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EDEXCEL A LEVEL FURTHER MATHEMATICS: QUESTIONS BY TOPIC

Algorithms and Graph Theory (1)

Decision Mathematics 1 · Topic 1

25 questions · 245 marks · Mark schemes included

Pearson Edexcel A Level Further Mathematics (9FM0), June 2019 - June 2025

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 Piccadilly Business Centre, Unit C, Aldow Enterprise Park
Blackett Street, Manchester, M12 6AE

HOW TO USE THIS BOOKLET

Questions are grouped by specification sub-topic and run from the lowest mark tariff to the highest within each one.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme.

Diagrams, graphs, tables and data are printed exactly as they appeared in the original papers.

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QUESTIONS

25 questions · 245 marks · 10 sub-topics



SUB-TOPIC 1.1

ALGORITHMS AND ORDER

Sub-topic 1.1 · 1 question · 12 marks · lowest tariff first



6. The following algorithm determines the number of comparisons made when Prim's algorithm is applied to K_n

Step 1	Start
Step 2	Input the value of n
Step 3	Let $a = 1$
Step 4	Let $b = n - 2$
Step 5	Let $c = b$
Step 6	Let $a = a + 1$
Step 7	Let $b = b - 1$
Step 8	Let $c = c + (a \times b) + (a - 1)$
Step 9	If $b > 0$ go to Step 6
Step 10	Output c
Step 11	Stop

- (a) For K_5 , complete the table in the answer book to show the results obtained at each step of the algorithm.

(3)



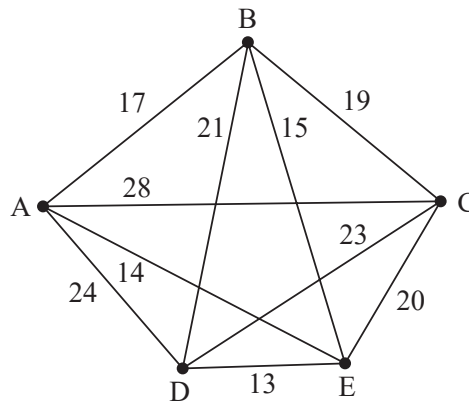


Figure 4

The weights of the ten arcs in Figure 4 are

17 21 24 14 23 13 15 19 28 20

- (b) (i) Starting at the left-hand end of the above list, sort the list into ascending order using bubble sort. You need only write down the state of the list at the end of each pass.

- (ii) Find the total number of comparisons performed during the sort.

(5)

- (c) Find the maximum total number of comparisons required to sort the weights of the 10 arcs of K_5 into ascending order using bubble sort.

(1)

It is given that the maximum total number of comparisons required to sort the weights of the arcs of K_n into ascending order using bubble sort is

$$\lambda n(n-1)(n+1)(n-2)$$

where λ is a constant.

- (d) Determine the maximum total number of comparisons required to sort the weights of the arcs of K_{50} into ascending order using bubble sort. You must make your method and working clear.

(3)

(Total for Question 6 is 12 marks)

SUB-TOPIC 1.2

BIN PACKING AND SORTING

Sub-topic 1.2 · 7 questions · 57 marks · lowest tariff first



1. A gardener needs the following lengths of string. All lengths are in metres.

4.3 6.1 5.1 4.7 2.5 5.9 3.4 1.7 2.1 0.4 1.3

She cuts the lengths from balls of string. Each ball contains 10m of string.

- (a) Calculate a lower bound for the number of balls of string the gardener needs.
You must make your method clear.

(2)

- (b) Use the first-fit bin packing algorithm to determine how the lengths could be cut
from the balls of string.

(3)

(Total for Question 1 is 5 marks)



5. The nine distinct numbers in the following list are to be packed into bins of size 50

23 17 19 x 24 8 18 10 21

When the first-fit bin packing algorithm is applied to the numbers in the list it results in the following allocation.

Bin 1: 23 17 8

Bin 2: 19 x 10

Bin 3: 24 18

Bin 4: 21

- (a) Explain why $13 < x < 21$

(3)

The same list of numbers is to be sorted into descending order. A bubble sort, starting at the left-hand end of the list, is to be used to obtain the sorted list. After the first complete pass the list is

23 19 17 24 x 18 10 21 8

- (b) Using this information, write down the smallest interval that must contain x , giving your answer as an inequality.

(2)

When the first-fit decreasing bin packing algorithm is applied to the nine distinct numbers it results in the following allocation.

Bin 1: 24 23

Bin 2: 21 19 10

Bin 3: 18 17 x

Bin 4: 8

Given that only one of the bins is full and that x is an integer,

- (c) calculate the value of x . You must give reasons for your answer.

(2)

(Total for Question 5 is 7 marks)



1. 2.1 1.7 3.0 1.9 3.2 1.2 3.3 1.4 1.5 0.2

- (a) Use the first-fit bin packing algorithm to determine how the numbers listed above can be packed into bins of size 5 (2)

The list of numbers is now to be sorted into descending order.

- (b) Perform a quick sort on the original list to obtain the sorted list. You should show the result of each pass and identify your pivots clearly. (4)

For a list of n numbers, the quick sort algorithm has, on average, order $n \log n$.

Given that it takes 2.32 seconds to run the algorithm when $n = 450$

- (c) calculate approximately how long it will take, to the nearest tenth of a second, to run the algorithm when $n = 11\,250$. You should make your method and working clear. (2)

(Total for Question 1 is 8 marks)



5. 30 12 5 2 23 18 36 10 15 24

- (a) The list of ten numbers above is to be sorted into descending order. Use a quick sort to obtain the sorted list. You should show the result of each pass and identify your pivots clearly.

(4)

The ten numbers are to be packed into bins of size n , where n is a positive integer.

When the first-fit bin packing algorithm is applied to the original list of ten numbers, the following allocation is obtained.

Bin 1: 30 12 2

Bin 2: 5 23 10

Bin 3: 18 15

Bin 4: 36

Bin 5: 24

- (b) Explain why the value of the integer n must be either 44 or 45

(3)

- (c) Use the first-fit decreasing bin packing algorithm to determine how the numbers can be packed into bins of size 45

(3)

(Total for Question 5 is 10 marks)



4. The eleven distinct numbers listed below are to be packed into bins of size 40

15 22 3 9 23 x 5 4 18 20 13

It is known that x

- is an integer less than 40
- is the largest number in the list

(a) Explain why it **is not** possible to pack the numbers into 3 bins of size 40 (1)

Given that it **is** possible to pack the numbers into 4 bins of size 40

(b) determine the range of values for x (2)

(c) Use the first-fit bin packing algorithm to determine how the numbers can be packed into bins of size 40 (3)

(d) Carry out a quick sort to produce a list of the numbers in descending order. You should show the result of each pass and identify your pivots clearly. (4)

When the first-fit decreasing bin packing algorithm is applied to the list, neither the 15 nor the 13 is placed in the first bin.

(e) Determine the value of x . You must give reasons for your answer. (2)

(Total for Question 4 is 12 marks)



1.

4.7 2.9 5.5 1.4 5.8 2.8 3.8 6.5 5.1 6.3 4.1

- (a) Use the first-fit bin packing algorithm to determine how the numbers listed above can be packed into bins of size 13

(3)

- (b) Use the first-fit decreasing bin packing algorithm to determine how the numbers listed above can be packed into bins of size 13

(2)

(Total for Question 1 is 5 marks)



2.

271 828 182 845 904 523 536 028 747 135

The list of ten numbers above is to be sorted into descending order.

- (a) Perform a quick sort on the list to obtain the sorted list. You should show the result of each pass and identify the pivots clearly.

(4)

A list of n numbers is to be sorted into descending order using bubble sort.

- (b) Determine, in the worst case, the total number of comparisons required to sort the list. Give your answer as a simplified expression in terms of n .

(2)

The following algorithm determines an approximation to the value of e

Step 1 Start
 Step 2 Let $a = 1$
 Step 3 Let $b = 1$
 Step 4 Let $c = 1$
 Step 5 Let $d = a$
 Step 6 Let $c = c \times b$
 Step 7 Let $d = d + \frac{1}{c}$
 Step 8 If $b = 6$ go to Step 11
 Step 9 Let $b = b + 1$
 Step 10 Go to Step 6
 Step 11 Output d
 Step 12 Stop

- (c) Complete the table in the answer book to show the results obtained at each step of the algorithm.

(3)

- (d) Calculate, to 3 significant figures, the percentage error in using the value found in (c) to approximate the value of e

(1)

(Total for Question 2 is 10 marks)



SUB-TOPIC 1.3

EULERIAN GRAPHS

Sub-topic 1.3 · 1 question · 7 marks · lowest tariff first



1.

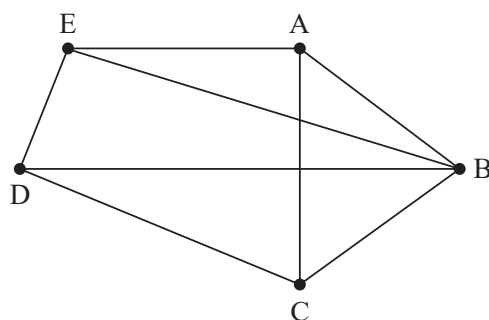
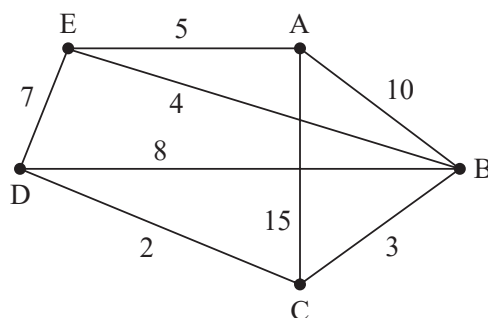


Figure 1

Figure 1 shows the graph G .

