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EDEXCEL A LEVEL FURTHER MATHEMATICS: QUESTIONS BY TOPIC

Hyperbolic Functions (8)

Core Pure Mathematics 2 · Topic 8

10 questions · 73 marks · Mark schemes included

Pearson Edexcel A Level Further Mathematics (9FM0), June 2019 - June 2025

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HOW TO USE THIS BOOKLET

Questions are grouped by specification sub-topic and run from the lowest mark tariff to the highest within each one.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme.

Diagrams, graphs, tables and data are printed exactly as they appeared in the original papers.

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QUESTIONS

10 questions · 73 marks · 6 sub-topics



SUB-TOPIC 1.4

INDUCTION: DERIVATIVES

Sub-topic 1.4 · 1 question · 6 marks · lowest tariff first



6. Given that

$$y = e^{2x} \sinh x$$

prove by induction that for $n \in \mathbb{N}$

$$\frac{d^n y}{dx^n} = e^{2x} \left(\frac{3^n + 1}{2} \sinh x + \frac{3^n - 1}{2} \cosh x \right)$$

(6)



(Total for Question 6 is 6 marks)



SUB-TOPIC 7.3

AREAS AND TANGENTS OF POLAR CURVES

Sub-topic 7.3 · 1 question · 4 marks · lowest tariff first



Question 1 continued

(Total for Question 1 is 4 marks)



SUB-TOPIC 8.1

HYPERBOLIC FUNCTIONS AND GRAPHS

Sub-topic 8.1 · 1 question · 7 marks · lowest tariff first



(Total for Question 1 is 7 marks)



SUB-TOPIC 8.2

DIFFERENTIATING AND INTEGRATING HYPERBOLIC FUNCTIONS

Sub-topic 8.2 · 4 questions · 31 marks · lowest tariff first



1. The curve C has equation

$$y = 31 \sinh x - 2 \sinh 2x \quad x \in \mathbb{R}$$

Determine, in terms of natural logarithms, the exact x coordinates of the stationary points of C .

(7)



Question 1 continued

(Total for Question 1 is 7 marks)



2.

$$f(x) = \tanh^{-1}\left(\frac{3-x}{6+x}\right) \quad |x| < \frac{3}{2}$$

(a) Show that

$$f'(x) = -\frac{1}{2x+3} \quad (4)$$

(b) Hence determine $f''(x)$

(1)

(c) Hence show that the Maclaurin series for $f(x)$, up to and including the term in x^2 , is

$$\ln p + qx + rx^2$$

where p , q and r are constants to be determined.

(3)



(Total for Question 2 is 8 marks)



9.

$$y = \cosh^n x \quad n \geq 5$$

(a) (i) Show that

$$\frac{d^2y}{dx^2} = n^2 \cosh^n x - n(n-1) \cosh^{n-2} x \quad (4)$$

(ii) Determine an expression for $\frac{d^4y}{dx^4}$

(2)

(b) Hence determine the first three non-zero terms of the Maclaurin series for y , giving each coefficient in simplest form. (2)



8. Given that

$$y = \cos x \sinh x \quad x \in \mathbb{R}$$

(a) show that

$$\frac{d^4 y}{dx^4} = ky$$

where k is a constant to be determined.

(5)

(b) Hence determine the first three non-zero terms of the Maclaurin series for y , giving each coefficient in simplest form.

(3)





SUB-TOPIC 8.4

LOGARITHMIC FORMS OF INVERSE HYPERBOLIC FUNCTIONS

Sub-topic 8.4 · 2 questions · 19 marks · lowest tariff first



7. Solutions based entirely on graphical or numerical methods are not acceptable.

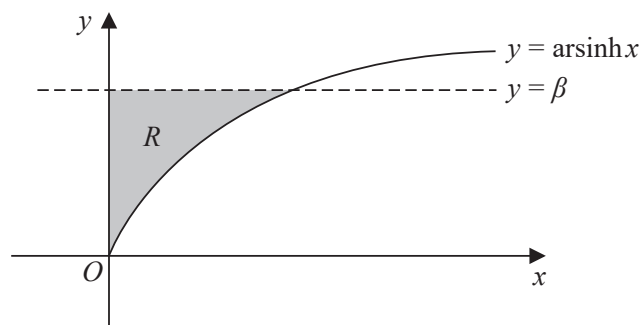


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = \operatorname{arsinh} x \quad x \geqslant 0$$

and the straight line with equation $y = \beta$

The line and the curve intersect at the point with coordinates (α, β)

Given that $\beta = \frac{1}{2} \ln 3$

(a) show that $\alpha = \frac{1}{\sqrt{3}}$

(3)

The finite region R , shown shaded in Figure 1, is bounded by the curve with equation $y = \operatorname{arsinh} x$, the y -axis and the line with equation $y = \beta$

The region R is rotated through 2π radians about the y -axis.

(b) Use calculus to find the exact value of the volume of the solid generated.

(6)





(Total for Question 7 is 9 marks)



1. (a) Prove that

$$\tanh^{-1}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad -k < x < k$$

stating the value of the constant k .

(5)

(b) Hence, or otherwise, solve the equation

$$2x = \tanh\left(\ln \sqrt{2 - 3x}\right)$$

(5)

Question 1 continued

(Total for Question 1 is 10 marks)



SUB-TOPIC 8.5

INTEGRATION USING HYPERBOLIC SUBSTITUTIONS

Sub-topic 8.5 · 1 question · 6 marks · lowest tariff first



Question 3 continued

(Total for Question 3 is 6 marks)



MARK SCHEMES

official mark schemes



Question	Scheme	Marks	AOs
6	$\frac{dy}{dx} = 2e^{2x} \sinh x + e^{2x} \cosh x = ae^{2x} \sinh x + be^{2x} \cosh x$ or $e^{2x} (a \sinh x + b \cosh x)$	M1	2.2a
	$\frac{dy}{dx} = e^{2x} (2 \sinh x + \cosh x)$ $n = 1$ then $\frac{dy}{dx} = e^{2x} \left(\frac{3+1}{2} \sinh x + \frac{3-1}{2} \cosh x \right)$ {so the result is true for $n = 1$ }	A1	2.4
	(Assume the result is true for $n = k$, then) Must be an attempt at the product rule, with k 's in all terms $\frac{d^{k+1}y}{dx^{k+1}} = Ae^{2x} (f(k) \sinh x + g(k) \cosh x) + e^{2x} (f(k) \cosh x + g(k) \sinh x)$ $\frac{d^{k+1}y}{dx^{k+1}} = 2e^{2x} \left(\frac{3^k+1}{2} \sinh x + \frac{3^k-1}{2} \cosh x \right) + e^{2x} \left(\frac{3^k+1}{2} \cosh x + \frac{3^k-1}{2} \sinh x \right)$	M1	2.1
	$= e^{2x} \left(\left(3^k + 1 + \frac{3^k-1}{2} \right) \sinh x + \left(3^k - 1 + \frac{3^k+1}{2} \right) \cosh x \right)$ or $= e^{2x} \left(\frac{3 \times 3^k + 1}{2} \sinh x + \frac{3 \times 3^k - 1}{2} \cosh x \right)$	dM1	1.1b
	$= e^{2x} \left(\frac{3^{k+1}+1}{2} \sinh x + \frac{3^{k+1}-1}{2} \cosh x \right)$	A1	2.1
	If true for $n = k$ then true for $n = k + 1$, and as also true for $n = 1$, so the result is true for all positive integers or true $n \in \mathbb{N}$	A1	2.4
		(6)	
	Alternative using exponential definitions $y = e^{2x} \sinh x \Rightarrow y = e^{2x} \left(\frac{e^x - e^{-x}}{2} \right) = \frac{1}{2} (e^{3x} - e^x)$ $\frac{dy}{dx} = \frac{1}{2} (3e^{3x} - e^x)$	M1	2.2a
	$n = 1$ then	A1	2.4

