1.1b
		(4)	
(9 marks)			
Notes			
(c) This is not for finding their shaded area of their diagram in part (a) but is for a correct process for finding the correct required area. M1: Correct strategy for identifying both coordinates of the point of intersection. Award for substituting a line of the form $y = kx$ into the equation of a circle $(x-4)^2 + y^2 = 16$ then proceeds to find a value for x from a quadratic, where $x \neq 0$ and then find their y coordinate. A1: Correct coordinates, either written separately, or as $(2, 2\sqrt{3})$ or $2 + 2\sqrt{3}i$ dM1: Fully correct strategy for the area, must be consistent with a half line making an angle of $\frac{\pi}{3}$ with the real axis. Can be found by subtracting the area of a segment from the area of a semicircle or by adding the area of a sector to the area of a triangle. The candidate may do combinations of semicircles, triangles and sectors so look carefully for the method. A1: Correct answer in the required form. Alternative 1 M1: Deduces that since the half line makes an angle of $\frac{\pi}{3}$ with the real axis, the horizontal and vertical distances from the origin to the point of intersection are $4 \cos \frac{\pi}{3}$ and $4 \sin \frac{\pi}{3}$. A1: $4 \cos \frac{\pi}{3}$ and $4 \sin \frac{\pi}{3}$ are seen or used in their workings dM1: Fully correct strategy for the area, must be consistent with a half line making an angle of $\frac{\pi}{3}$ with the real axis. Can be found by subtracting the area of a segment from the area of a semicircle or by adding the area of a sector to the area of a triangle.			



The candidate may do combinations of semicircles, triangles and sectors so look carefully for the method.

A1: Correct answer in the required form.

Alternative 2

M1: Alternatively deduces that the angle implies that there is an equilateral triangle of radius 4

A1: 4 and $\sin \frac{\pi}{3}$ are seen or used in their workings

dM1: Fully correct strategy for the area, must be consistent with a half line making an angle of $\frac{\pi}{3}$ with the real axis.

Can be found by subtracting the area of a segment from the area of a semicircle or by adding the area of a sector to the area of a triangle.

The candidate may do combinations of semicircles, triangles and sectors so look carefully for the method.

A1: Correct answer in the required form.

Alternative 3: using polar coordinates

M1: Achieves a polar equation of the form $r = k \cos \theta$ and uses $\left(\frac{1}{2}\right) \int r^2 d\theta$ to obtain $k \int \cos^2 \theta d\theta$

A1: Obtains $\int (16 + 16 \cos 2\theta) d\theta$ or, $d\theta$ may be missing

dM1: Integrates to obtain an expression of the form $a\theta + b \sin 2\theta$, substitutes in limits of 0 and $\frac{\pi}{3}$ and subtracts. If they reach a correct answer with no integration seen then withhold this mark.

A1: Correct answer in the form required

e.g. $(x-4)^2 + y^2 = 16, \Rightarrow r^2 = 8r \cos \theta \Rightarrow r = 8 \cos \theta$

$$\text{Area} = \int \frac{1}{2} r^2 d\theta = \int \frac{1}{2} 64 \cos^2 \theta d\theta = \int \frac{32(1 + \cos 2\theta)}{2} d\theta = \int (16 + 16 \cos 2\theta) d\theta = [16\theta + 8 \sin 2\theta]$$

$$[16\theta + 8 \sin 2\theta]_0^{\frac{\pi}{3}} = \left[16 \left(\frac{\pi}{3} \right) + 8 \sin 2 \left(\frac{\pi}{3} \right) \right] - [0] = \frac{16}{3} \pi + 4\sqrt{3}$$

Question	Scheme	Marks	AOs
4(i)	$z_1 + z_2 = a + bi + c + di$ leading to $b + d = 0$ (oe) or $z_2 - z_1 = (c + di) - (a + bi)$ leading to $c - a = 2$ (oe)	B1	1.1b
	Forms equations using modulus information $ z_1 = \sqrt{13} \Rightarrow a^2 + b^2 = 13$ and $ z_2 = 5 \Rightarrow c^2 + d^2 = 25$	M1	3.1a
	Uses their equations to solve simultaneously to find at least one value <u>e.g. Method 1</u> $c = 2 + a$, $d = -b$ leading to $(2 + a)^2 + (-b)^2 = 25 \Rightarrow a^2 + 4a + 4 + b^2 = 25$ Solve with $a^2 + b^2 = 13$ gives $13 + 4a + 4 = 25 \Rightarrow 4a = 8 \Rightarrow a = \dots$ <u>e.g. Method 2</u> $\left. \begin{matrix} a^2 + b^2 = 13 \\ c^2 + d^2 = 25 \end{matrix} \right\} \Rightarrow a^2 - c^2 = -12$ and solves simultaneous with $c - a = 2$ to find a value for a or c .	M1	3.1a
	Uses their equations to find values for all the constants	ddM1	2.1
	$a = 2, b = 3, c = 4, d = -3$	A1	2.3
		(5)	
(ii)(a)		B1	1.1b
		B1	1.1b
		(2)	
(ii)(b)	Distance between centres $= \sqrt{5^2 + 12^2} = 13$	M1	1.1b
	$13 \pm (7 + 4)$	dM1	3.1a
	$2 \leq z - w \leq 24$	A1	1.1b
		(3)	
(10 marks)			

Notes:**(i)****B1:** States either $b + d = 0$ or $c - a = 2$ (oe)**M1:** Uses the modulus information to write down two more equations (need not be squared). Accept if they forget to square one of the moduli but the sum of squares must be correct.**M1:** Uses their equations to solve simultaneously to find at least one value. If they assume $a = 2$, you may allow this mark for finding at least one of b or d .**ddM1:** Dependent on both previous method marks. Uses their equations to find values for all of the constants. This mark is not available if $a = 2$ is assumed. All must be found from algebra.**A1:** Correct values from correct work.**(ii)(a)****B1:** One circle drawn with correct intercepts OR both circles drawn in correct position, not overlapping, but with no intercepts shown. Be tolerant on the shape but must be a clear attempt at a circle – a closed loop with no obvious kinks. Labels on imaginary axis may be just numbers or i and $9i$.**B1:** Both circles drawn with correct intercepts and the circles does not intersect each other. Again be tolerant as per first B mark.**(ii)(b)****M1:** Finds the distance between the centres of the circles.**dM1:** Dependent on previous method mark. Full method to find the nearest and furthest points of circles. E.g. Uses their distance $\pm$ sum of radii. If their distance between circles is less than 11, there must a check seen to show this has been considered. If by error they show the circles touch (not cross) then note that finding twice the sum of radii is a valid method for the greatest distance.**A1:** Correct answer. Allow if the centres were on the negative axes.