- (a) State whether G is Eulerian, semi-Eulerian, or neither, giving a reason for your answer. (1)
- (b) Write down an example of a Hamiltonian cycle on G . (1)
- (c) State whether or not G is planar, justifying your answer. (1)
- (d) State the number of arcs that would need to be added to G to make the graph K_5 . (1)

**Figure 2**

Direct roads between five villages, A, B, C, D and E, are represented in Figure 2. The weight on each arc is the time, in minutes, required to travel along the corresponding road. Floyd's algorithm is to be used to find the complete network of shortest times between the five villages.

- (e) For the network represented in Figure 2, complete the initial time matrix in the answer book.

(1)

The time matrix after four iterations of Floyd's algorithm is shown in Table 1.

	A	B	C	D	E
A	–	10	13	15	5
B	10	–	3	5	4
C	13	3	–	2	7
D	15	5	2	–	7
E	5	4	7	7	–

Table 1

- (f) Perform the final iteration of Floyd's algorithm that follows from Table 1, showing the time matrix for this iteration.

(2)**(Total for Question 1 is 7 marks)**

SUB-TOPIC 1.4

PLANARITY

Sub-topic 1.4 · 1 question · 4 marks · lowest tariff first



1.

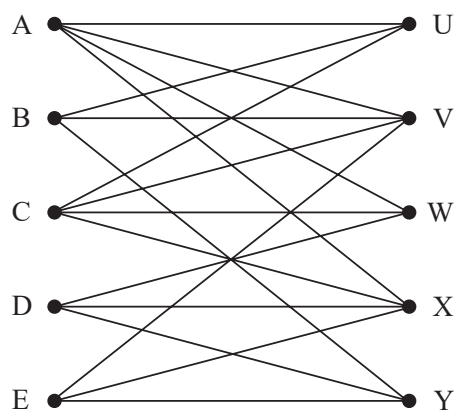


Figure 1

A Hamiltonian cycle for the graph in Figure 1 begins C, V, E, X, A, W, ...

(a) Complete the Hamiltonian cycle.

(1)

(b) Hence use the planarity algorithm to determine whether the graph shown in Figure 1 is planar. You must make your working clear and justify your answer.

(3)

(Total for Question 1 is 4 marks)



SUB-TOPIC 2.1

MINIMUM SPANNING TREES

Sub-topic 2.1 · 2 questions · 14 marks · lowest tariff first



1. The table below shows the lengths, in km, of the roads in a network connecting seven towns, A, B, C, D, E, F and G.

	A	B	C	D	E	F	G
A	–	24	–	22	35	–	–
B	24	–	25	27	–	–	–
C	–	25	–	33	31	36	26
D	22	27	33	–	–	42	–
E	35	–	31	–	–	37	29
F	–	–	36	42	37	–	40
G	–	–	26	–	29	40	–

- (a) By adding the arcs from vertex D along with their weights, complete the drawing of the network on Diagram 1 in the answer book. (2)
- (b) Use Kruskal's algorithm to find a minimum spanning tree for the network. You should list the arcs in the order that you consider them. In each case, state whether you are adding the arc to your minimum spanning tree. (3)
- (c) State the weight of the minimum spanning tree. (1)

(Total for Question 1 is 6 marks)



3.

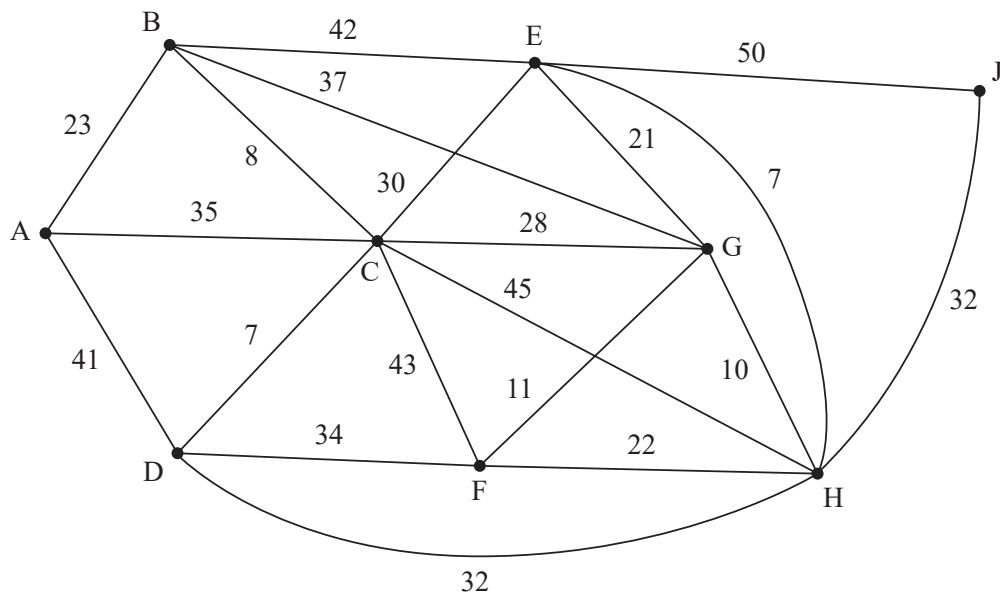
**Figure 4**

Figure 4 represents a network with nodes, A, B, C, D, E, F, G, H and J. The number on each edge gives the length of the corresponding edge.

(a) (i) Use Dijkstra's algorithm to find the shortest path from A to J.

(ii) State the length of the shortest path from A to J.

(6)

One application of Dijkstra's algorithm has order n^2 , where n is the number of nodes in the network.

It takes a computer 0.0312 seconds to find the shortest path from a given start node to a given end node in a network of 9 nodes.

(b) Calculate approximately how long it would take, in minutes, for the computer to find the shortest path from a given start node to a given end node for a network of 9000 nodes.

(2)**(Total for Question 3 is 8 marks)**

SUB-TOPIC 2.2

SHORTEST PATHS: DIJKSTRA AND FLOYD

Sub-topic 2.2 · 4 questions · 43 marks · lowest tariff first



3.

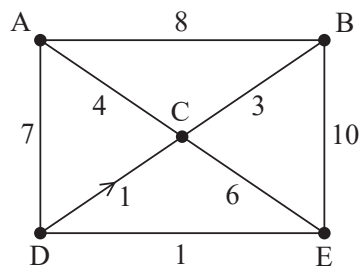


Figure 2

Direct roads between five villages, A, B, C, D and E, are shown in Figure 2. The weight on each arc is the time, in minutes, it takes to travel along the corresponding road. The road from D to C is one-way as indicated by the arrow on the corresponding arc.

Floyd's algorithm is to be used to find the complete network of shortest times between the five villages.

(a) Set up initial time and route matrices.

(2)

The matrices after two iterations of Floyd's algorithm are shown below.

Time matrix

	A	B	C	D	E
A	–	8	4	7	18
B	8	–	3	15	10
C	4	3	–	11	6
D	7	15	1	–	1
E	18	10	6	1	–

Route matrix

	A	B	C	D	E
A	A	B	C	D	B
B	A	B	C	A	E
C	A	B	C	A	E
D	A	A	C	D	E
E	B	B	C	D	E

(b) Perform the next two iterations of Floyd's algorithm that follow from the tables above. You should show the time and route matrices after each iteration.

(4)

The final time matrix after completion of Floyd's algorithm is shown below.

Final time matrix

	A	B	C	D	E
A	–	7	4	7	8
B	7	–	3	10	9
C	4	3	–	7	6
D	5	4	1	–	1
E	6	5	2	1	–



- (c) (i) Use the nearest neighbour algorithm, starting at A, to find a Hamiltonian cycle in the complete network of shortest times.
- (ii) Find the time taken for this cycle.
- (iii) Interpret the cycle in terms of the actual villages visited.

(3)**(Total for Question 3 is 9 marks)**

3. The initial distance matrix (Table 1) shows the lengths, in metres, of the corridors connecting six classrooms, A, B, C, D, E and F, in a school. For safety reasons, some of the corridors are one-way only.

	A	B	C	D	E	F
A	–	12	32	24	29	11
B	12	–	17	8	∞	∞
C	32	17	–	4	12	∞
D	24	∞	4	–	∞	13
E	∞	∞	12	18	–	12
F	11	∞	∞	13	12	–

Table 1

- (a) By adding the arcs from vertex A, along with their weights, complete the drawing of this network on Diagram 1 in the answer book.

(2)

Floyd's algorithm is to be used to find the complete network of shortest distances between the six classrooms.

The distance matrix after **two** iterations of Floyd's algorithm is shown in Table 2.

	A	B	C	D	E	F
A	–	12	29	20	29	11
B	12	–	17	8	41	23
C	29	17	–	4	12	40
D	24	36	4	–	53	13
E	∞	∞	12	18	–	12
F	11	23	40	13	12	–

Table 2

- (b) Perform the next two iterations of Floyd's algorithm that follow from Table 2. You should show the distance matrix after each iteration.