(Assume the result is true for $n = k$, then) $\frac{d^k y}{dx^k} = e^{2x} \left(\frac{3^k + 1}{2} \left(\frac{e^x - e^{-x}}{2} \right) + \frac{3^k - 1}{2} \left(\frac{e^x + e^{-x}}{2} \right) \right)$ $= e^{2x} \left(\frac{3^k}{2} e^x - \frac{1}{2} e^{-x} \right) = \frac{3^k}{2} e^{3x} - \frac{1}{2} e^x$ Then differentiates $\frac{d^{k+1} y}{dx^{k+1}} = e^{2x} \left(\frac{3^{k+1} + 1}{2} \left(\frac{e^x - e^{-x}}{2} \right) + \frac{3^{k+1} - 1}{2} \left(\frac{e^x + e^{-x}}{2} \right) \right)$	M1	2.1
Simplifies $\frac{d^{k+1} y}{dx^{k+1}} = e^{2x} \left(\frac{3^{k+1}}{2} e^x - \frac{1}{2} e^{-x} \right) = \frac{3^{k+1}}{2} e^{3x} - \frac{1}{2} e^x$	dM1	1.1b
Using the given result $\frac{d^{k+1} y}{dx^{k+1}} = e^{2x} \left(\frac{3^{k+1} + 1}{2} \left(\frac{e^x - e^{-x}}{2} \right) + \frac{3^{k+1} - 1}{2} \left(\frac{e^x + e^{-x}}{2} \right) \right)$ $= 3 \times \frac{3^k}{2} e^{3x} - \frac{1}{2} e^x = \frac{3^{k+1}}{2} e^{3x} - \frac{1}{2} e^x$	A1	2.1
If true for $n = k$ then true for $n = k + 1$, and as also true for $n = 1$, so the result is true for all positive integers or true $n \in \mathbb{N}$	A1	2.4
	(6)	

(6 marks)

Notes:
M1: Differentiates to a correct form

A1: Correct derivative and reaches appropriate form to deduce the result is true for $n = 1$, minimum

$$\frac{dy}{dx} = e^{2x} \left(\frac{4}{2} \sinh x + \frac{2}{2} \cosh x \right)$$

M1: (Makes the inductive assumption and) attempts the $(k + 1)$ th derivative from the k th derivative. Allow slips in coefficients but must be evidence of the use of the product rule. Must be of the form

$$\frac{d^{k+1} y}{dx^{k+1}} = Ae^{2x} (f(k) \sinh x + g(k) \cosh x) + e^{2x} (f(k) \cosh x + g(k) \sinh x)$$

dM1: Dependent on the previous method mark. Factors out the exponential and gathers the $\sinh x$ and $\cosh x$ terms. Accept either form shown or equivalent.


A1: Reaches the correct form from correct work. Must have the “ $k+1$ ” showing. Depends on the previous two method marks.

A1: Makes appropriate concluding sentence covering the points indicated in scheme. Depends on all method marks having been scored. Must have reached at least the second line shown in the dM mark, and made an attempt at checking $n=1$ (though the first A mark need not have been scored if insufficient detail shown).

Maybe seen as a narrative throughout their solution.

This mark requires all necessary brackets throughout.

Alternative: using exponentials

M1: Uses the exponential definition of $\sinh x$ and differentiates to a correct form

A1: Correct derivative and uses the exponential definitions of $\sinh x$ and $\cosh x$ to deduce the result is true for $n=1$

M1: (Makes the inductive assumption and) attempts the $(k+1)$ th derivative from the k th derivative. Uses the exponential definition of $\sinh x$ first and then differentiates to a correct form

dM1: Dependent on the previous method mark. Collects exponential terms and simplifies

A1: Reaches the $k+1$ th derivative. Uses the exponential definitions of $\sinh x$ and $\cosh x$ in the result, simplifies and achieves the correct result

A1: Makes appropriate concluding sentence covering the points indicated in scheme. Depends on all method marks having been scored. Must have reached at least the second line shown in the dM mark, and made an attempt at checking $n=1$ (though the first A mark need not have been scored if insufficient detail shown).

Maybe seen as a narrative throughout their solution.

This mark requires all necessary brackets throughout.

Question	Scheme	Marks	AOs
1	$\text{Area} = \frac{1}{2} \int_0^{\pi} r^2 d\theta = \frac{1}{2} \int_0^{\pi} \left[2\sqrt{\sinh \theta + \cosh \theta} \right]^2 \{d\theta\}$ or $\frac{1}{2} \int_0^{\pi} 4(\sinh \theta + \cosh \theta) \{d\theta\}$	B1	1.1b
	$= 2 \left[\cosh \theta + \sinh \theta \right]_0^{\pi}$ or $2 \int_0^{\pi} \left(\frac{e^{\theta} - e^{-\theta}}{2} + \frac{e^{\theta} + e^{-\theta}}{2} \right) d\theta = 2 \int_0^{\pi} e^{\theta} d\theta = 2 \left[e^{\theta} \right]_0^{\pi}$	M1	1.1b
	$= 2 \left((\cosh \pi + \sinh \pi) - (\cosh 0 + \sinh 0) \right)$ $= 2 \left(\left(\frac{e^{\pi} + e^{-\pi}}{2} + \frac{e^{\pi} - e^{-\pi}}{2} \right) - (1 + 0) \right)$ or $= 2(e^{\pi} - e^0)$	M1	3.1a
	$= 2e^{\pi} - 2 \text{ or } 2(e^{\pi} - 1)$	A1	2.1
		(4)	
(4 marks)			
Notes:			
B1: Correct area formula applied, including the $\frac{1}{2}$ and correct limits, may be seen later, $d\theta$ may be implied. M1: Attempts the integration $\sinh \theta \rightarrow \pm \cosh \theta$ and $\cosh \theta \rightarrow \pm \sinh \theta$ or in terms of exponentials $\int e^{\lambda \theta} d\theta = \frac{1}{\lambda} e^{\lambda \theta}$ M1: Applies their limits to the integral, subtracts (there must be an attempt to integrate) and uses exponential definitions to achieve answer in suitable form. Condone the inclusion of i or a missing $\frac{1}{2}$ from the definitions. This can be implied. A1: Correct exact answer, no i			