Alt: There may be attempts via finding the line through the centres.

M1: Full method to find the equation of the line through the two centres:

$$m = \frac{0-5}{12-0} = -\frac{5}{12} \Rightarrow y = -\frac{5}{12}(x-12) \text{ (oe)}$$

dM1: Full method to find the nearest and furthest points of circles, so finds the intersection points of this line with the circles and finds distances between the relevant points.

$$\left. \begin{aligned} x^2 + \left(-\frac{5}{12}x + 5 - 5\right)^2 &= 16 \\ (x-12)^2 + \left(-\frac{5}{12}x + 5\right)^2 &= 49 \end{aligned} \right\} \Rightarrow \begin{aligned} \frac{169}{144}x^2 &= 16 \Rightarrow x = \pm \frac{48}{13} \Rightarrow y = \frac{45}{13}, \frac{85}{13} \\ \frac{169}{144}x^2 - \frac{169}{6}x + 120 &= 0 \Rightarrow x = \frac{72}{13}, \frac{240}{13} \Rightarrow y = \pm \frac{35}{13} \end{aligned}$$

$$\text{nearest} = \sqrt{\left(\frac{72}{13} - \frac{48}{13}\right)^2 + \left(\frac{35}{13} - \frac{45}{13}\right)^2} = 2, \text{ furthest} = \sqrt{\left(\frac{240}{13} + \frac{48}{13}\right)^2 + \left(-\frac{35}{13} - \frac{85}{13}\right)^2} = 24$$

A1: Correct answer.

Question	Scheme	Marks	AOs
9(a)	$\frac{1}{1-z}$	B1	2.2a
		(1)	
(b)(i)	$1+z+z^2+z^3+\dots$ $=1+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)^3+\dots$ $=1+\frac{1}{2}(\cos\theta+i\sin\theta)+\frac{1}{4}(\cos 2\theta+i\sin 2\theta)+\frac{1}{8}(\cos 3\theta+i\sin 3\theta)+\dots$	M1	3.1a
	$\frac{1}{1-z} = \frac{1}{1-\frac{1}{2}(\cos\theta+i\sin\theta)} \times \frac{1-\frac{1}{2}\cos\theta+\frac{1}{2}i\sin\theta}{1-\frac{1}{2}\cos\theta+\frac{1}{2}i\sin\theta}$ or $\frac{1}{1-z} = \frac{2}{2-(\cos\theta+i\sin\theta)} \times \frac{2-(\cos\theta-i\sin\theta)}{2-(\cos\theta-i\sin\theta)}$	M1	3.1a
	$\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin 2\theta)+\frac{1}{8}(\sin 3\theta)+\dots\right\} = \frac{\frac{1}{2}\sin\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2}$ or $\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin 2\theta)+\frac{1}{8}(\sin 3\theta)+\dots\right\} = \frac{2\sin\theta}{(2-\cos\theta)^2+(\sin\theta)^2}$	M1	2.1
	$\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2 = 1-\cos\theta+\frac{1}{4}\cos^2\theta+\frac{1}{4}\sin^2\theta$ $=\frac{5}{4}-\cos\theta$ or $(2-\cos\theta)^2+(\sin\theta)^2 = 4-4\cos\theta+\cos^2\theta+\sin^2\theta$ $=5-4\cos\theta$	M1	1.1b
	$\frac{1}{2}\sin\theta+\frac{1}{4}\sin 2\theta+\frac{1}{8}\sin 3\theta+\dots = \frac{\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta} = \frac{2\sin\theta}{5-4\cos\theta}^*$	A1*	1.1b
	Alternative $1+z+z^2+z^3+\dots$ $=1+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)^3+\dots$ $=1+\frac{1}{2}(\cos\theta+i\sin\theta)+\frac{1}{4}(\cos 2\theta+i\sin 2\theta)+\frac{1}{8}(\cos 3\theta+i\sin 3\theta)+\dots$	M1	3.1a

	$\frac{1}{1-z} = \frac{1}{1-\frac{1}{2}e^{i\theta}} \times \frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{2}e^{-i\theta}}$	M1	3.1a
	$\frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{4}e^{i\theta}-\frac{1}{4}e^{-i\theta}+\frac{1}{4}} = \frac{4-2e^{-i\theta}}{5-2(e^{i\theta}+e^{-i\theta})} = \frac{4-2(\cos\theta-i\sin\theta)}{5-2(2\cos\theta)}$	M1	2.1
	Select the imaginary part $\frac{2\sin\theta}{5-4\cos\theta}$	M1	1.1b
	$\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \dots = \frac{2\sin\theta}{5-4\cos\theta}^*$	A1*	1.1b
		(5)	
(b)(ii)	$\frac{1-\frac{1}{2}\cos\theta}{\frac{5}{4}-\cos\theta} = 0 \Rightarrow \cos\theta = 2$	M1	3.1a
	As $(-1 \leq) \cos\theta \leq 1$ therefore there is no solution to $\cos\theta = 2$ so there will also be a real part, hence the sum cannot be purely imaginary.	A1	2.4
	Alternative 1 Imaginary part is $\frac{4-2\cos\theta}{5-4\cos\theta} = \frac{1}{2} + \frac{3}{2(5-4\cos\theta)}$	M1	3.1a
	$-1 \leq \cos\theta \leq 1$ therefore $\frac{1}{6} < \frac{3}{2(5-4\cos\theta)} < \frac{3}{2}$ so sum must contain real part	A1	2.4
	Alternative 2 $\frac{1}{1-z} = ki \Rightarrow z = 1 + \frac{i}{k}$	M1	3.1a
	mod $ z > 1$ contradiction hence cannot be purely imaginary	A1	2.4
		(2)	
(8 marks)			
Notes:			
(a) B1: See scheme			
(b)(i) M1: Substitutes $z = \frac{1}{2}(\cos\theta + i\sin\theta)$ into at least 3 terms of the series and applies de Moivre's theorem. M1: Substitutes $z = \frac{1}{2}(\cos\theta + i\sin\theta)$ into their answer to part (a) and rationalises the denominator. M1: Equates the imaginary terms. M1: Multiplies out the denominator and simplifies by using the identity $\cos^2\theta + \sin^2\theta = 1$			

A1*: cso. Achieves the printed answer having substituted $z = \frac{1}{2}(\cos \theta + i \sin \theta)$ into 4 terms of the series.