(4)



The final distance matrix after completion of Floyd's algorithm is shown in Table 3.

	A	B	C	D	E	F
A	–	12	24	20	23	11
B	12	–	12	8	24	21
C	28	17	–	4	12	17
D	24	21	4	–	16	13
E	23	29	12	16	–	12
F	11	23	17	13	12	–

Table 3

Yinka must visit each classroom. He will start and finish at E and wishes to minimise the total distance travelled.

- (c) (i) Use the nearest neighbour algorithm, starting at E, to find two Hamiltonian cycles in the completed network of shortest distances.
- (ii) Find the length of each of the two cycles.
- (iii) State, with a reason, which of the two cycles provides the better upper bound for the length of Yinka's route.

(4)

(Total for Question 3 is 10 marks)



6.

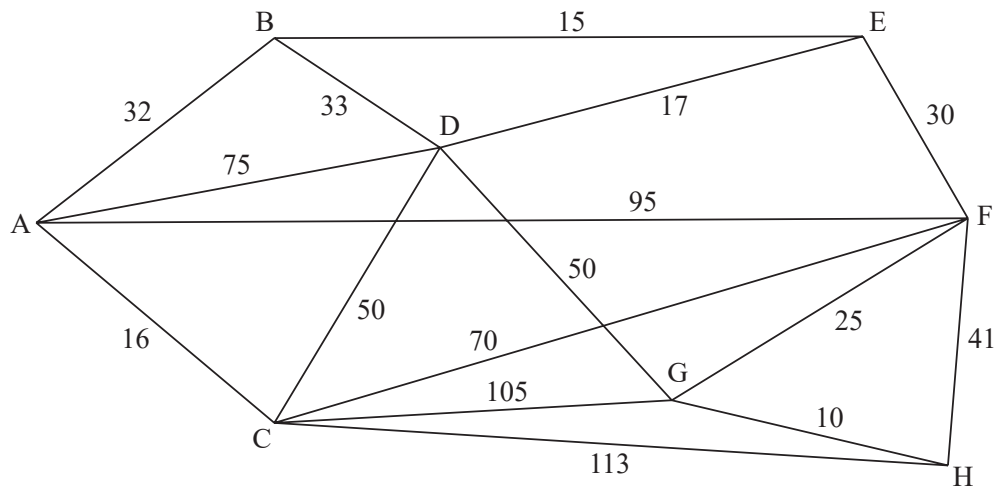


Figure 4

In Figure 4 the weights on the arcs represent distances.

(a) (i) Use Dijkstra's algorithm to find the shortest path from A to H.

(ii) State the length of the shortest path from A to H.

(6)

One application of Dijkstra's algorithm has order n^2 , where n is the number of nodes in the network. A computer produces a table of shortest distances between any two different nodes by repeatedly applying Dijkstra's algorithm from each node of the network.

It takes the computer 0.082 seconds to produce a table of shortest distances for a network of 10 nodes.

(b) Calculate approximately how long it will take, in seconds, for the computer to produce a table of shortest distances for a network with 200 nodes. You must give a reason for your answer.

(3)

(c) Explain why your answer to part (b) can only be an approximation.

(1)

(Total for Question 6 is 10 marks)



7.

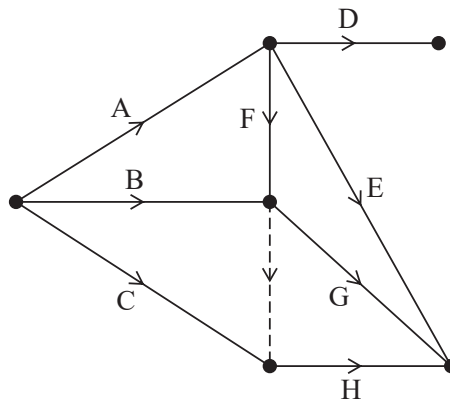


Figure 5

Figure 5 shows a partially completed activity network for a project that consists of 14 activities.

(a) Complete the precedence table in the answer book for the 8 activities in Figure 5.

(2)

The precedence table for the remaining 6 activities is given below.

Activity	Immediately preceding activities
I	D, E, G, H
J	D, E, G, H
K	E, G, H
L	I, J, K
M	J, K
N	J, K

(b) Complete the activity network in the answer book for the project. Your completed activity network must contain only the minimum number of dummies.

(4)

Given that all 14 activities have the same duration,

(c) explain why activity D cannot be critical.

(2)

(Total for Question 7 is 8 marks)

8. Susie is preparing for a triathlon event that is taking place next month. A triathlon involves three activities: swimming, cycling and running.

Susie decides that in her training next week she should

- maximise the total time spent cycling and running
- train for at most 39 hours
- spend at least 40% of her time swimming
- spend a total of at least 28 hours of her time swimming and running

Susie needs to determine how long she should spend next week training for each activity. Let

- x represent the number of hours swimming
- y represent the number of hours cycling
- z represent the number of hours running

- (a) Formulate the information above as a linear programming problem. State the objective and list the constraints as simplified inequalities with integer coefficients.

(5)

Susie decides to solve this linear programming problem by using the two-stage Simplex method.

- (b) Set up an initial tableau for solving this problem using the two-stage Simplex method.

As part of your solution you must show how

- the constraints have been made into equations using slack variables, exactly one surplus variable and exactly one artificial variable
- the rows for the two objective functions are formed

(6)

The following tableau T is obtained after one iteration of the second stage of the two-stage Simplex method.

b.v.	x	y	z	s_1	s_2	s_3	Value
y	0	1	0	1	0	1	11
s_2	0	0	5	-2	1	-5	62
x	1	0	1	0	0	-1	28
P	0	0	-1	1	0	1	11

- (c) Obtain a suitable pivot for a second iteration. You must give reasons for your answer.

(2)

- (d) Starting from tableau T , solve the linear programming problem by performing one further iteration of the second stage of the two-stage Simplex method. You should make your method clear by stating the row operations you use.

(5)

(Total for Question 8 is 18 marks)

TOTAL FOR PAPER IS 75 MARKS



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Pearson Edexcel		Centre Number	Candidate Number
Level 3 GCE			
Time 1 hour 30 minutes	Paper reference	9FM0/3D	
Further Mathematics			
Advanced			
PAPER 3D: Decision Mathematics 1			
Answer book		Total Marks	
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1.

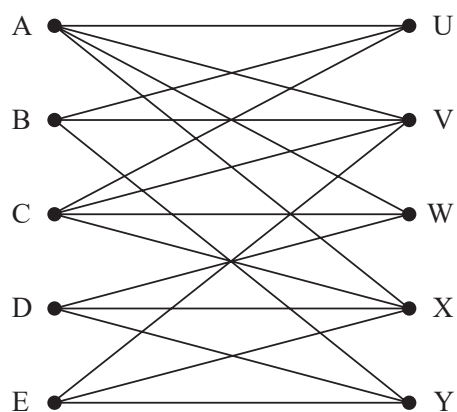


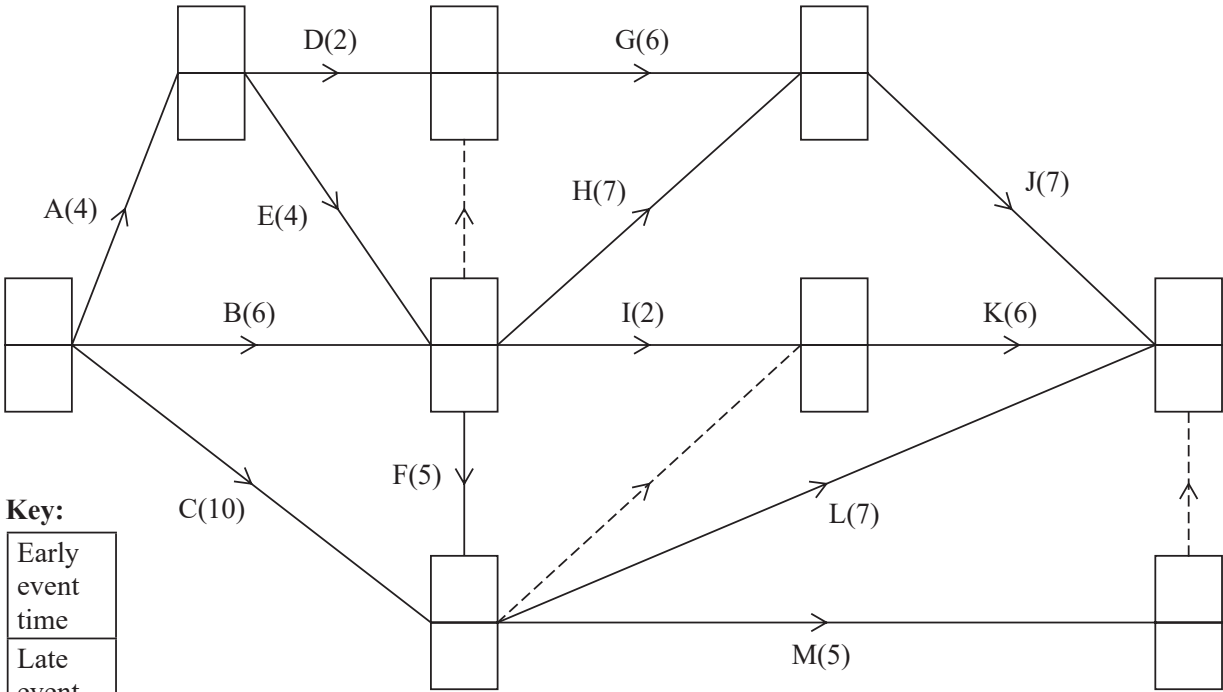
Figure 1

Question 1 continued

(Total for Question 1 is 4 marks)



2.