Question	Scheme	Marks	AOs
1(a)	$4\sinh^3 x + 3\sinh x \equiv 4\left(\frac{e^x - e^{-x}}{2}\right)^3 + 3\left(\frac{e^x - e^{-x}}{2}\right)$ $\equiv 4\left(\frac{e^{3x} - 3e^x + 3e^{-x} - e^{-3x}}{8}\right) + 3\left(\frac{e^x - e^{-x}}{2}\right)$	M1	2.1
	$\equiv \frac{e^{3x} - e^{-3x}}{2} \equiv \sinh 3x^*$	A1*	1.1b
		(2)	
(b)	$\sinh 3x = 19\sinh x \Rightarrow 4\sinh^3 x + 3\sinh x = 19\sinh x$ $\sinh 3x = 19\sinh x \Rightarrow 4\sinh^3 x - 16\sinh x = 0$ $4\sinh x(\sinh^2 x - 4) = 0$	M1	3.1a
	$\sinh x = 0 \Rightarrow x = 0$	B1	2.2a
	$\sinh^2 x = 4 \Rightarrow \sinh x = \pm 2$ $\Rightarrow x = \ln\left(\pm 2 + \sqrt{(\pm 2)^2 + 1}\right)$	M1	1.1b
	$x = \ln(2 + \sqrt{5}) \text{ or } x = \ln(-2 + \sqrt{5}) \text{ oe e.g. } x = -\ln(2 + \sqrt{5})$	A1	1.1b
	$x = \ln(2 + \sqrt{5}) \text{ and } x = \ln(-2 + \sqrt{5})$ $\text{Alternatively, } x = \ln(\sqrt{5} \pm 2) \text{ oe e.g. } x = \pm \ln(2 + \sqrt{5})$ $\text{or } \frac{1}{2} \ln(9 \pm 4\sqrt{5})$	A1	1.1b
		(5)	
(7 marks)			
Notes			
(a) M1: Begins the proof by expressing $\sinh x$ correctly in terms of exponentials, substitutes and makes progress in cubing the bracket. Award for obtaining an expression of the form $Ae^{3x} + Be^x + Ce^{-x} + De^{-3x}$ but terms do not need to be collected. Note that $(e^{2x} - 2 + e^{-2x})(e^x - e^{-x}) = e^{3x} - e^x - 2e^x + 2e^{-x} + e^{-x} - e^{-3x}$ A1*: Fully correct proof with no errors. Must see $= \sinh 3x$ or e.g. LHS = RHS (b) M1: Uses part (a), collects terms and attempts to factorise or cancel $\sinh x$. This can be implied if they go straight from their cubic to writing all correct answers for $\sinh x$ including zero from their calculator. B1: Deduces the root $x = 0$, allow $\ln 1$ but not $\ln[0 + \ln \sqrt{0+1}]$ M1: Proceeds to $\sinh x = \alpha$ and uses the correct logarithmic form of arsinh to obtain at least one exact value for x Alternatively, candidates proceed from $\sinh x = \alpha$ to substitute the exponential form and then solve a 3TQ in e^x to obtain at least one exact value for x A1: One correct non-zero solution A1: Both correct non-zero solutions and no incorrect other solutions, but isw if they go on to evaluate these answers incorrectly. Allow $x = \ln(\sqrt{5} \pm 2)$ for listing both solutions			

Alternative if candidates substitute the exponential form at the start:

M1: Candidates substitutes the exponential form for each term and proceeds to find a four term cubic equation in $e^{2x} = 0$, which may not be correct.

B1: Deduces the root $x = 0$, allow $\ln 1$ but not $\ln[0 + \ln \sqrt{0+1}]$

M1: They factorise their cubic, proceed to obtain exact values for e^{2x} , then take logs to obtain at least one exact value for x

If they go directly to decimal answers this will usually score M0 unless they recover to exact form.

A1: One correct non-zero solution

A1: Both correct non-zero solutions and no incorrect other solutions, but isw if they go on to evaluate these answers incorrectly.

This is how this would be awarded:

$$\frac{(e^{3x} - e^{-3x})}{2} = 19 \left(\frac{e^x - e^{-x}}{2} \right)$$

$$e^{3x} - e^{-3x} - 19e^x + 19e^{-x} = 0$$

$$e^{6x} - 1 - 19e^{4x} + 19e^{2x} = 0 \quad \text{M1 (for cubic in } e^{2x} = 0)$$

$$e^{6x} + 19e^{4x} + 19e^{2x} - 1 = 0$$

$$\Rightarrow (e^{2x} - 1)(e^{4x} - 18e^{2x} + 1) = 0$$

$$e^{2x} = 1, 9 \pm 4\sqrt{5}$$

$$2x = \ln 1, 2x = \ln(9 \pm 4\sqrt{5})$$

$$x = 0, x = \frac{1}{2} \ln(9 \pm 4\sqrt{5}) \quad \text{B1, M1, A1, A1}$$



Question	Scheme	Marks	AOs
1	$\frac{dy}{dx} = 31 \cosh x - 4 \cosh 2x$	B1	1.1b
	$\frac{dy}{dx} = 31 \cosh x - 4(2 \cosh^2 x - 1)$	M1	3.1a
	$8 \cosh^2 x - 31 \cosh x - 4 = 0$	A1	1.1b
	$(8 \cosh x + 1)(\cosh x - 4) = 0 \Rightarrow \cosh x = \dots$	M1	1.1b
	$\cosh x = 4, \left(-\frac{1}{8}\right)$	A1	1.1b
	$\cosh x = \alpha \Rightarrow x = \ln(\alpha + \sqrt{\alpha^2 - 1}) \text{ or } \ln(\alpha - \sqrt{\alpha^2 - 1})$ or $-\ln(\alpha + \sqrt{\alpha^2 - 1}) \text{ or } \ln(\alpha - \sqrt{\alpha^2 - 1})$ or $\frac{e^x + e^{-x}}{2} = 4 \text{ or } \frac{e^x + e^{-x}}{2} = -\frac{1}{8} \Rightarrow e^{2x} - 8e^x + 7 = 0 \text{ or } e^x = \dots \text{ or } x = \ln(\dots)$	M1	1.2
	$\pm \ln(4 + \sqrt{15}) \text{ or } \ln(4 \pm \sqrt{15})$	A1	2.2a
		(7)	
	Alternative		
	$\frac{dy}{dx} = 31 \cosh x - 4 \cosh 2x \text{ or } 31\left(\frac{e^x + e^{-x}}{2}\right) - 4\left(\frac{e^{2x} + e^{-2x}}{2}\right)$	B1	1.1b
	Using $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$ as required $31\left(\frac{e^x + e^{-x}}{2}\right) - 4\left(\frac{e^{2x} + e^{-2x}}{2}\right) = 0$ leading to $4e^{4x} - 31e^{3x} - 31e^x + 4 = 0$ o.e.	M1 A1	3.1a 1.1b
	Solves $4e^{4x} - 31e^{3x} - 31e^x + 4 = 0$ or $e^x = \dots$	M1	1.1b
	$e^x = 4 \pm \sqrt{15}$ or awrt 7.87, 0.13	A1	1.1b
	$x = \ln(b)$ where b is a real exact value	M1	1.2
	$\ln(4 \pm \sqrt{15})$	A1	2.2a
		(7)	
(7 marks)			
Notes			
B1: Correct differentiation M1: Identifies a correct approach by using a correct identity to make progress to obtain a quadratic in $\cosh x$ A1: Correct 3 term quadratic obtained M1: Solves their 3TQ A1: Correct values (may only see 4 here) M1: Correct process to reach at least one value for x from their $\cosh x$			