Alternative

M1: Substitutes $z = \frac{1}{2}(\cos \theta + i \sin \theta)$ into at least 3 terms of the series and applies de Moivre's theorem.

M1: Substitutes $z = \frac{1}{2}e^{i\theta}$ into their answer to part (a) and rationalises the denominator.

M1: Uses $e^{-i\theta} = \cos \theta - i \sin \theta$ and $e^{i\theta} + e^{-i\theta} = 2 \cos \theta$ to express in terms of $\sin \theta$ and $\cos \theta$

M1: Select the imaginary terms.

A1*: cso Achieves the printed answer having substituted $z = \frac{1}{2}(\cos \theta + i \sin \theta)$ into 4 terms of the series.

(b)(ii)

M1: Setting the real part of the series = 0 and rearranges to find $\cos \theta = \dots$

A1: See scheme

Alternative 1

M1: Rearranges imaginary part so that $\cos \theta$ only appears once

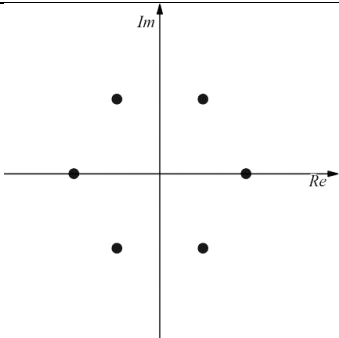
A1: Uses $-1 \leq \cos \theta \leq 1$ to show that the sum must always be positive so must contain a real part

Alternative 2

M1: Sets sum as purely imaginary and rearranges to make z the subject

A1: Shows a contradiction and draws an appropriate conclusion

Question	Scheme	Marks	AOs
4(a)	$(\cos \theta + i \sin \theta)^7 = \cos^7 \theta + \binom{7}{1} \cos^6 \theta (i \sin \theta) + \binom{7}{2} \cos^5 \theta (i \sin \theta)^2 + \dots$ Some simplification may be done at this stage e.g. $c^7 + 7c^6 is - 21c^5 s^2 - 35c^4 is^3 + 35c^3 s^4 + 21c^2 is^5 - 7cs^6 - is^7$	M1	1.1b
	$i \sin 7\theta = {}^7C_1 c^6 is + {}^7C_3 c^4 i^3 s^3 + {}^7C_5 c^2 i^5 s^5 + i^7 s^7$ or $= 7c^6 is + 35c^4 i^3 s^3 + 21c^2 i^5 s^5 + i^7 s^7$	M1	2.1
	$\sin 7\theta = 7c^6 s - 35c^4 s^3 + 21c^2 s^5 - s^7$	A1	1.1b
	$= 7(1-s^2)^3 s - 35(1-s^2)^2 s^3 + 21(1-s^2)s^5 - s^7$ $= 7(1-3s^2+3s^4-s^6)s - 35(1-2s^2+s^4)s^3 + 21(1-s^2)s^5 - s^7$	M1	2.1
	$\{7s - 21s^3 + 21s^5 - 7s^7 - 35s^3 + 70s^5 - 35s^7 + 21s^5 - 21s^7 - s^7\}$ leading to $\sin 7\theta = 7 \sin \theta - 56 \sin^3 \theta + 112 \sin^5 \theta - 64 \sin^7 \theta *$	A1*	1.1b
		(5)	
(b)	$1 + \sin 7\theta = 0 \Rightarrow \sin 7\theta = -1$	M1	3.1a
	$7\theta = -450, -90, 270, 630, \dots$ or $7\theta = -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \dots$	A1	1.1b
	$\theta = -\frac{450}{7}, -\frac{90}{7}, \frac{270}{7}, \frac{630}{7}, \dots \Rightarrow \sin \theta = \dots$ or $\theta = -\frac{5\pi}{14}, -\frac{\pi}{14}, \frac{3\pi}{14}, \frac{7\pi}{14}, \dots \Rightarrow \sin \theta = \dots$	M1	2.2a
	$x = \sin \theta = -0.901, -0.223, 0.623, 1$	A1	1.1b
		A1	2.3
		(5)	
(10 marks)			
Notes			
(a) M1: Attempts to expand $(\cos \theta + i \sin \theta)^7$ including a recognisable attempt at binomial coefficients Some simplification may be done at this stage. (May only see imaginary terms) M1: Identifies imaginary terms with $\sin 7\theta$ A1: Correct expression with coefficients evaluated and i's dealt with correctly M1: Replaces $\cos^2 \theta$ with $1 - \sin^2 \theta$ and applies the expansions of $(1 - \sin^2 \theta)^2$ and $(1 - \sin^2 \theta)^3$ to their expression A1*: Reaches the printed answer with no errors and expansion of brackets seen. (b) M1: Makes the connection with part (a) and realises the need to solve $\sin 7\theta = -1$ A1: At least one correct value for 7θ M1: Divides by 7 and deduces that x values are found by finding at least one value for $\sin \theta$ A1: Awrt 2 correct values for x A1: Awrt all 4 x values correct and no extras			