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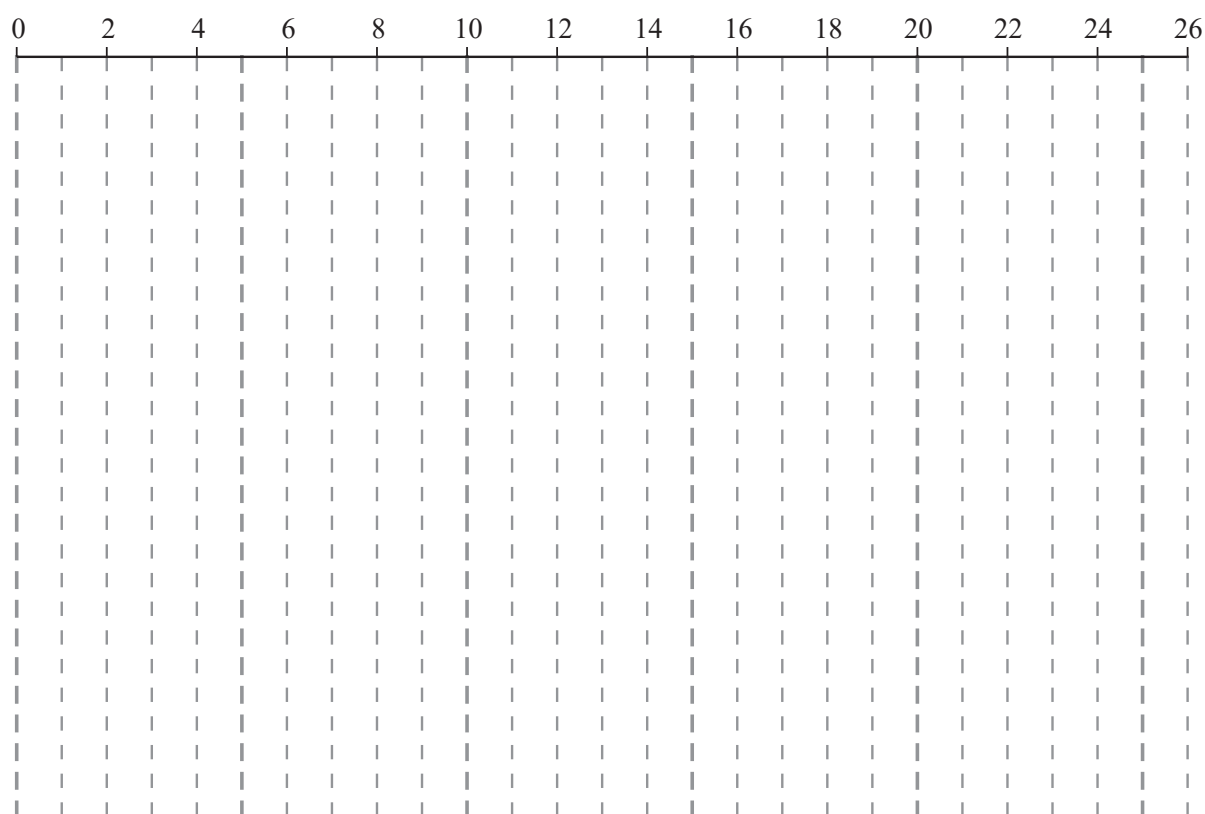
Early event time
Late event time

Diagram 1

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Question 2 continued

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Grid 1

(Total for Question 2 is 9 marks)

3.

	A	B	C	D	E	F	G	H
A	–	24	42	48	34	37	32	22
B	24	–	40	35	30	41	39	44
C	42	40	–	21	26	45	38	36
D	48	35	21	–	32	37	29	27
E	34	30	26	32	–	34	40	28
F	37	41	45	37	34	–	43	41
G	32	39	38	29	40	43	–	38
H	22	44	36	27	28	41	38	–

Question 3 continued

	A	B	C	D	E	F	G	H
A	–	24	42	48	34	37	32	22
B	24	–	40	35	30	41	39	44
C	42	40	–	21	26	45	38	36
D	48	35	21	–	32	37	29	27
E	34	30	26	32	–	34	40	28
F	37	41	45	37	34	–	43	41
G	32	39	38	29	40	43	–	38
H	22	44	36	27	28	41	38	–

(Total for Question 3 is 8 marks)



4. (a)

Initial time matrix

	A	B	C	D	E	F
A						
B						
C						
D						
E						
F						

Initial route matrix

	A	B	C	D	E	F
A	A	B	C	D	E	F
B	A	B	C	D	E	F
C	A	B	C	D	E	F
D	A	B	C	D	E	F
E	A	B	C	D	E	F
F	A	B	C	D	E	F

(b) 1st iteration

Time matrix

	A	B	C	D	E	F
A						
B						
C						
D						
E						
F						

Route matrix

	A	B	C	D	E	F
A						
B						
C						
D						
E						
F						



Question 4 continued

(Total for Question 4 is 8 marks)



5. 30 12 5 2 23 18 36 10 15 24



Question 5 continued

(Total for Question 5 is 10 marks)



Question 6 continued

(Total for Question 6 is 10 marks)



3.

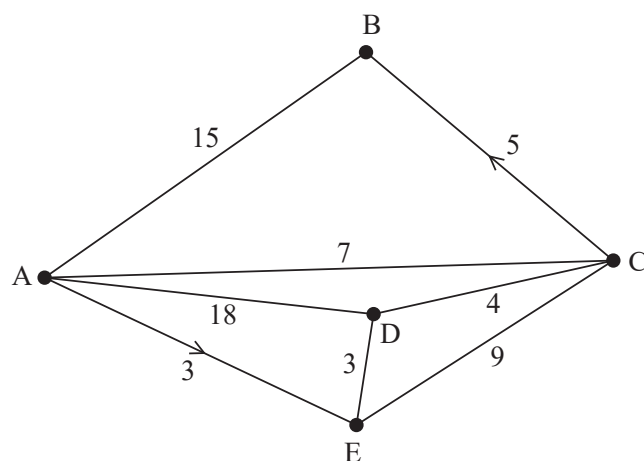


Figure 2

The network in Figure 2 shows the direct roads linking five villages, A, B, C, D and E. The number on each arc represents the length, in miles, of the corresponding road. The roads from A to E and from C to B are one-way, as indicated by the arrows.

(a) Complete the initial distance and route tables for the network provided in the answer book. (2)

(b) Perform the first three iterations of Floyd's algorithm. You should show the distance table and the route table after each of the three iterations. (5)

After five iterations of Floyd's algorithm the final distance table and partially completed final route table are shown below.

	Distance table				
	A	B	C	D	E
A	–	12	7	6	3
B	15	–	22	21	18
C	7	5	–	4	7
D	11	9	4	–	3
E	14	12	7	3	–

	Route table				
	A	B	C	D	E
A	A				
B	A	B			
C	A	B	C		
D	C	C	C	D	
E	D	D	D	D	E

(c) (i) Explain how the partially completed final route table can be used to find the shortest route from E to A.

(ii) State this route.

(3)

Mabintou decides to use the distance table to try to find the shortest cycle that passes through each vertex. Starting at D, she applies the nearest neighbour algorithm to the final distance table.

(d) (i) State the cycle obtained using the nearest neighbour algorithm.

(ii) State the length of this cycle.

(iii) Interpret the cycle in terms of the actual villages visited.

(iv) Prove that Mabintou's cycle is not optimal.

(4)

(Total for Question 3 is 14 marks)



SUB-TOPIC 3.1

ROUTE INSPECTION

Sub-topic 3.1 · 5 questions · 59 marks · lowest tariff first



4.

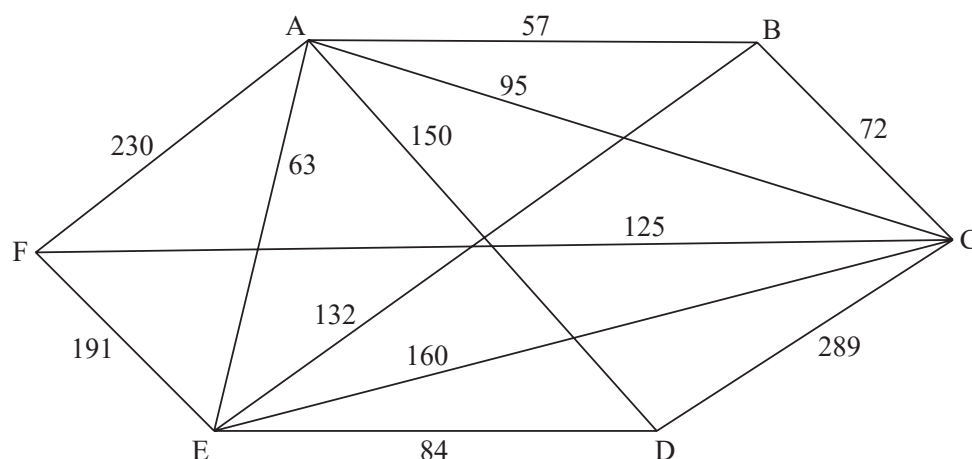


Figure 3

[The total weight of the network is 1648]

Direct roads between six cities, A, B, C, D, E and F, are represented in Figure 3. The weight on each arc is the time, in minutes, required to travel along the corresponding road.