A1: Deduces the correct 2 values with no incorrect values or work involving $\cosh x = -\frac{1}{8}$

Alternative

B1: Correct differentiation

M1: Using the exponential form for $\cosh x$, and $\sinh x$ if required, and forms a quartic equation for e^x with all terms simplified and all on one side

A1: Correct quartic equation for e^x

M1: Solves their quartic equation in e^x

A1: Correct values to two decimal places or exact values

M1: $x = \ln(b)$ where b is a real exact value

A1: Deduces the correct 2 values only



Question	Scheme	Marks	AOs
2(a)	$\frac{d\left(\frac{3-x}{6+x}\right)}{dx} = \frac{-(6+x)-(3-x)}{(6+x)^2}$	M1 A1	3.1a 1.1b
	$f(x) = \tanh^{-1}\left(\frac{3-x}{6+x}\right) \Rightarrow f'(x) = \frac{1}{1-\left(\frac{3-x}{6+x}\right)^2} \times \frac{-9}{(6+x)^2}$	dM1	3.1a
	$= \frac{(6+x)^2}{36+12x+x^2-9+6x-x^2} \times \frac{-9}{(6+x)^2} = \frac{-9}{18x+27} = \frac{-1}{2x+3}^*$	A1*	2.1
		(4)	
	Alternative 1 for part (a)		
	$\frac{d\left(\frac{3-x}{6+x}\right)}{dx} = \frac{-(6+x)-(3-x)}{(6+x)^2}$	M1 A1	3.1a 1.1b
	$y = \tanh^{-1}\left(\frac{3-x}{6+x}\right) \Rightarrow \tanh y = \frac{3-x}{6+x} \Rightarrow \operatorname{sech}^2 y \frac{dy}{dx} = \frac{-9}{(6+x)^2}$ $\frac{dy}{dx} = \frac{1}{1-\tanh^2 y} \times \frac{-9}{(6+x)^2} = \frac{1}{1-\left(\frac{3-x}{6+x}\right)^2} \times \frac{-9}{(6+x)^2}$	dM1	3.1a
	$= \frac{(6+x)^2}{36+12x+x^2-9+6x-x^2} \times \frac{-9}{(6+x)^2} = \frac{-9}{18x+27} = \frac{-1}{2x+3}^*$	A1*	2.1
		(4)	
	Alternative 2 for part (a)		
	$f(x) = \tanh^{-1}\left(\frac{3-x}{6+x}\right) = \frac{1}{2} \ln \left(\frac{1+\frac{3-x}{6+x}}{1-\frac{3-x}{6+x}} \right) = \frac{1}{2} \ln \left(\frac{9}{3+2x} \right)$	M1 A1	3.1a 1.1b
	$f(x) = \tanh^{-1}\left(\frac{3-x}{6+x}\right) \Rightarrow f'(x) = \frac{1}{2} \times \frac{3+2x}{9} \times \frac{-18}{(3+2x)^2}$	dM1	3.1a
	$= \frac{-1}{2x+3}^*$	A1*	1.1b
		(4)	
	Alternative 3 for part (a)		
	$\tanh y = f(x) = \left(\frac{3-x}{6+x}\right) \Rightarrow \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{3-x}{6+x} \text{ or } \frac{e^{2y} - 1}{e^{2y} + 1} = \frac{3-x}{6+x}$ $e^{2y} = \frac{9}{2x+3}$	M1 A1	3.1a 1.1b
	$y = \frac{1}{2} \ln \left(\frac{9}{2x+3} \right) \Rightarrow f'(x) = \frac{1}{2} \times \frac{2x+3}{9} \times \frac{-18}{(2x+3)^2}$	dM1	3.1a
	$= \frac{-1}{2x+3}^*$	A1	1.1b
		(4)	

Notes

(a) **Do not allow misreads of tanx for tanhx in this question; this would be a maximum of M1A1 for part (a)**

M1: Attempts to use the quotient (or product) rule on $\frac{3-x}{6+x}$ to obtain an expression of the form

$$\frac{A(6+x) - B(3-x)}{(6+x)^2}, \quad B > 0 \quad \text{or} \quad C(3-x)(6+x)^{-2} + D(6+x)^{-1}$$

Alternatively, candidates may also write $\frac{3-x}{6+x}$ as $-1 + \frac{9}{6+x}$ and then differentiate to find an

expression of the form $\frac{E}{(6+x)^2}$ which may be seen embedded in their working.

They may also write $\frac{3-x}{6+x}$ as $\frac{3}{6+x} - \frac{x}{6+x}$ and then differentiate to find an expression of the

form $\frac{E}{(6+x)^2}$ or

A1: Correct expression in any form.

dM1: A complete method to find the derivative using the chain rule to obtain

$$\frac{1}{1 - \left(\frac{3-x}{6+x}\right)^2} \times \left(\text{their } \frac{-9}{(6+x)^2} \right)$$

This is dependent on the first M mark.

A1*: Reaches the printed answer with sufficient working and no errors, with at least one intermediate line.

Alternative 1:

M1: see main scheme

A1: see main scheme

dM1: A complete method to find the derivative using the chain rule:

Rearranges the equation and uses implicit differentiation. Must proceed from $\text{sech}^2 y$ or equivalent to $1 - \tanh^2 y$ and then substitute for $\tanh y$

This is dependent on the first M mark.

A1*: see main scheme

Alternative 2:

M1: Uses the logarithmic form of artanh to obtain $k \ln \left(\frac{1 + \frac{3-x}{6+x}}{1 - \frac{3-x}{6+x}} \right)$

A1: Correct simplified expression

dM1: A complete method to find the derivative to obtain

$$f'(x) = k \times \left(1 \div \left(\text{their } \frac{9}{3+2x} \right) \right) \times \left(\text{their } \frac{9}{3+2x} \right)^2$$

Alternatively uses log rules to partition their expression and then differentiates each term.

This is dependent on the first M mark.



A1*: Reaches the printed answer with sufficient working and no errors.

Alternative 3:

M1: Takes $\tanh$ of both sides to obtain $\tanh y$ in terms of x and expresses $\tanh y$ correctly in terms of exponentials and makes e^{2y} the subject.

A1: A correct expression for e^{2y}

dM1: Rearranges their function to the form $k \ln \left(\frac{a}{bx+c} \right)$ **and** uses the chain rule to find a derivative of the form $f'(x) = \frac{1}{2} \times \frac{bx+c}{a} \times \frac{k}{(bx+c)^2}$ or uses log rules to partition their expression and then differentiates each term.

This is dependent on the first M mark.