Question	Scheme	Marks	AOs
7(a)	$z = e^{\frac{k\pi}{3}i}, k = 0, 1, 2, 3, 4, 5$	M1	1.1b
		A1	1.1b
		(2)	
(b)		B1	2.2a
		dB1	1.1b
		(2)	
(c)	e.g. $(\sqrt{3} + i)^6 = \left(2e^{\frac{\pi}{6}i}\right)^6 = 64e^{i\pi} = -64^*$ or $\left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^6 = 2^6(\cos\pi + i\sin\pi) = 64(-1) = -64^*$ or $(\sqrt{3} + i)^6 = (\sqrt{3})^6 + 6(\sqrt{3})^5i - 15(\sqrt{3})^4 - 20(\sqrt{3})^3i + 15(\sqrt{3})^2 + 6\sqrt{3}i + i^6$ $= 27 - 135 + 45 - 1 = -64^*$ or $(\sqrt{3} + i)^6 = 27 + 54\sqrt{3}i + 135i^2 + 60\sqrt{3}i^3 + 45i^4 + 6\sqrt{3}i^5 + i^6$ $= 27 + 54\sqrt{3}i - 135 - 60\sqrt{3}i + 45 + 6\sqrt{3}i - 1 = -64^*$	M1	1.1b
		A1*	2.1
		(2)	
(d)	$r = 2$	B1	2.2a
	$z = 2e^{\frac{\pi}{6}i} \times e^{\frac{k\pi}{3}i}, k = 0, 1, 2, 3, 4, 5$	M1	3.1a
	$z = 2e^{\left(\frac{\pi}{6} + \frac{k\pi}{3}\right)i}, k = 0, 1, 2, 3, 4, 5$	A1	1.1b
		(3)	
(9 marks)			
Notes			

(a)

M1: For sight of $e^{\frac{k\pi}{3}i}$
 Accept any value for k

A1: All six roots fully defined as shown or listed separately with their values of θ within the given range with no incorrect or extra values. Ensure i and π are present in each term.

Note: Roots if listed are $e^0, e^{\frac{\pi}{3}i}, e^{\frac{2\pi}{3}i}, e^{\pi i}, e^{\frac{4\pi}{3}i}, e^{\frac{5\pi}{3}i}$, condone 1 for e^0 and/or -1 for $e^{\pi i}$



(b)

B1: Plots 6 points that form a hexagon, with a point on the positive real axis and a point on the negative real axis, and one point in each quadrant.

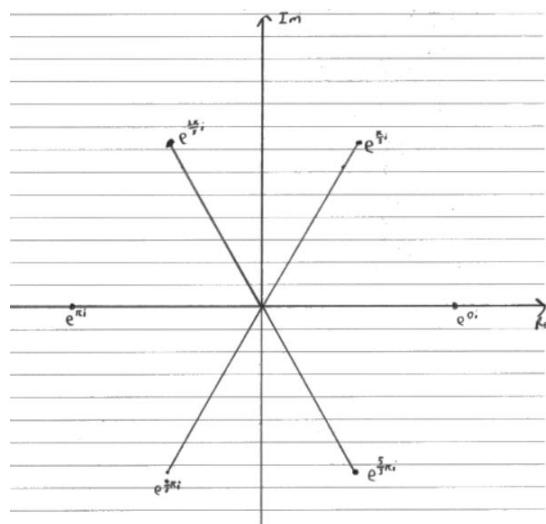
Do not be concerned about the position of each point from the centre, however the sketch must convey a hexagon.

dB1: The points form a hexagon, centre the origin (see diagram), axes need not be labelled.

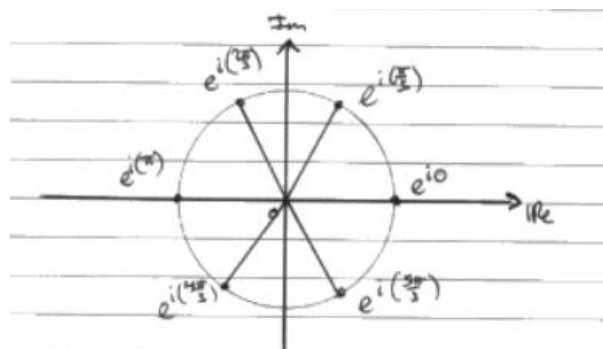
Look for the axes acting as lines of symmetry.

(Drawing line/vectors to each point is acceptable but not necessary for either mark)

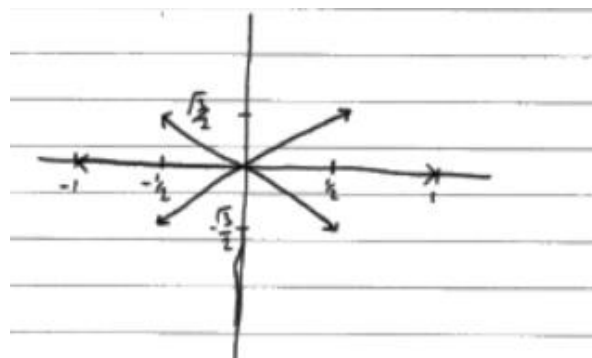
Examples



B1dB1



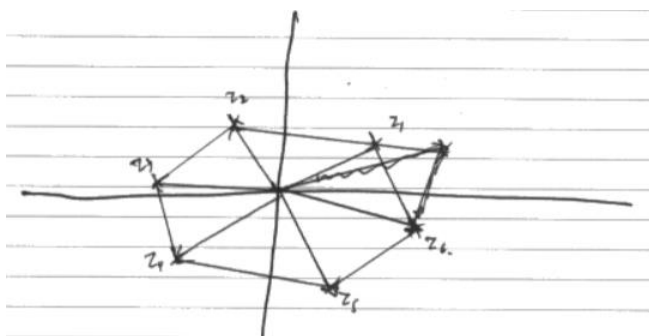
B1dB1



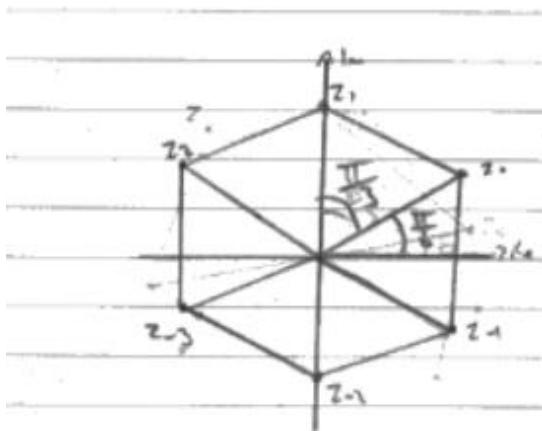
B1dB1



B1dB1



B0dB0



B0dB0

(c)

M1: Converts $\sqrt{3} + i$ to polar form to obtain $re^{i\theta}$ with at least $r = 2$ or $\theta = \frac{\pi}{6}$ and applies the power of 6 correctly to obtain $r^6 e^{6\theta i}$

A1*: Obtains the given answer with sufficient working shown.