Floyd's algorithm is to be used to find the complete network of shortest times between the six cities.

An initial route matrix is given in the answer book.

(a) Set up the initial time matrix.

(1)

(b) Perform the first iteration of Floyd's algorithm. You should show the time and route matrices after this iteration.

(2)

The final time matrix after completion of Floyd's algorithm is shown below.

	A	B	C	D	E	F
A	–	57	95	147	63	220
B	57	–	72	204	120	197
C	95	72	–	242	158	125
D	147	204	242	–	84	275
E	63	120	158	84	–	191
F	220	197	125	275	191	–

A route is needed that minimises the total time taken to traverse each road at least once.

The route must start at B and finish at E.

(c) Use an appropriate algorithm to find the roads that will need to be traversed twice. You should make your method and working clear.

(4)

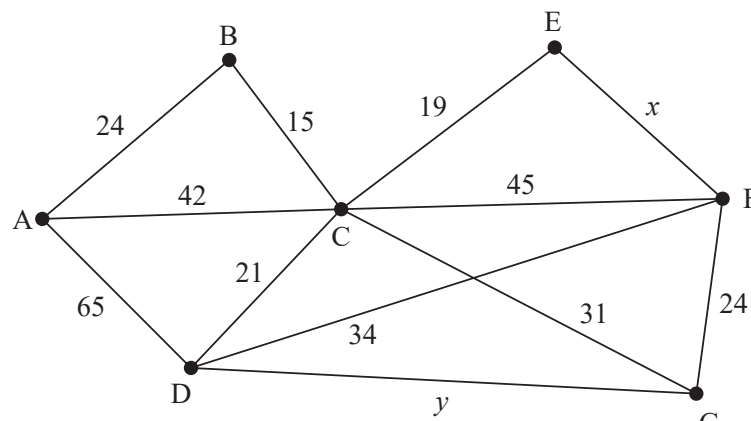
(d) Write down the length of the route.

(1)

(Total for Question 4 is 8 marks)



6.

**Figure 4**

[The total weight of the network is $320 + x + y$]

- (a) State, with justification, whether the graph in Figure 4 is Eulerian, semi-Eulerian or neither. (2)

The weights on the arcs in Figure 4 represent distances. The weight on arc EF is x where $12 < x < 26$ and the weight on arc DG is y where $0 < y < 10$

An inspection route of minimum length that traverses each arc at least once is found. The inspection route starts and finishes at A and has a length of 409

It is also given that the length of the shortest route from F to G via A is 140

- (b) Using appropriate algorithms, find the value of x and the value of y . (9)

(Total for Question 6 is 11 marks)

2.

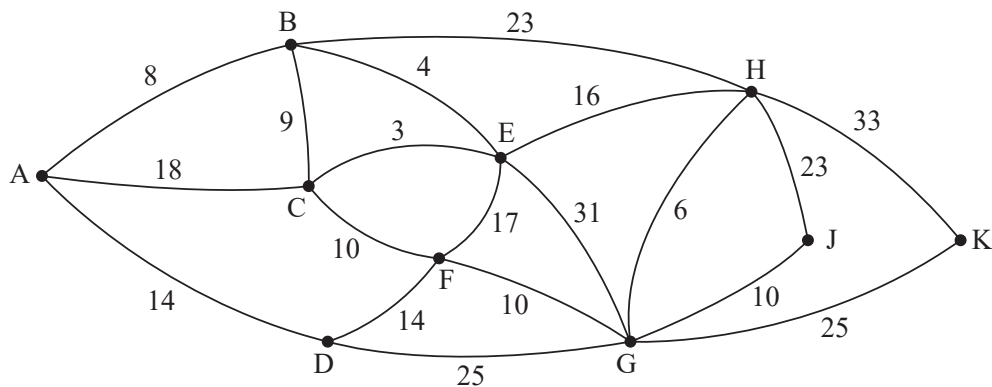


Figure 1

[The total weight of the network is 299]

Figure 1 represents a network of cycle tracks between 10 landmarks, A, B, C, D, E, F, G, H, J and K. The number on each edge represents the length, in kilometres, of the corresponding track.

One day, Blanche wishes to cycle from A to K. She wishes to minimise the distance she travels.

(a) (i) Use Dijkstra's algorithm to find the shortest path from A to K.

(ii) State the length of the shortest path from A to K.

(6)

The cycle tracks between the landmarks now need to be inspected. Blanche must travel along each track at least once. She wishes to minimise the length of her inspection route. Blanche will start her inspection route at D and finish at E.

(b) (i) State the edges that will need to be traversed twice.

(ii) Find the length of Blanche's route.

(2)

It is now decided to start the inspection route at A and finish at K. Blanche must minimise the length of her route and travel along each track at least once.

(c) By considering the pairings of all relevant nodes, find the length of Blanche's new route. You must make your method and working clear.

(5)

(Total for Question 2 is 13 marks)



5.

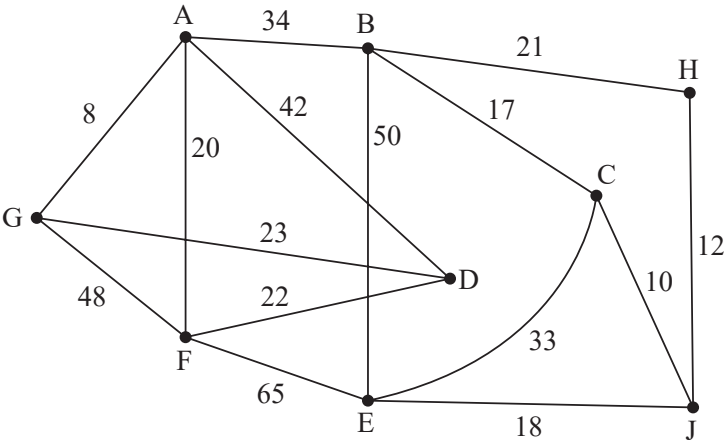


Figure 5

[The total weight of the network is 423]

Direct roads between nine towns, A, B, C, D, E, F, G, H and J, are represented in Figure 5. The number on each arc represents the length, in miles, of the corresponding road.

The table below shows the shortest distances, in miles, between the nine towns.

	A	B	C	D	E	F	G	H	J
A	–	34	51	31	79	20	8	55	61
B	34	–	17	65	45	54	42	21	27
C	51	17	–	82	28	71	59	22	10
D	31	65	82	–	87	22	23	86	92
E	79	45	28	87	–	65	87	30	18
F	20	54	71	22	65	–	28	75	81
G	8	42	59	23	87	28	–	63	69
H	55	21	22	86	30	75	63	–	12
J	61	27	10	92	18	81	69	12	–

Table of shortest distances



A route is needed that minimises the total distance required to traverse each road at least once.

The route must start at F and finish at J.

(a) (i) By considering the pairings of all relevant nodes, find the roads that would need to be traversed twice.

(ii) State the total length of this route.

(5)

(b) Starting at A, use Prim's algorithm to find the minimum spanning tree for the table of shortest distances. You must state the order in which you select the arcs of your tree.

(3)

Pete needs to visit all nine towns, starting and finishing in the same town, and wishes to minimise the total distance he travels.

(c) Starting at G, use the nearest neighbour algorithm on the table of shortest distances to find an upper bound for the length of Pete's route. Write down the route that gives this upper bound.

(2)

(d) By deleting G and all of its arcs, find a lower bound for the length of Pete's route.

(2)

Pete decides to take the route he found in (c).

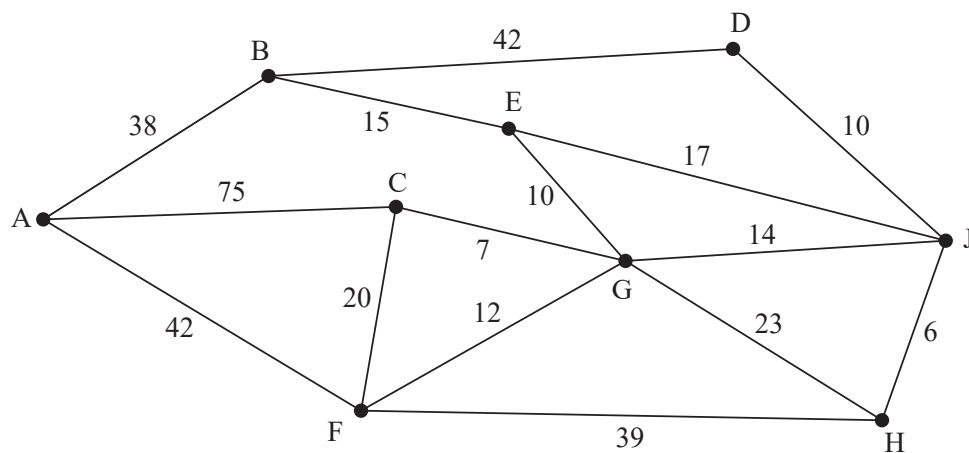
(e) Interpret the route in terms of the actual towns visited.