A1*: Reaches the printed answer with sufficient working and no errors

(b)	$[f''(x) =] \frac{2}{(2x+3)^2}$	B1	1.1b
		(1)	
(c)	$f(0) = \tanh^{-1} \left(\frac{1}{2} \right) \left(= \frac{1}{2} \ln 3 \right), f'(0) = -\frac{1}{3}, f''(0) = \frac{2}{9}$	M1	1.1b
	$[f(x)] = f(0) + xf'(0) + \frac{x^2}{2} f''(0)$	M1	1.1b
	$= \ln \sqrt{3} - \frac{1}{3}x + \frac{1}{9}x^2$	A1	1.1b
		(3)	

(8 marks)

Notes

(b)

B1: Correct second derivative in any form such as e.g. $\frac{2}{(4x^2+12x+9)}$ or $2(2x+3)^{-2}$

(c)

M1: Attempts at least two of the values of $f(0)$, $f'(0)$ and $f''(0)$

M1: Correct application of the Maclaurin series for **all** of $f(0)$, $f'(0)$ and $f''(0)$ where all are non-zero values. Substitutes their values into a correct expression. This mark is not dependent.

A1: Correct expansion.

Award if written as a single expression, or p , q and r written separately.

Ignore any extra terms written as powers of x^3 and above, isw when a correct answer is seen.



Question	Scheme	Marks	AOs
9(a)(i)	$\frac{dy}{dx} = \dots \cosh^{n-1} x \sinh x$ $\frac{d^2y}{dx^2} = \dots \cosh^{n-2} x \sinh^2 x + \dots \cosh^{n-1} x \cosh x$ Alternatively $y = \left(\frac{e^x + e^{-x}}{2}\right)^n \text{ leading to } \frac{dy}{dx} = \dots \left(\frac{e^x + e^{-x}}{2}\right)^{n-1} \left(\frac{e^x - e^{-x}}{2}\right)$ $\frac{d^2y}{dx^2} = \dots \left(\frac{e^x + e^{-x}}{2}\right)^{n-2} \left(\frac{e^x - e^{-x}}{2}\right)^2 + \dots \left(\frac{e^x + e^{-x}}{2}\right)^n$	M1	1.1b
	$\frac{dy}{dx} = n \cosh^{n-1} x \sinh x$ $\frac{d^2y}{dx^2} = n(n-1) \cosh^{n-2} x \sinh^2 x + n \cosh^n x$ Alternatively $\frac{dy}{dx} = n \left(\frac{e^x + e^{-x}}{2}\right)^{n-1} \left(\frac{e^x - e^{-x}}{2}\right)$ $\frac{d^2y}{dx^2} = n(n-1) \left(\frac{e^x + e^{-x}}{2}\right)^{n-2} \left(\frac{e^x - e^{-x}}{2}\right)^2 + n \left(\frac{e^x + e^{-x}}{2}\right)^n$	A1	2.1
	$\frac{d^2y}{dx^2} = n(n-1) \cosh^{n-2} x (\cosh^2 x - 1) + n \cosh^n x$	M1	2.1
	$\frac{d^2y}{dx^2} = n^2 \cosh^n x - n(n-1) \cosh^{n-2} x \text{ * cso}$	A1*	1.1b
		(4)	
(a)(ii)	$\frac{d^3y}{dx^3} = \dots \cosh^{n-1} x \sinh x - \dots \cosh^{n-3} x \sinh x$ $\frac{d^4y}{dx^4} = \dots \cosh^{n-2} x \sinh^2 x + \dots \cosh^n x - \dots \cosh^{n-4} x \sinh^2 x - \dots \cosh^n x$	M1	1.1b
	$\frac{d^3y}{dx^3} = n^3 \cosh^{n-1} x \sinh x - n(n-1)(n-2) \cosh^{n-3} x \sinh x$ $\frac{d^4y}{dx^4} = n^3(n-1) \cosh^{n-2} x \sinh^2 x + n^3 \cosh^n x - n(n-1)(n-2)(n-3) \cosh^{n-4} x \sinh^2 x - n(n-1)(n-2) \cosh^{n-2} x$	A1	1.1b
		(2)	
	Alternative 1 using $\frac{d^2y}{dx^2} = n^2 y - n(n-1) \cosh^{n-2} x$ leading to $\frac{d^3y}{dx^3} = n^2 \frac{dy}{dx} - \dots \cosh^{n-3} x \sinh x$ $\frac{d^4y}{dx^4} = n^2 \frac{d^2y}{dx^2} - \dots \cosh^{n-4} x \sinh^2 x - \dots \cosh^{n-2} x$	M1	1.1b

	$\frac{d^3y}{dx^3} = n^2 \frac{dy}{dx} - n(n-1)(n-2) \cosh^{n-3} x \sinh x$ $\frac{d^4y}{dx^4} = n^2 \frac{d^2y}{dx^2} - n(n-1)(n-2)(n-3) \cosh^{n-4} x \sinh^2 x$ $- n(n-1)(n-2) \cosh^{n-2} x$	A1	1.1b
		(2)	
	Alternative 2 $y = \cosh^n x \Rightarrow \frac{d^2y}{dx^2} = n^2 \cosh^n x - n(n-1) \cosh^{n-2} x$ $y = \cosh^{n-2} x \Rightarrow \frac{d^2y}{dx^2} = \dots \cosh^{n-2} x - \dots \cosh^{n-4} x$ $\frac{d^4y}{dx^4} = n^2 [n^2 \cosh^n x - n(n-1) \cosh^{n-2} x]$ $- n(n-1) [\dots \cosh^{n-2} x - \dots \cosh^{n-4} x]$	M1	1.1b
	$y = \cosh^n x \Rightarrow \frac{d^2y}{dx^2} = n^2 \cosh^n x - n(n-1) \cosh^{n-2} x$ $y = \cosh^{n-2} x \Rightarrow \frac{d^2y}{dx^2} = (n-2)^2 \cosh^{n-2} x - (n-2)(n-3) \cosh^{n-4} x$ $\frac{d^4y}{dx^4} = n^2 [n^2 \cosh^n x - n(n-1) \cosh^{n-2} x]$ $- n(n-1) [(n-2)^2 \cosh^{n-2} x - (n-2)(n-3) \cosh^{n-4} x]$	A1	1.1b
		(2)	
	Alternative 3 Using $\frac{d^2y}{dx^2} = n^2 \left(\frac{e^x + e^{-x}}{2}\right)^n - n(n-1) \left(\frac{e^x + e^{-x}}{2}\right)^{n-2}$ leading to $\frac{d^3y}{dx^3} = \dots \left(\frac{e^x + e^{-x}}{2}\right)^{n-1} \left(\frac{e^x - e^{-x}}{2}\right) - \dots \left(\frac{e^x + e^{-x}}{2}\right)^{n-3} \left(\frac{e^x - e^{-x}}{2}\right)$ $\frac{d^4y}{dx^4} = \dots \left(\frac{e^x + e^{-x}}{2}\right)^{n-2} \left(\frac{e^x - e^{-x}}{2}\right)^2 + \dots \left(\frac{e^x + e^{-x}}{2}\right)^{n-2}$ $- \dots \left(\frac{e^x + e^{-x}}{2}\right)^{n-4} \left(\frac{e^x - e^{-x}}{2}\right)^2 - \dots \left(\frac{e^x + e^{-x}}{2}\right)^{n-2}$	M1	1.1b
	$\frac{d^3y}{dx^3} = n^3 \left(\frac{e^x + e^{-x}}{2}\right)^{n-1} \left(\frac{e^x - e^{-x}}{2}\right) - n(n-1)(n-2) \left(\frac{e^x + e^{-x}}{2}\right)^{n-3} \left(\frac{e^x - e^{-x}}{2}\right)$	A1	1.1b