As a minimum need to see $2^6 e^{\frac{6\pi i}{6}} = -64$ or $2^6 e^{\pi i} = -64$

If $r = -2$ is seen in their workings withhold this mark.

OR

M1: Converts $\sqrt{3} + i$ to modulus-argument form $r(\cos \theta + i \sin \theta)$ with at least $r = 2$ or $\theta = \frac{\pi}{6}$ and applies the power of 6 correctly to obtain $r^6(\cos 6\theta + i \sin 6\theta)$

A1*: Obtains the given answer with sufficient working shown.

OR

M1: Attempts to expand $(\sqrt{3} + i)^6$ fully using an attempt at the binomial expansion. Must have 7 terms for $(a + b)^n$ and correct binomial coefficients with $a = \sqrt{3}$, $b = i$ and $n = 6$
A1*: Obtains the given answer with at least one intermediate line.

OR

M1: Attempts the full expansion of $(\sqrt{3} + i)^6 = (\sqrt{3} + i)(\sqrt{3} + i)(\sqrt{3} + i) \dots (\sqrt{3} + i) =$
 There must be no brackets, no irrational numbers and no terms in i in their simplified answer.

A1*: Obtains the given answer with sufficient working shown including correct full expansion, with at least one intermediate line.

(d)

B1: Deduces $r = 2$ (only)

M1: Obtains at least one value of z in the form $re^{i\theta}$ with their consistent value of r , and θ taking one of

$$\left\{ \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6} \right\}$$

A1: For $2e^{\frac{\pi i}{6}}, 2e^{\frac{\pi i}{2}}, 2e^{\frac{5\pi i}{6}}, 2e^{\frac{7\pi i}{6}}, 2e^{\frac{3\pi i}{2}}, 2e^{\frac{11\pi i}{6}}$ with no incorrect or extra values. Accept unsimplified arguments such as having a solution of $2e^{\frac{9\pi i}{6}}$. Ensure i and π are present in each term.

Accept $2e^{\frac{\pi i}{2}}$ as $2i$ and $2e^{\frac{3\pi i}{2}}$ as $-2i$

Question	Scheme	Marks	AOs
5(a)	$\alpha = \frac{z_1 + z_2}{2} = \frac{35 - 25i - 29 + 39i}{2} = \dots$ $\alpha = z_1 + \frac{1}{2} \overline{z_1 z_2} = 35 - 25i + \frac{1}{2}(-64 + 64i) = \dots$ $\alpha = z_2 + \frac{1}{2} \overline{z_2 z_1} = -29 + 39i + \frac{1}{2}(64 - 64i) = \dots$	M1	1.1b
	$= 3 + 7i *$	A1*	1.1b
		(2)	
(b)	$\beta(z_1 - \alpha) = \left(\frac{1+i}{64}\right)(35 - 25i - (3 + 7i)) = \left(\frac{1+i}{64}\right)(32 - 32i) =$ $= \frac{1}{64}(32 - 32i + 32i - 32i^2) = \frac{1}{64}(32 - 32i + 32i + 32)$	M1	1.1b
	$= \frac{1}{64}(64) = 1 *$	A1*	1.1b
		(2)	
(c)(i)	Roots are $\left\{e^0 \text{ (or 1 or } e^{i2\pi})\right\}, e^{\frac{i\pi}{3}}, e^{\frac{2i\pi}{3}}, e^{i\pi}, e^{\frac{4i\pi}{3}}, e^{\frac{5i\pi}{3}} \text{ or } e^{\frac{k\pi}{3}}, k = 0, 1, 2, 3, 4, 5$	B1	1.1b
	$\left\{e^0 \text{ (or 1 or } e^{i2\pi})\right\}, e^{\frac{i\pi}{3}}, e^{\frac{2i\pi}{3}}, e^{i\pi}, e^{-\frac{i\pi}{3}}, e^{-\frac{2i\pi}{3}} \text{ or } e^{\frac{k\pi}{3}}, k = -2, -1, 0, 1, 2, 3,$		
		(1)	
(c)(ii)	$w = \beta(z - \alpha) = e^{\frac{k\pi}{3}} \Rightarrow z = \frac{e^{\frac{k\pi}{3}}}{\beta} + \alpha$	M1	3.1a
	$\Rightarrow z = \frac{64\left(\cos \frac{k\pi}{3} + i \sin \frac{k\pi}{3}\right)(1-i)}{(1+i)(1-i)} + 3 + 7i = \dots$	M1	1.1b
	Two of $\begin{array}{cc} (19 + 16\sqrt{3}) + (-9 + 16\sqrt{3})i & (-13 - 16\sqrt{3}) + (23 - 16\sqrt{3})i \\ (-13 + 16\sqrt{3}) + (23 + 16\sqrt{3})i & (19 - 16\sqrt{3}) - (9 + 16\sqrt{3})i \end{array}$ Or four correct decimal answers $46.7 + 18.7i \quad -40.7 - 4.7i \quad 14.7 + 50.7i \quad -8.7 - 36.7i$	A1	2.5

	$\begin{aligned} & (19+16\sqrt{3})+(-9+16\sqrt{3})i \\ \text{All four of } & (-13+16\sqrt{3})+(23+16\sqrt{3})i \\ & (-13-16\sqrt{3})+(23-16\sqrt{3})i \\ & (19-16\sqrt{3})-(9+16\sqrt{3})i \end{aligned}$	A1	2.2a
		(4)	
	Alternative 1 $(\beta(z-\alpha))^6 = 1^6 \Rightarrow (z-\alpha)^6 = \frac{1}{\beta^6} = 8589934459i$ $r = \sqrt[6]{858993459} = 32\sqrt{2} \text{ or } 45.25\dots \text{ and } \theta = \frac{\pi}{12} + \frac{k\pi}{3} \text{ or}$ $\theta = -\frac{\pi}{4} + \frac{k\pi}{3}$	M1	3.1a
	$z = r(\cos \theta - i \sin \theta) + 3 + 7i = \dots$	M1	1.1b
	$\begin{aligned} & \begin{matrix} (19+16\sqrt{3})+(-9+16\sqrt{3})i & (-13-16\sqrt{3})+(23-16\sqrt{3})i \\ (-13+16\sqrt{3})+(23+16\sqrt{3})i & (19-16\sqrt{3})-(9+16\sqrt{3})i \end{matrix} \\ \text{Two of } & \\ \text{Or four correct decimal answers} & \\ 46.7 + 18.7i & -40.7 - 4.7i & 14.7 + 50.7i & -8.7 - 36.7i \end{aligned}$	A1	2.5
	$\begin{aligned} & (19+16\sqrt{3})+(-9+16\sqrt{3})i \\ \text{All four of } & (-13+16\sqrt{3})+(23+16\sqrt{3})i \\ & (-13-16\sqrt{3})+(23-16\sqrt{3})i \\ & (19-16\sqrt{3})-(9+16\sqrt{3})i \end{aligned}$	A1	2.2a
		(4)	
	Alternative 2 $\text{Rotation matrix } \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \text{ and } \begin{pmatrix} 35 \\ -25 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \end{pmatrix} \text{ or } \begin{pmatrix} -29 \\ 39 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \end{pmatrix}$ Or $\text{find the exponential form for } \begin{pmatrix} 35 \\ -25 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \end{pmatrix} \text{ or } \begin{pmatrix} -29 \\ 39 \end{pmatrix} - \begin{pmatrix} 3 \\ 7 \end{pmatrix}$	M1	3.1a