(1)

(Total for Question 5 is 13 marks)



2.

**Figure 1**

[The total weight of the network is 370]

Figure 1 represents a network of corridors in a building. The number on each arc represents the length, in metres, of the corresponding corridor.

- (a) Use Dijkstra's algorithm to find the shortest path from A to D, stating the path and its length.

(6)

On a particular day, Naasir needs to check the paintwork along each corridor. Naasir must find a route of minimum length. It must traverse each corridor at least once, starting at B and finishing at G.

- (b) Use an appropriate algorithm to find the arcs that will need to be traversed twice. You must make your method and working clear.

(4)

- (c) Find the length of Naasir's route.

(1)

On a different day, all the corridors that start or finish at B are closed for redecorating. Naasir needs to check all the remaining corridors and may now start at any vertex and finish at any vertex. A route is required that excludes all those corridors that start or finish at B.

- (d) (i) Determine the possible starting and finishing points so that the length of Naasir's route is minimised. You must give reasons for your answer.
(ii) Find the length of Naasir's new route.

(3)**(Total for Question 2 is 14 marks)**

SUB-TOPIC 3.2

TRAVELLING SALESMAN PROBLEM

Sub-topic 3.2 · 1 question · 8 marks · lowest tariff first



3.

	A	B	C	D	E	F	G	H
A	–	24	42	48	34	37	32	22
B	24	–	40	35	30	41	39	44
C	42	40	–	21	26	45	38	36
D	48	35	21	–	32	37	29	27
E	34	30	26	32	–	34	40	28
F	37	41	45	37	34	–	43	41
G	32	39	38	29	40	43	–	38
H	22	44	36	27	28	41	38	–

Table 1

Table 1 shows the shortest distances, in miles, between eight towns, A, B, C, D, E, F, G and H.

- (a) Use Prim's algorithm, starting at A, to find the minimum spanning tree for this table of distances. You must clearly state the order in which you select the edges of your tree. (3)
- (b) State the weight of the minimum spanning tree. (1)

	A	B	C	D	E	F	G	H
J	31	27	50	29	43	25	49	35

Table 2

Table 2 shows the distances, in miles, between town J and towns A, B, C, D, E, F, G and H.

Pranav needs to visit all of the towns, starting and finishing at J, and wishes to minimise the total distance he travels.

- (c) Starting at J, use the nearest neighbour algorithm to obtain an upper bound for the length of Pranav's route. You must state your route and its length. (2)
- (d) Starting by deleting J, and all of its edges, find a lower bound for the length of Pranav's route. (2)

(Total for Question 3 is 8 marks)



SUB-TOPIC 5.2

GRAPHICAL SOLUTIONS

Sub-topic 5.2 · 1 question · 15 marks · lowest tariff first



7.

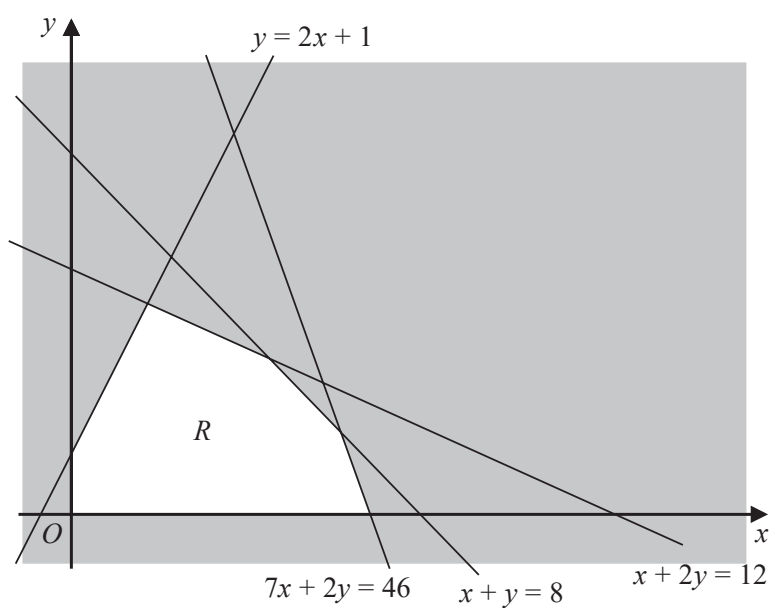
**Figure 5**

Figure 5 shows the constraints of a linear programming problem in x and y where R is the feasible region.

The objective is to maximise $P = x + ky$, where k is a positive constant.

The optimal vertex of R is to be found using the Simplex algorithm.

- (a) Set up an initial tableau for solving this linear programming problem using the Simplex algorithm.

(5)

After two iterations of the Simplex algorithm a possible tableau T is

b.v.	x	y	s_1	s_2	s_3	s_4	Value
s_1	0	0	1	$-\frac{3}{5}$	0	$\frac{1}{5}$	1
x	1	0	0	$\frac{1}{5}$	0	$-\frac{2}{5}$	2
s_3	0	0	0	$-\frac{11}{5}$	1	$\frac{12}{5}$	22
y	0	1	0	$\frac{2}{5}$	0	$\frac{1}{5}$	5
P	0	0	0	$\frac{1}{5} + \frac{2}{5}k$	0	$-\frac{2}{5} + \frac{1}{5}k$	$5k + 2$

(b) State the value of each variable after the second iteration.

(1)

It is given that T does not give an optimal solution to the linear programming problem.

After a third iteration of the Simplex algorithm the resulting tableau does give an optimal solution to the problem.

(c) Perform the third iteration of the Simplex algorithm and hence determine the range of possible values for P . You should state the row operations you use and make your method and working clear.

(9)

(Total for Question 7 is 15 marks)



SUB-TOPIC 5.3

THE SIMPLEX ALGORITHM

Sub-topic 5.3 · 2 questions · 26 marks · lowest tariff first



6. A linear programming problem in x , y and z is described as follows.

$$\text{Maximise } P = 2x + 2y - z$$

$$\text{subject to } 3x + y + 2z \leq 30$$

$$x - y + z \geq 8$$

$$4y + 2z \geq 15$$

$$x, y, z \geq 0$$

(a) Explain why the Simplex algorithm cannot be used to solve this linear programming problem.

(1)

(b) Set up the initial tableau for solving this linear programming problem using the big-M method.

(7)

After a first iteration of the big-M method, the tableau is

b.v.	x	y	z	s_1	s_2	s_3	a_1	a_2	Value
s_1	3	0	1.5	1	0	0.25	0	-0.25	26.25
a_1	1	0	1.5	0	-1	-0.25	1	0.25	11.75
y	0	1	0.5	0	0	-0.25	0	0.25	3.75
P	$-(2 + M)$	0	$2 - 1.5M$	0	M	$-0.5 + 0.25M$	0	$0.5 + 0.75M$	$7.5 - 11.75M$

(c) State the value of each variable after the first iteration.

(1)

(d) Explain why the solution given by the first iteration is not feasible.

(1)

Taking the most negative entry in the profit row to indicate the pivot column,

(e) obtain the most efficient pivot for a second iteration. You must give reasons for your answer.

(2)

(Total for Question 6 is 12 marks)



7. A maximisation linear programming problem in x , y and z is to be solved using the Simplex method.

The tableau after the 1st iteration is shown below.

b.v.	x	y	z	s_1	s_2	s_3	Value
s_1	0	$-\frac{1}{2}$	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	30
x	1	$\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	10
s_3	0	1	1	0	0	1	26
P	0	$-\frac{1}{4}$	$-\frac{11}{4}$	0	$\frac{3}{4}$	0	30

- (a) State the column that contains the pivot value for the 1st iteration. You must give a reason for your answer. (2)
- (b) By considering the equations represented in the above tableau, formulate the linear programming problem in x , y and z only. State the objective and list the constraints as inequalities with integer coefficients. (5)
- (c) Taking the most negative number in the profit row to indicate the pivot column, perform the 2nd iteration of the Simplex algorithm, to obtain a new tableau, T. Make your method clear by stating the row operations you use. (4)
- (d) (i) Explain, using T, how you know that an optimal solution to the original linear programming problem has **not** been found after the 2nd iteration. (2)
- (ii) State the values of the basic variables after the 2nd iteration.



A student attempts the 3rd iteration of the Simplex algorithm and obtains the tableau below.

b.v.	x	y	z	s_1	s_2	s_3	Value
z	0	0	1	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{43}{2}$
x	1	0	0	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{1}{8}$	$\frac{57}{4}$
y	0	1	0	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{9}{2}$
P	0	1	0	$\frac{5}{4}$	$\frac{1}{8}$	$\frac{7}{8}$	$\frac{361}{4}$

(e) Explain how you know that the student's attempt at the 3rd iteration is **not** correct.