	$\frac{d^4 y}{dx^4} = n^3(n-1)\left(\frac{e^x + e^{-x}}{2}\right)^{n-2}\left(\frac{e^x - e^{-x}}{2}\right)^2 + n^3\left(\frac{e^x + e^{-x}}{2}\right)^{n-2}$ $-n(n-1)(n-2)(n-3)\left(\frac{e^x + e^{-x}}{2}\right)^{n-4}\left(\frac{e^x - e^{-x}}{2}\right)^2 - n(n-1)(n-2)\left(\frac{e^x + e^{-x}}{2}\right)^{n-2}$		
		(2)	
(b)	When $x = 0$ $y = 1, y' = 0, y'' = n^2 - n(n-1), y^{(3)} = 0,$ $y^{(4)} = n^3 - n(n-1)(n-2)$ Uses their values in the expansion $y = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y^{(3)}(0) + \frac{x^4}{4!}y^{(4)}(0) + \dots$	M1	1.1b
	$y = 1 + \frac{nx^2}{2} + \frac{(3n^2-2n)x^4}{24} + \dots$ CSO	A1	2.5
		(2)	

(8 marks)

Notes:
(a)(i)

M1: Uses the chain rule and product rule to find the first and second derivatives which must be of the required form, condone sign slips

Alternatively uses the exponential definition and uses the chain rule and product rule to find the first and second derivatives which must be of the required form.

A1: Correct unsimplified first and second derivatives, may be in exponential form.

M1: Uses the identity $\pm \cosh^2 x \pm \sinh^2 x = 1$

A1*: Achieves the printed answer with no errors or omissions e.g. missing x 's

(a)(ii)

M1: Uses the chain rule and product rule to find the third and fourth derivatives which must be of the required form, condone sign slips

A1: Correct fourth derivative, does not need to be simplified ISW

Alternative 1

M1: Using $\frac{d^2 y}{dx^2} = n^2 y - n(n-1) \cosh^{n-2} x$ to find the third and fourth derivatives which must be of the required form, condone sign slips

A1: Correct fourth derivative, does not need to be simplified ISW

Alternative 2

M1: Using $y = \cosh^n x \Rightarrow \frac{d^2 y}{dx^2} = n^2 \cosh^n x - n(n-1) \cosh^{n-2} x$

$y = \cosh^{n-2} x \Rightarrow \frac{d^2 y}{dx^2} = \dots \cosh^{n-2} x - \dots \cosh^{n-4} x$ leading to

$$\frac{d^4 y}{dx^4} = n^2 [n^2 \cosh^n x - n(n-1) \cosh^{n-2} x] - n(n-1) \left[\text{their } \frac{d(\cosh^{n-2} x)}{dx} \right]$$

A1: Correct fourth derivative, does not need to be simplified ISW

Alternative 3

M1: Uses the exponential definition and uses the chain rule and product rule to find the third and fourth derivatives which must be of the required form.

A1: Correct fourth derivative, does not need to be simplified ISW

(b)

M1: Attempts the evaluation of all four of their derivatives at $x = 0$ and applies the Maclaurin formula with their values. Note that $y^{(1)}(0) = 0$ and $y^{(3)}(0) = 0$ may be implied as they will have a multiple of $\sinh 0$. If their $y^{(3)}(0) \neq 0$ they allow this mark for their first 3 non-zero terms

A1: Correct simplified expansion from correct derivatives cso

Question	Scheme	Marks	AOs
8(a)	$\frac{dy}{dx} = A \sin x \sinh x + B \cos x \cosh x$	M1	1.1b
	$\frac{dy}{dx} = -\sin x \sinh x + \cos x \cosh x$ and $\frac{d^2y}{dx^2} = -2 \sin x \cosh x$	A1	1.1b
	$\frac{d^2y}{dx^2} = C \sin x \cosh x$ $\frac{d^3y}{dx^3} = D \cos x \cosh x + E \sin x \sinh x$ $\frac{d^4y}{dx^4} = F \cos x \sinh x$ (unsimplified)	M1	2.1
	$\frac{d^3y}{dx^3} = -2 \cos x \cosh x - 2 \sin x \sinh x$ and $\frac{d^4y}{dx^4} = -4 \cos x \sinh x$	A1	1.1b
	$\frac{d^4y}{dx^4} = -4y$	A1	2.1
		(5)	
(b)	$\frac{d^5y}{dx^5} = -4 \frac{dy}{dx}$ or $\frac{d^5y}{dx^5} = -4 \cos x \cosh x + 4 \sin x \sinh x$ or $y^{(5)} = -4 \times \text{their } y'$	B1ft	2.2a
	When $x = 0$ $y = 0, y' = 1, y'' = 0, y^{(3)} = -2, y^{(4)} = 0, y^{(5)} = -4$ Uses their values in the expansion $y = y(0) + xy'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y^{(3)}(0) + \frac{x^4}{4!} y^{(4)}(0) + \frac{x^5}{5!} y^{(5)}(0) + \dots$	M1	1.1b
	$(y =) x - \frac{x^3}{3} - \frac{x^5}{30} + \dots$	A1	2.5
		(3)	
(8 marks)			
Notes:			
(a) M1: Uses the product rule to find the correct form for at least the first derivative. A1: Achieves correct first and second derivatives. Need not be simplified. M1: Uses the product rule to find the correct form for the second, third and fourth derivatives. A1: Correct (unsimplified) expressions for the third and fourth derivatives. A1: Correct fourth derivative in terms of y from correct work. Note do not accept just $k = -4$, we must see the $\frac{d^4y}{dx^4} = -4y$			

(b)

B1ft: Correct fifth derivative or value for the fifth derivative deduced following through on their k from part (a). If they have stated y' incorrectly allow for $y^{(5)} = -4$ × their y' provided their y' is non-zero

M1: Attempts the evaluation of all the derivatives at $x = 0$ and applies the Maclaurin formula correctly with their values to achieve at least three non-zero terms.

A1: Correct simplified expansion. Accept with $f(x) = \dots$ or with nothing before – mark the expression. Accept as a list of terms.

8(a) Alt	$u = \cos x \quad v = \sinh x$ $u' = -\sin x \quad v' = \cosh x$ $u'' = -\cos x \quad v'' = \sinh x$ $u''' = \sin x \quad v''' = \cosh x$ $u^{iv} = \cos x \quad v^{iv} = \sinh x$	M1 A1	1.1b 1.1b
	$\frac{d^4 y}{dx^4} = uv^{iv} + 4u'v''' + 6u''v'' + 4u'''v' + u^{iv}v = \dots$	M1	2.1
	$= \cos x \sinh x - 4 \sin x \cosh x - 6 \cos x \sinh x + 4 \sin x \cosh x + \cos x \sinh x$	A1	1.1b
	$\frac{d^4 y}{dx^4} = (-4 \cos x \sinh x) - 4y$	A1	2.1
		(5)	

There may be attempts via Leibnitz theorem.