	$32\sqrt{2}e^{\frac{\pi}{4}} \text{ or } 32\sqrt{2}e^{-\frac{\pi}{4}}$		
	$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 32 \\ -32 \end{pmatrix} + \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \dots \text{ or } \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -32 \\ 32 \end{pmatrix} + \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \dots$ Or $32\sqrt{2}e^{\frac{\pi}{4}} \times e^{\frac{\pi}{3}} = \dots \text{ then applies } r(\cos \theta - i \sin \theta) + 3 + 7i = \dots$ $32\sqrt{2}e^{-\frac{\pi}{4}} \times e^{\frac{\pi}{3}} = \dots \text{ then applies } r(\cos \theta - i \sin \theta) + 3 + 7i = \dots$	M1	1.1b
	Two of $\begin{matrix} (19+16\sqrt{3})+(-9+16\sqrt{3})i & (-13-16\sqrt{3})+(23-16\sqrt{3})i \\ (-13+16\sqrt{3})+(23+16\sqrt{3})i & (19-16\sqrt{3})-(9+16\sqrt{3})i \end{matrix}$ Or four correct decimal answers $46.7 + 18.7i \quad -40.7 - 4.7i \quad 14.7 + 50.7i \quad -8.7 - 36.7i$ Or as coordinates	A1	2.5
	All four of $\begin{matrix} (19+16\sqrt{3})+(-9+16\sqrt{3})i \\ (-13+16\sqrt{3})+(23+16\sqrt{3})i \\ (-13-16\sqrt{3})+(23-16\sqrt{3})i \\ (19-16\sqrt{3})-(9+16\sqrt{3})i \end{matrix}$	A1	2.2a
		(4)	
(11 marks)			
Notes:			
(a) M1: Attempts the midpoint of z_1 and z_2 A1*: Correct point. (b) M1: Substitutes into the equation with z_1 and α and β, simplifies and expands and applies $i^2 = -1$, this may be implied by their working. A1*: Completes the proof to find the correct answer with no errors seen, all necessary brackets as required Mark (c) as one (c) (i) B1: Correct roots, accept all 6 listed or given in general form as in the scheme. Need not show the 1. (c)(ii)			

M1: Realises the need to set the roots of unity equal to $\beta(z - \alpha)$ and solve for z . Must be attempted at least once with any of their roots.

M1: Finds the Cartesian form for their equation for at least one of the roots other than z_1 and z_2

A1: At least two correct other roots than z_1 and z_2 in Cartesian form.

A1: Deduces all four correct in Cartesian form and no extra solutions

Alternative 1

M1: Finds the modulus and argument of $(z - \alpha)^6$

M1: Finds the Cartesian form for one of their modulus and arguments

A1A1: same as above

Alternative 2

M1: Finds the rotation matrix and subtracts the centre from z_1 or z_2 . Or finds the exponential form for $z_1 - \alpha$ or $z_2 - \alpha$

M1: Finds the Cartesian form by multiplying by the rotation matrix and adding the centre. Or multiplies by $e^{\frac{\pi i}{3}}$ write in Cartesian form and adds on the centre

A1A1: same as above

Note all four correct decimal answers or written as coordinates score **A1A0**

46.7 + 18.7i - 40.7 - 4.7i 14.7 + 50.7i - 8.7 - 36.7i

Note: Correct answers implies the method marks

Question	Scheme	Marks	AOs
6(a)	Examples: $\begin{pmatrix} \cos 120 & -\sin 120 \\ \sin 120 & \cos 120 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \dots \text{or } (6 + 2i) \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right)$ or $\sqrt{40} \left(\cos \arctan\left(\frac{2}{6}\right) + i \sin \arctan\left(\frac{2}{6}\right) \right) \left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right)$ or $\sqrt{40} \left(\cos\left(\arctan\left(\frac{2}{6}\right) + \frac{2\pi}{3}\right) + i \sin\left(\arctan\left(\frac{2}{6}\right) + \frac{2\pi}{3}\right) \right)$ or $\sqrt{40} e^{i \arctan\left(\frac{2}{6}\right)} e^{i\left(\frac{2\pi}{3}\right)}$	M1	3.1a
	$(-3 - \sqrt{3}) \text{ or } (3\sqrt{3} - 1)i$	A1	1.1b
	$(-3 - \sqrt{3}) + (3\sqrt{3} - 1)i$	A1	1.1b
	Examples: $\begin{pmatrix} \cos 240 & -\sin 240 \\ \sin 240 & \cos 240 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \dots \text{or } (6 + 2i) \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right)$ or $\sqrt{40} \left(\cos \arctan\left(\frac{2}{6}\right) + i \sin \arctan\left(\frac{2}{6}\right) \right) \left(\cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) \right)$ or $\sqrt{40} \left(\cos\left(\arctan\left(\frac{2}{6}\right) + \frac{4\pi}{3}\right) + i \sin\left(\arctan\left(\frac{2}{6}\right) + \frac{4\pi}{3}\right) \right)$ or $\sqrt{40} e^{i \arctan\left(\frac{2}{6}\right)} e^{i\left(\frac{4\pi}{3}\right)}$	M1	3.1a
	$(-3 + \sqrt{3}) \text{ or } (-3\sqrt{3} - 1)i$	A1	1.1b
	$(-3 + \sqrt{3}) + (-3\sqrt{3} - 1)i$	A1	1.1b
		(6)	
	Area $ABC = 3 \times \frac{1}{2} \sqrt{6^2 + 2^2} \sqrt{6^2 + 2^2} \sin 120^\circ$ or Area $AOB = \frac{1}{2} \sqrt{6^2 + 2^2} \sqrt{6^2 + 2^2} \sin 120^\circ$	M1	2.1
	Area $DEF = \frac{1}{4} ABC \text{ or } \frac{3}{4} AOB$	dM1	3.1a
	$= \frac{3}{8} \times 40 \times \frac{\sqrt{3}}{2} = \frac{15\sqrt{3}}{2}$	A1	1.1b
		(3)	
(b) Way 1			