(1)

(Total for Question 7 is 14 marks)



MARK SCHEMES

official mark schemes



Qu	Scheme										Marks	AOs							
6(a)	<table><tr><td></td><td>3</td><td>1</td><td>13</td></tr><tr><td>4</td><td>0</td><td>16</td></tr></table>											3	1	13	4	0	16	A1	1.1b
												3	1	13					
											4	0	16						
Output: 16	A1	2.2a																	
											(3)								
(b) (i)	17	21	24	14	23	13	15	19	28	20	M1 A1 A1ft A1	1.1b 1.1b 1.1b 1.1b							
	17	21	14	23	13	15	19	24	20	28									
	17	14	21	13	15	19	23	20	24	28									
	14	17	13	15	19	21	20	23	24	28									
	14	13	15	17	19	20	21	23	24	28									
	13	14	15	17	19	20	21	23	24	28									
	13	14	15	17	19	20	21	23	24	28									
(b) (ii)	Total number of comparisons: $9 + 8 + 7 + 6 + 5 + 4 = 39$										B1	2.2a							
											(5)								
(c)	45										B1	2.2a							
											(1)								
(d)	e.g. when $n = 3$ the total number of comparisons is 3 therefore $\lambda(3)(3-1)(3+1)(3-2) = 3$ e.g. when $n = 5$ the total number of comparisons is 45 (from (c)) therefore $\lambda(5)(5-1)(5+1)(5-2) = 45$										M1	3.1a							
	For K_{50} total number of comparisons is ' $\lambda(50)(50-1)(50+1)(50-2)$ '										dM1	3.4							
	749 700										A1	2.2a							
											(3)								
(12 marks)																			

Notes for Question 6

a1M1: At least three rows of cells completed with a correct first row – condone repeated values in all columns or a single value in each row

a1A1: cao – the values in the second and third rows correct

a2A1: cao – correct output following a correct fourth row (with no extra rows) – the output **must** either be stated on the given answer line or ‘output 16’ must be clearly written somewhere near the table (do not bod the 16 circled, underlined, written twice, etc.)

bi1M1: Bubble sort. Consistent direction, end number (28) in place, the list containing ten numbers with the list beginning with the correct first five numbers (17 21 14 23 13). Do check these carefully as some candidates show the result of each comparison and swap in their first pass. Consider the placement of the candidate’s numbers, rather than what the candidate labels each line of their pass. For example, assume that the first time that the 28 appears at the end of the list is the end of their first pass

bi1A1: Second and third passes correct – so end three numbers in place

bi2A1ft: Fourth and fifth passes correct following through from the candidate’s third pass – so end five numbers in place. After their third pass their list must contain the correct 10 numbers

bi3A1: cso (correct solution only – so previous three marks must have been awarded in this part). Must show a 6th pass showing no swaps/changes (give bod if the passes are not labelled but do not award this mark if it is clear that after the 5th pass the list is simply being written out again (rather than a genuine 6th pass taking place)). Condone if the sort continues until a 9th pass has been completed (but there must be no changes in the 6th to 9th passes)

bii1B1: cao (for total number of comparisons)

In (b) starting at the right-hand end of the list is M0. Quick sort (or any other sorting algorithm e.g. shuttle) is M0. No misreads in this part – mark exactly to the scheme.

If sorting into descending order, then award M1 for 21 24 17 23 14 15 19 28 20 13 and the first A1 for **both** 24 21 23 17 15 19 28 20 14 13 **and** 24 23 21 17 19 28 20 15 14 13 **ONLY** (so two out of the first four marks) even if the list is re-ordered after the sort is complete

c1B1: cao (45)

d1M1: Considering the total number of comparisons for any positive integer value of n (greater than 2) and substitute into the given expression (if correct $\lambda = \frac{1}{8}$). The correct value of λ implies this mark. If using $n = 5$ then follow through their value from (c). If any other value of n used, then the maximum number of comparisons must be correct e.g.

n	3	4	5	6	7	8	9	10
Comparisons	3	15	45	105	210	378	630	990

d2dM1: Using their λ and $n = 50$ to calculate the maximum total number of comparisons (dependent on the previous M mark). Writing $\frac{1}{8}(50)(50-1)(50+1)(50-2)$ implies the first two M marks

d1A1: cao (749 700) – no marks for the correct answer with **no** working



Alternative solution to (d) for those who are clearly using an algebraic method to derive the quartic expression for the maximum total number of comparisons:

If there are N values then

1st pass of bubble sort requires $N - 1$ comparisons

2nd pass of bubble sort requires $N - 2$ comparisons

3rd pass of bubble sort requires $N - 3$ comparisons and so on

Therefore, the total number of comparisons is $\sum_{r=1}^{N-1} r = \frac{1}{2}(N-1)N$ M1

Maximum number of comparisons in K_n is therefore

$$\begin{aligned} & \frac{1}{2} \left(\frac{1}{2} n(n-1) - 1 \right) \left(\frac{1}{2} n(n-1) \right) \\ &= \frac{1}{8} n(n-1) [n(n-1) - 2] \\ &= \frac{1}{8} n(n-1) (n^2 - n - 2) = \frac{1}{8} n(n-1)(n-2)(n+1) \text{ so } k = \frac{1}{8} \end{aligned} \quad \text{dM1}$$

Therefore, maximum number of comparisons for K_{50} is 749 700 A1

M1: Considering the total number of comparisons that are required to sort a list containing N values (possibly will see $1 + 2 + 3 + \dots + (n-1)$) and using the standard series result that $\sum r = \frac{1}{2}n(n+1)$ to get the correct quadratic expression for the total number of comparisons when sorting N values (allow any letter)

dM1: Dependent on previous M mark – substituting the correct quadratic expression into the correct quadratic expression with correct algebraic working leading to $\frac{1}{8}n(n-1)(n-2)(n+1)$

A1: Correct answer of 749 700



Question	Scheme	Marks	AOs
1(a)	$\frac{4.3+6.1+\dots+1.3}{10} = \frac{37.5}{10} = 3.75$ so lower bound is four (balls of string)	M1 A1	1.1b 2.2a
		(2)	
(b)	Bin 1: 4.3 5.1 0.4 Bin 2: 6.1 2.5 1.3 Bin 3: 4.7 3.4 1.7 Bin 4: 5.9 2.1	M1 A1 A1	1.1b 1.1b 1.1b
		(3)	
(5 marks)			

Notes for Question 1

a1M1: Attempt to find the lower bound $(37.5 \pm 6.1)/10$ (a value of 3.75 seen with no working can imply this mark)

a1A1: cso – a lower bound of 4 with either a correct calculation seen or 3.75 or ‘total is 37.5 and if each ball contains 10 this gives a lower bound of 4’. An answer of 4 with no working (or from part **(b)**) scores M0A0. Any incorrect working loses this mark e.g. a correct calculation followed by an incorrect value followed by 4 is A0

b1M1: First four items placed correctly and at least eight items placed in bins – condone cumulative totals for M1 only (the boxed values)

b1A1: First eight items placed correctly (the boxed **and** bold values), and all eleven correct values only placed in bins (so no additional/repeated values)

b2A1: cso (no additional/repeated values)

Condone working in cm provided consistent

No MR in this question – mark according to the scheme



Question	Scheme	Marks	AOs
5(a)	If x has been placed in Bin 2 then $10 < x \leq 31$ - this is because Bin 1 at this stage only contains 40 and before x had been placed in Bin 2 it only contained 19		
	As the 18 has been placed in Bin 3 this implies that $x > 50 - (19 + 18)$ so $x > 13$	B1	3.1a
	As the 10 has been placed in Bin 2 after the x then $x \leq 50 - (19 + 10)$ so $x \leq 21$	B1	2.4
	However, the number are all distinct and therefore $13 < x < 21$	B1	2.2a
		(3)	
(b)	$13 < x < 17$	B1	2.2a
		B1	2.2a
		(2)	
(c)	If x has been placed in Bin 3 then this implies that $x \leq 15$ So x is either 14 or 15 – but as Bin 2 is full $\Rightarrow x = 14$	M1	2.4
		A1	2.2a
		(2)	
(7 marks)			
Notes for Question 5			
(a) B1: Correct reasoning of why $x > 13$ accept $x > 50 - (19 + 18)$ B1: Correct explanation of why $x \leq 21$ accept $x \leq 50 - (19 + 10)$ B1: Correct deduction that $13 < x < 21$ must mention that the numbers are distinct (oe) (b) B1: Use first complete pass to deduce that $x < 17$ B1: Correct lower bound of $x > 13$ (c) M1: Using first-fit decreasing in an attempt to derive new upper bound for x (so either for stating both x could equal 14 or 15, $x \leq 15$ or $x < 15$) A1: Correct deduction that $x = 14$ (must clearly state or imply that Bin 2 is full)			