M1: Attempts first four derivatives for each of $\cos x$ and $\sinh x$

A1: Correct derivatives.

M1: Applies Leibnitz theorem correctly with their derivatives.

A1: Correct unsimplified expression.

A1: Correct fourth derivative in terms of y .

There may be rare cases where exponential forms are used in part (a). For reference here are the derivatives they should get.

$$y = \frac{1}{2}(e^x - e^{-x})\cos x$$

$$\frac{dy}{dx} = \frac{1}{2}(e^x + e^{-x})\cos x - \frac{1}{2}(e^x - e^{-x})\sin x = \frac{1}{2}e^x(\cos x - \sin x) + \frac{1}{2}e^{-x}(\cos x + \sin x)$$

$$\frac{d^2 y}{dx^2} = -(e^x + e^{-x})\sin x \left[= \frac{1}{2}e^x(\cos x - \sin x) + \frac{1}{2}e^x(-\sin x - \cos x) - \frac{1}{2}e^{-x}(\cos x + \sin x) + \frac{1}{2}e^{-x}(-\sin x + \cos x) \right]$$

$$\frac{d^3 y}{dx^3} = -(e^x - e^{-x})\sin x - (e^x + e^{-x})\cos x$$

$$\frac{d^4 y}{dx^4} = -(e^x + e^{-x})\sin x - (e^x - e^{-x})\cos x - (e^x - e^{-x})\cos x + (e^x + e^{-x})\sin x = -2(e^x - e^{-x})\cos x = -4\sinh x \cos x$$

NB Some are misreading the function as $y = \cosh x \sinh x$ or $y = \cos x \sin x$. These can be score method marks as long as work is equivalent (ie via product rule), but reduction of the problem to triviality via an identity is less demand and so will score M0. Allow SC A1 for the first A if the equivalent form is reached via the misread to that shown in the scheme.





Question	Scheme	Marks	AOs
7(a)	Using $\operatorname{arsinh} \alpha = \frac{1}{2} \ln 3$ $\alpha = \frac{e^{\frac{1}{2} \ln 3} - e^{-\frac{1}{2} \ln 3}}{2}$	B1	1.2
	$\alpha = \frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{2} \Rightarrow \alpha = \dots$ $\alpha + \sqrt{\alpha^2 + 1} = \sqrt{3}$ $\sqrt{\alpha^2 + 1} = \sqrt{3} - \alpha$ $\alpha^2 + 1 = 3 - 2\sqrt{3}\alpha + \alpha^2 \Rightarrow \alpha = \dots$	M1	1.1b
	$\alpha = \frac{\sqrt{3}}{3} \text{ or } \frac{1}{\sqrt{3}}$	A1	2.2a
		(3)	
(b)	Volume = $\pi \int_0^{\frac{1}{2} \ln 3} \sinh^2 y \, dy$	B1	2.5
	$\{\pi\} \int \left(\frac{e^y - e^{-y}}{2} \right)^2 dy = \{\pi\} \int \left(\frac{e^{2y} - 2 + e^{-2y}}{4} \right) dy$ or $\{\pi\} \int \frac{1}{2} \cosh 2y - \frac{1}{2} dy$	M1	3.1a
	$\frac{1}{4} \left(\frac{1}{2} e^{2y} - 2y - \frac{1}{2} e^{-2y} \right)$ or $\frac{1}{4} \sinh 2y - \frac{1}{2} y$	dM1 A1	1.1b 1.1b
	Use limits $y = 0$ and $y = \frac{1}{2} \ln 3$ and subtracts the correct way round	M1	1.1b
	$\frac{\pi}{4} \left(\frac{4}{3} - \ln 3 \right)$ or exact equivalent	A1	1.1b
		(6)	
(9 marks)			

Notes:

(a)

 B1: Recalls the definition for $\sinh \left(\frac{1}{2} \ln 3 \right)$ or forms an equation for $\operatorname{arsinh} x$

 M1: Uses logarithms to find a value for α or forms and solves a correct equation without log

 A1: Deduces the correct exact value for α

Note using the result

$$\ln \left(\frac{1}{\sqrt{3}} + \sqrt{\left(\frac{1}{\sqrt{3}} \right)^2 + 1} \right) = \ln \left(\frac{1}{\sqrt{3}} + \sqrt{\frac{4}{3}} \right) = \ln \sqrt{3} = \frac{1}{2} \ln 3 \text{ therefore } \operatorname{arsinh} \left(\frac{1}{\sqrt{3}} \right) = \frac{1}{2} \ln 3$$



B1 for substituting in α into $\operatorname{arcsinh} x$, M1 for rearranging to show $\frac{1}{2} \ln 3$, A1 for conclusion

(b)

B1: Correct expression for the volume $\pi \int_0^{\frac{1}{2} \ln 3} \sinh^2 y \, dy$ requires integration signs, dy and correct limits.

M1: Uses the exponential formula for $\sinh y$ or the identity $\cosh 2y = \pm 1 \pm 2 \sinh^2 y$ to write in a form which can be integrated at least one term

dM1: Dependent of previous method mark, integrates.

A1: Correct integration.

M1: Correct use of the limits $y = 0$ and $y = \frac{1}{2} \ln 3$

A1: Correct exact volume.



Question	Scheme	Marks	AOs
1(a)	$y = \tanh^{-1}(x) \Rightarrow \tanh y = x \Rightarrow x = \frac{\sinh y}{\cosh y} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$	M1 A1	2.1 1.1b
	Note that some candidates only have one variable and reach e.g. $x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ or $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ Allow this to score M1A1		
	$x(e^{2y} + 1) = e^{2y} - 1 \Rightarrow e^{2y}(1 - x) = 1 + x \Rightarrow e^{2y} = \frac{1+x}{1-x}$	M1	1.1b
	$e^{2y} = \frac{1+x}{1-x} \Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right) \Rightarrow y = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)^*$	A1*	2.1
	Note that $e^{2y}(x-1) + x + 1 = 0$ can be solved as a quadratic in e^y : $e^y = \frac{-\sqrt{0-4(x-1)(x+1)}}{2(x-1)} = \frac{-\sqrt{4(1-x)(x+1)}}{2(x-1)} = \frac{2\sqrt{(1-x)(x+1)}}{2(1-x)}$ $= \frac{\sqrt{(x+1)}}{\sqrt{(1-x)}} \Rightarrow y = \frac{1}{2} \ln\left(\frac{x+1}{1-x}\right)^*$ Score M1 for an attempt at the quadratic formula to make e^y the subject (condone $\pm \sqrt{\dots}$) and A1* for a correct solution that rejects the positive root at some point and deals with the $(x-1)$ bracket correctly		
	$k = 1$ or $-1 < x < 1$	B1	1.1b
		(5)	
(a) Way 2	$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \Rightarrow x = \tanh\left(\frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)\right) = \frac{e^{\frac{\ln(1+x)}{1-x}} - 1}{e^{\frac{\ln(1+x)}{1-x}} + 1}$	M1 A1	2.1 1.1b
	$x = \frac{e^{\frac{\ln(1+x)}{1-x}} - 1}{e^{\frac{\ln(1+x)}{1-x}} + 1} = \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} = x$ Hence true, QED, tick etc.	M1 A1	1.1b 2.1
(b)	$2x = \tanh(\ln \sqrt{2-3x}) \Rightarrow \tanh^{-1}(2x) = \ln \sqrt{2-3x}$	M1	3.1a
	$\frac{1}{2} \ln\left(\frac{1+2x}{1-2x}\right) = \frac{1}{2} \ln(2-3x) \Rightarrow \frac{1+2x}{1-2x} = 2-3x$	M1	2.1
	$6x^2 - 9x + 1 = 0$	A1	1.1b
	$6x^2 - 9x + 1 = 0 \Rightarrow x = \dots$	M1	1.1b
	$x = \frac{9 - \sqrt{57}}{12}$	A1	3.2a
		(5)	