(b) Way 2	$D\left(\frac{3-\sqrt{3}}{2}, \frac{3\sqrt{3}+1}{2}\right)$ $OD = \sqrt{\left(\frac{3-\sqrt{3}}{2}\right)^2 + \left(\frac{3\sqrt{3}+1}{2}\right)^2} = \sqrt{10}$ $\text{Area } DOF = \frac{1}{2}\sqrt{10}\sqrt{10} \sin 120^\circ$	M1	2.1
	$\text{Area } DEF = 3DOF$	dM1	3.1a
	$= 3 \times \frac{1}{2} \times \sqrt{10}\sqrt{10} \times \frac{\sqrt{3}}{2} = \frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 3	$AB = \sqrt{(9+\sqrt{3})^2 + (3-3\sqrt{3})^2} = \sqrt{120}$ $\text{Area } ABC = \frac{1}{2}\sqrt{120}\sqrt{120} \sin 60^\circ (= 30\sqrt{3})$	M1	2.1
	$\text{Area } DEF = \frac{1}{4}ABC$	dM1	3.1a
	$= \frac{1}{4} \times 30\sqrt{3} = \frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 4	$D\left(\frac{3-\sqrt{3}}{2}, \frac{3\sqrt{3}+1}{2}\right), E(-3, -1), F\left(\frac{3+\sqrt{3}}{2}, \frac{-3\sqrt{3}+1}{2}\right)$ $DE = \sqrt{\left(\frac{3-\sqrt{3}}{2} + 3\right)^2 + \left(\frac{3\sqrt{3}+1}{2} + 1\right)^2} (= \sqrt{30})$ $\text{Area } DEF = \frac{1}{2}\sqrt{30}\sqrt{30} \sin 60^\circ$	M1 dM1	2.1 3.1a
	$= \frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 5	$\text{Area } ABC = \frac{1}{2} \begin{vmatrix} 6 & -3-\sqrt{3} & \sqrt{3}-3 & 6 \\ 2 & 3\sqrt{3}-1 & -3\sqrt{3}-1 & 2 \end{vmatrix} = 30\sqrt{3}$	M1	2.1
	$\text{Area } DEF = \frac{1}{4}ABC$	dM1	3.1a
	$= \frac{1}{4} \times 30\sqrt{3} = \frac{15\sqrt{3}}{2}$	A1	1.1b

(9 marks)

Notes

(a)

 M1: Identifies a suitable method to rotate the given point by 120° (or equivalent) about the origin. May see equivalent work with modulus/argument or exponential form e.g. an attempt to multiply

 by $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$ or $e^{\frac{2\pi}{3}i}$

A1: Correct real part or correct imaginary part

A1: Completely correct complex number

 M1: Identifies a suitable method to rotate the given point by 240° (or equivalent e.g. rotate their B by 120°) about the origin


May see equivalent work with modulus/argument or exponential form e.g. an attempt to multiply $6 + 2i$ by $\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$ or $e^{\frac{4\pi}{3}i}$ or their B by $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$ or $e^{\frac{2\pi}{3}i}$

A1: Correct real part or correct imaginary part

A1: Completely correct complex number

(b)

In general, the marks in (b) should be awarded as follows:

M1: Attempts to find the area of a relevant triangle

dM1: completes the problem by multiplying by an appropriate factor to find the area of DEF

Dependent on the first method mark

A1: Correct exact area

In some cases it may not be possible to distinguish the 2 method marks. In such cases they can be awarded together for a direct method that finds the area of DEF

Examples:

Way 1

M1: A correct strategy for the area of a relevant triangle such as ABC or AOB

dM1: Completes the problem by linking the area of DEF correctly with ABC or with AOB

A1: Correct value

Way 2

M1: A correct strategy for the area of a relevant triangle such as DOF

dM1: Completes the problem by linking the area of DEF correctly with DOF

A1: Correct value

Way 3

M1: A correct strategy for the area of a relevant triangle such as ABC

dM1: Completes the problem by linking the area of DEF correctly with ABC

A1: Correct value

Way 4

M1dM1: A correct strategy for the area of DEF . Finds 2 midpoints and attempts one side of DEF and uses a correct triangle area formula. By implication this scores both M marks.

A1: Correct value

Way 5

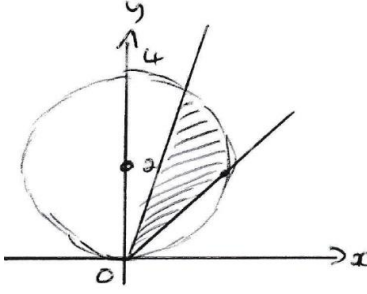
M1: A correct strategy for the area of ABC using the “shoelace” method.

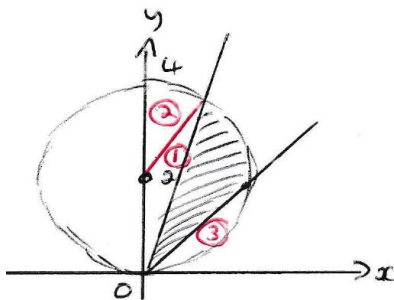
dM1: Completes the problem by linking the area of DEF correctly with ABC

A1: Correct value

Note the marks in (b) can be scored using inexact answers from (a) and the A1 scored if an exact area is obtained.



Question	Scheme	Marks	AOs
8(i)	$ z = \sqrt{6^2 + 6^2} = \dots 6\sqrt{2} \text{ or } \sqrt{72} \text{ and } \arg z = \tan^{-1}\left(\frac{6}{6}\right) = \dots \left\{\frac{\pi}{4}\right\}$ Can be implied by $r = 6\sqrt{2}e^{\frac{\pi}{4}i}$	M1 A1	3.1a 1.1b
	Adding multiples of $\frac{2\pi}{5}$ to their argument $z = 6\sqrt{2}e^{\frac{\pi}{4}i} \times e^{\frac{2\pi k}{5}i} \text{ or } z = 6\sqrt{2} \left[\cos\left(\frac{\pi}{4} + \frac{2\pi k}{5}\right) + i \sin\left(\frac{\pi}{4} + \frac{2\pi k}{5}\right) \right]$	M1	1.1b
	$z = re^{\left(\theta + \frac{2\pi}{5}\right)i}, re^{\left(\theta + \frac{4\pi}{5}\right)i}, re^{\left(\theta + \frac{6\pi}{5}\right)i}, re^{\left(\theta + \frac{8\pi}{5}\right)i} \text{ o.e.}$ or $z = re^{\left(\theta + \frac{2\pi}{5}\right)i}, re^{\left(\theta - \frac{2\pi}{5}\right)i}, re^{\left(\theta - \frac{6\pi}{5}\right)i}, re^{\left(\theta - \frac{8\pi}{5}\right)i} \text{ o.e.}$	A1ft	1.1b
	$z = 6\sqrt{2}e^{\frac{13\pi}{20}i}, 6\sqrt{2}e^{\frac{21\pi}{20}i}, 6\sqrt{2}e^{\frac{29\pi}{20}i}, 6\sqrt{2}e^{\frac{37\pi}{20}i} \text{ o.e.}$ or $z = 6\sqrt{2}e^{\frac{13\pi}{20}i}, 6\sqrt{2}e^{-\frac{19\pi}{20}i}, 6\sqrt{2}e^{-\frac{11\pi}{20}i}, 6\sqrt{2}e^{-\frac{3\pi}{20}i} \text{ o.e.}$	A1	1.1b
		(5)	
(ii)(a)	Circle centre (0, 2) and radius 2 or with the point on the origin	B1	1.1b
	Fully correct 	B1	1.1b
		(2)	
(ii)(b)	$\text{area} = \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 4 \sin^2 \theta \, d\theta \text{ or } \text{area} = \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \alpha \sin^2 \theta \, d\theta$	M1	3.1a
	Uses $\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$ and integrates to the form $A\theta + B \sin 2\theta$ $\text{area} = 8 \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^2 \theta \, d\theta = 4 \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1 - \cos 2\theta \, d\theta = 4\theta - 2 \sin 2\theta$	M1	3.1a
	Uses the limits of $\frac{\pi}{4}$ and $\frac{\pi}{3}$ and subtracts the correct way around $\left[4\left(\frac{\pi}{3}\right) - 2 \sin\left(\frac{2\pi}{3}\right) \right] - \left[4\left(\frac{\pi}{4}\right) - 2 \sin\left(\frac{2\pi}{4}\right) \right]$	M1	1.1b

Area = $\frac{\pi}{3} - \sqrt{3} + 2$	A1	1.1b
	(4)	
Alternative 		
Finds either the areas 1 or 2 $\text{Area 1} = \frac{1}{2} \times 2^2 \times \sin\left(\frac{2\pi}{3}\right) \{ = \sqrt{3} \}$ $\text{Area 2} = \frac{1}{2} \times 2^2 \times \frac{\pi}{3} \left\{ = \frac{2\pi}{3} \right\}$	M1	1.1b
A complete method to find area 3 $\text{Area 3} = \frac{1}{4} \pi \times 2^2 - \frac{1}{2} \times 2^2 \{ = \pi - 2 \}$	M1	3.1a
A complete method to find the required area Shaded area = Area of semi circle – area 1 – area 2 – area 3 $= \left[\frac{1}{2} \pi \times 2^2 \right] - \left[\frac{1}{2} \times 2^2 \times \sin\left(\frac{2\pi}{3}\right) \right] - \left[\frac{1}{2} \times 2^2 \times \frac{\pi}{3} \right] - \left[\frac{1}{4} \pi \times 2^2 - \frac{1}{2} \times 2^2 \right]$ $= 2\pi - \sqrt{3} - \frac{2\pi}{3} - (\pi - 2)$ Or Shaded area = Area of sector – area 1 – area 3 $= \left[\frac{1}{2} \times 4 \times \left(\frac{2\pi}{3}\right) \right] - \left[\frac{1}{2} \times 2^2 \times \sin\left(\frac{2\pi}{3}\right) \right] - \left[\frac{1}{4} \pi \times 2^2 - \frac{1}{2} \times 2^2 \right]$ $= \frac{4\pi}{3} - \sqrt{3} - (\pi - 2)$	M1	3.1a
Area = $\frac{\pi}{3} - \sqrt{3} + 2$	A1	1.1b
	(4)	
(11 marks)		
Notes:		
(i) M1: Finds the modulus and argument of z A1: Correct modulus and argument of z		

M1: Uses a correct method to find to all the other 4 vertices of the pentagon. Must be doing the equivalent of adding/ subtracting multiplies of $\frac{2\pi}{5}$ to the argument.

A1ft: All 4 vertices following through on their modulus and argument. Does not need to be simplified for this mark.

A1: All 4 vertices correct in the required form

(ii)(a)

B1: Circle centre (0, 2) and radius 2 or  with the vertex on the origin.

B1: Fully correct region shaded.

(ii) (b)

M1: Writes the required area using polar coordinates

M1: Uses $\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$ and integrates to the form $A\theta + B \sin 2\theta$

M1: Uses the limits of $\frac{\pi}{4}$ and $\frac{\pi}{3}$ and subtracts the correct way around. Must be some attempt at

area $= \frac{1}{2} \int \alpha \sin \theta^2 d\theta$ and integration.

A1: Correct exact area $= \frac{\pi}{3} - \sqrt{3} + 2$

Alternative

M1: Finds either area 1 or area 2

M1: A complete method to find the area 3

M1: A complete method to find the required area = Area of semi circle – area 1 – area 2 – area 3 or
= Area of sector – area 1 – area 3

A1: Correct exact area $= \frac{\pi}{3} - \sqrt{3} + 2$