	Alternative for first 2 marks of (b)		
	$2x = \tanh\left(\ln\sqrt{2-3x}\right) \Rightarrow 2x = \frac{e^{2\ln\sqrt{2-3x}} - 1}{e^{2\ln\sqrt{2-3x}} + 1}$	M1	3.1a
	$\Rightarrow \frac{2-3x-1}{2-3x+1} = 2x$	M1	2.1
(10 marks)			
Notes			
(a) If you come across any attempts to use calculus to prove the result – send to review M1: Begins the proof by expressing tanh in terms of exponentials and forms an equation in exponentials. The exponential form can be any of $\frac{(e^y - e^{-y})}{2}$, $\frac{e^y - e^{-y}}{e^y + e^{-y}}$, $\frac{e^{2y} - 1}{e^{2y} + 1}$ Allow any variables to be used but the final answer must be in terms of x. Allow alternative notation for $\tanh^{-1}x$ e.g. artanh, arctanh. A1: Correct expression for “x” in terms of exponentials M1: Full method to make e^{2y} the subject of the formula. This must be correct algebra so allow sign errors only. A1*: Completes the proof by using logs correctly and reaches the printed answer with no errors. Allow e.g. $\frac{1}{2}\ln\left(\frac{x+1}{1-x}\right)$, $\frac{1}{2}\ln\frac{x+1}{1-x}$, $\frac{1}{2}\ln\left \frac{x+1}{1-x}\right$. Need to see $\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ as a conclusion but allow if the proof concludes that $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ with y defined as $\tanh^{-1}x$ earlier. B1: Correct value for k or writes $-1 < x < 1$ Way 2 M1: Starts with result, takes tanh of both sides and expresses in terms of exponentials A1: Correct expression M1: Eliminates exponentials and logs and simplifies A1: Correct result (i.e. $x = x$) with conclusion B1: Correct value for k or writes $-1 < x < 1$ (b) M1: Adopts a correct strategy by taking $\tanh^{-1}$ of both sides M1: Makes the link with part (a) by replacing artanh(2x) with $\frac{1}{2}\ln\left(\frac{1+2x}{1-2x}\right)$ and demonstrates the use of the power law of logs to obtain an equation with logs removed correctly. A1: Obtains the correct 3TQ M1: Solves their 3TQ using a correct method (see General Guidance – if no working is shown (calculator) and the roots are correct for their quadratic, allow M1) A1: Correct value with the other solution rejected (accept rejection by omission) so $x = \frac{9 \pm \sqrt{57}}{12}$ scores A0 unless the positive root is rejected Alternative for first 2 marks of (b) M1: Adopts a correct strategy by expressing tanh in terms of exponentials M1: Demonstrates the use of the power law of logs to obtain an equation with logs removed correctly			

Question	Scheme	Marks	AOs
3(a) Way 1	$x = \frac{3}{2} \sinh u$	B1	2.1
	$\int \frac{dx}{\sqrt{4x^2+9}} = \int \frac{1}{\sqrt{4(\frac{9}{4})\sinh^2 u + 9}} \times \frac{3}{2} \cosh u \, du$	M1	3.1a
	$= \int \frac{1}{2} du$	A1	1.1b
	$= \int \frac{1}{2} du = \frac{1}{2} u = \frac{1}{2} \sinh^{-1} \left(\frac{2x}{3} \right) + c$	A1	1.1b
		(4)	
(a) Way 2	$x = \frac{3}{2} \tan u$	B1	2.1
	$\int \frac{dx}{\sqrt{4x^2+9}} = \int \frac{1}{\sqrt{4(\frac{9}{4})\tan^2 u + 9}} \times \frac{3}{2} \sec^2 u \, du$	M1	3.1a
	$= \int \frac{1}{2} \sec u \, du$	A1	1.1b
	$= \frac{1}{2} \ln(\sec u + \tan u) = \frac{1}{2} \ln \left(\frac{2x}{3} + \sqrt{1 + \left(\frac{2x}{3} \right)^2} \right)$ $u = \frac{1}{2} \sinh^{-1} \left(\frac{2x}{3} \right) + c$	A1	1.1b
(a) Way 3	$x = \frac{1}{2} u$ or $x = ku$ where $k > 0$ $k \neq 1$	B1	2.1
	$\int \frac{dx}{\sqrt{4x^2+9}} = \int \frac{1}{\sqrt{4(\frac{1}{4})u^2+9}} \times \frac{1}{2} du$	M1	3.1a
	$= \frac{1}{2} \int \frac{1}{\sqrt{u^2+9}} du \left(\text{or } \frac{1}{2} \int \frac{1}{\sqrt{u^2 + \frac{9}{4k^2}}} du \text{ for } x = ku \right)$	A1	1.1b
	$= \frac{1}{2} \sinh^{-1} \frac{u}{3} = \frac{1}{2} \sinh^{-1} \frac{2x}{3} + c$	A1	1.1b
(b)	Mean value = $\frac{1}{3(-0)} \left[\frac{1}{2} \sinh^{-1} \left(\frac{2x}{3} \right) \right]_0^3 = \frac{1}{3} \times \frac{1}{2} \sinh^{-1} \left(\frac{2 \times 3}{3} \right) (-0)$	M1	2.1
	$= \frac{1}{6} \ln(2 + \sqrt{5})$ (Brackets are required)	A1ft	1.1b
		(2)	
(6 marks)			

Notes

(a)

B1: Selects an appropriate substitution leading to an integrable form

M1: Demonstrates a fully correct method for the substitution that includes substituting into the function and dealing with the “dx”. The substitution being substituted does not need to be

“correct” for this mark but the substitution must be an attempt at $\int \frac{1}{\sqrt{4[f(u)]^2 + 9}} \times f'(u) \, du$

with the $f'(u)$ correct for their substitution. E.g. if $x = \frac{1}{2}u$ is used, must see $dx = \frac{1}{2}du$ not $2du$.

A1: Correct simplified integral in terms of u from correct work and from a correct substitutionA1: Correct answer including “+ c”. Allow arcsinh or arsinh for $\sinh^{-1}$ from correct work and from a correct substitution

(b)

M1: Correctly applies the method for the mean value for their integration which must be of the form specified in part (a) and substitutes the limits 0 and 3 but condone omission of 0

A1: Correct exact answer (follow through their A and B). **Brackets are required if appropriate.**