Question	Scheme	Marks	AOs
1(a)	Bin 1: <u>2.1</u> <u>1.7</u> <u>1.2</u> Bin 2: <u>3.0</u> <u>1.9</u> Bin 3: <u>3.2</u> 1.4 0.2 Bin 4: 3.3 1.5	<u>M1</u> A1	1.1b 1.1b
		(2)	
(b)	e.g. middle right 2.1 1.7 3.0 1.9 3.2 <u>1.2</u> 3.3 1.4 1.5 0.2 Pivot: 1.2 2.1 1.7 3.0 1.9 <u>3.2</u> 3.3 1.4 1.5 <u>1.2</u> 0.2 Pivot(s): 3.2 (0.2) 3.3 <u>3.2</u> 2.1 1.7 3.0 <u>1.9</u> 1.4 1.5 <u>1.2</u> 0.2 Pivot(s): (3.3) 1.9 3.3 <u>3.2</u> 2.1 <u>3.0</u> <u>1.9</u> 1.7 <u>1.4</u> 1.5 <u>1.2</u> 0.2 Pivots: 3.0 1.4 3.3 <u>3.2</u> <u>3.0</u> 2.1 <u>1.9</u> 1.7 <u>1.5</u> <u>1.4</u> <u>1.2</u> 0.2 Pivot(s): (2.1) 1.5 3.3 <u>3.2</u> <u>3.0</u> 2.1 <u>1.9</u> 1.7 <u>1.5</u> <u>1.4</u> <u>1.2</u> 0.2	M1 A1 A1ft A1	1.1b 1.1b 1.1b 1.1b
		(4)	
(c)	$\frac{2.32(11\,250 \log 11\,250)}{450 \log 450}$ =88.6 seconds	M1 A1	1.1a 1.1b
		(2)	
(8 marks)			
Notes for Question 1			
PLEASE NOTE NO MISREADS IN PARTS (a) and (b) – MARK ACCORDING TO THE SCHEME AND THE SPECIAL CASE FOR ASCENDING ORDER IN (b)			
(a) M1 : First six items placed correctly and at least eight values placed in bins - condone cumulative totals for M1 only (the underlined values) A1 : CSO – all correct (so no additional/repeated values)			
(b) M1 : Quick sort, pivot, p, chosen (must be choosing middle left or right – choosing first/last item as the pivot is M0). After the first pass the list must read (values greater than the pivot), pivot, (values less than the pivot). If only choosing one pivot per iteration then max of M1A1 only – Bubble sort is not a MR and scores M0 A1 : First pass correct and next pivots chosen correctly for the second pass (but the second pass does not need to be correct) – so they must be choosing (if middle right) a pivot value of 3.2 for the second pass or (if middle left) a pivot value of 1.9 A1ft : Second and third passes correct (follow through from their first pass and choice of pivots). They do not need to be choosing a pivot for the fourth pass for this mark A1 : CSO (correct solution only – all previous marks in this part must have been awarded) including if middle right a fifth pass with the 1.5 used as a pivot or if middle left a fourth pass with the 1.7 used as a pivot			



Sorting list into ascending order in (b)

- If the candidate sorts the list into ascending order and reverses the list in this part then this can score full marks in **(b)**
- If the list is not reversed in **(b)** then remove the last two A marks earned in **(b)**. If the candidate says that the list needs reversing in **(b)** but does not actually show the reversed list in **(b)** then remove the last A mark earned
- **Note that if sorting into ascending order then a ‘sort complete’ statement is required – this could be shown by the final list being re-written or ‘sorted’ statement or each item being used as a pivot (which would therefore mean that the final list would have been written twice) BEFORE list is reversed**

Middle left

2.1	1.7	3.0	1.9	<u>3.2</u>	1.2	3.3	1.4	1.5	0.2
3.3	<u>3.2</u>	2.1	1.7	3.0	<u>1.9</u>	1.2	1.4	1.5	0.2
3.3	<u>3.2</u>	<u>2.1</u>	3.0	<u>1.9</u>	1.7	1.2	<u>1.4</u>	1.5	0.2
3.3	<u>3.2</u>	3.0	<u>2.1</u>	<u>1.9</u>	<u>1.7</u>	1.5	<u>1.4</u>	<u>1.2</u>	0.2
3.3	<u>3.2</u>	3.0	<u>2.1</u>	<u>1.9</u>	<u>1.7</u>	1.5	<u>1.4</u>	<u>1.2</u>	0.2

Middle right ascending (which requires a ‘sort complete’ statement – see above)

2.1	1.7	3.0	1.9	3.2	<u>1.2</u>	3.3	1.4	1.5	0.2
0.2	<u>1.2</u>	2.1	1.7	3.0	<u>1.9</u>	<u>3.2</u>	3.3	1.4	1.5
0.2	<u>1.2</u>	2.1	1.7	3.0	<u>1.9</u>	1.4	1.5	<u>3.2</u>	3.3
0.2	<u>1.2</u>	1.7	<u>1.4</u>	1.5	<u>1.9</u>	2.1	<u>3.0</u>	<u>3.2</u>	3.3
0.2	<u>1.2</u>	<u>1.4</u>	1.7	<u>1.5</u>	<u>1.9</u>	2.1	<u>3.0</u>	<u>3.2</u>	3.3
0.2	<u>1.2</u>	<u>1.4</u>	<u>1.5</u>	1.7	<u>1.9</u>	2.1	<u>3.0</u>	<u>3.2</u>	3.3

Middle left ascending (which required a ‘sort complete’ statement – see above)

2.1	1.7	3.0	1.9	<u>3.2</u>	1.2	3.3	1.4	1.5	0.2
2.1	1.7	3.0	<u>1.9</u>	1.2	1.4	1.5	0.2	<u>3.2</u>	3.3
1.7	1.2	<u>1.4</u>	1.5	0.2	<u>1.9</u>	<u>2.1</u>	3.0	<u>3.2</u>	3.3
<u>1.2</u>	0.2	<u>1.4</u>	<u>1.7</u>	1.5	<u>1.9</u>	<u>2.1</u>	3.0	<u>3.2</u>	3.3
0.2	<u>1.2</u>	<u>1.4</u>	1.5	<u>1.7</u>	<u>1.9</u>	<u>2.1</u>	3.0	<u>3.2</u>	3.3

(c) M1: Complete correct method – allow reciprocal – allow slips in values only e.g. 1250 for 11 250

A1: CAO – the exact value of 88.6 must be stated at some point (as question specifically asked for the answer to the nearest tenth of a second) – isw if 90 follows 88.6 seen. 90 with no working scores no marks. An answer of 88.6 with no working scores M1A0 – condone lack of units (but if present must be correct)



Question	Scheme	Marks	AOs																																																		
5(a)	e.g., middle right pivot(s)																																																				
	<table><tr><td>30</td><td>12</td><td>5</td><td>2</td><td>23</td><td>18</td><td>36</td><td>10</td><td>15</td><td>24</td></tr><tr><td>30</td><td>23</td><td>36</td><td>24</td><td>18</td><td>12</td><td>5</td><td>2</td><td>10</td><td>15</td></tr><tr><td>36</td><td>30</td><td>23</td><td>24</td><td>18</td><td>12</td><td>5</td><td>10</td><td>15</td><td>2</td></tr><tr><td>36</td><td>30</td><td>24</td><td>23</td><td>18</td><td>12</td><td>15</td><td>10</td><td>5</td><td>2</td></tr><tr><td>36</td><td>30</td><td>24</td><td>23</td><td>18</td><td>15</td><td>12</td><td>10</td><td>5</td><td>2</td></tr></table>	30	12	5	2	23	18	36	10	15	24	30	23	36	24	18	12	5	2	10	15	36	30	23	24	18	12	5	10	15	2	36	30	24	23	18	12	15	10	5	2	36	30	24	23	18	15	12	10	5	2	M1	1.1b
	30	12	5	2	23	18	36	10	15	24																																											
	30	23	36	24	18	12	5	2	10	15																																											
	36	30	23	24	18	12	5	10	15	2																																											
	36	30	24	23	18	12	15	10	5	2																																											
36	30	24	23	18	15	12	10	5	2																																												
		A1	1.1b																																																		
		A1ft	1.1b																																																		
		A1	1.1b																																																		
	Therefore, the sort is complete																																																				
		(4)																																																			
(b)	The 5 has been put in Bin 2 rather than Bin 1 which indicates that the size of the bins is less than $30 + 12 + 5 = 47$ and so therefore $(42 \leq n) \leq 46$	B1	3.1a																																																		
	The fact that there is still room for the 2 in Bin 1 indicates that $n \geq 44$	B1	2.4																																																		
	The 18 cannot fit in Bin 2 and so therefore $n < 5 + 23 + 18 = 46$ which implies that n is either 44 or 45	B1	2.2a																																																		
		(3)																																																			
(c)	Bin 1: <table><tr><td>36</td></tr></table> 5 2 Bin 2: <table><tr><td>30</td></tr></table> <table><tr><td>15</td></tr></table> Bin 3: <table><tr><td>24</td></tr></table> <table><tr><td>18</td></tr></table> Bin 4: <table><tr><td>23</td></tr></table> <table><tr><td>12</td></tr></table> <table><tr><td>10</td></tr></table>	36	30	15	24	18	23	12	10	<table><tr><td>M1</td></tr><tr><td>A1</td></tr><tr><td>A1</td></tr></table>	M1	A1	A1	1.1b 1.1b 1.1b																																							
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Notes:																																																					
(a)																																																					
M1: Quick sort, pivot, p, chosen (must be choosing middle left or right – choosing first/last item as the pivot is M0). After the first pass the list must read (values greater than the pivot), pivot, (values less than the pivot). If only choosing one pivot per iteration then max of M1 only – Bubble sort is not a MR and scores M0																																																					
A1: First pass correct and next pivots chosen correctly for the second pass (but the second pass does not need to be correct)																																																					
A1ft: Second and third passes correct (follow through from their first pass and choice of pivots). They do not need to be choosing a pivot for the fourth pass for this mark																																																					
A1: cso (correct solution only – all previous marks in this part must have been awarded) including ‘sort complete’ statement																																																					
If list sorted into ascending order then mark as a MR																																																					



(b)

B1: Correct reasoning why $n \leq 46$ or $n < 47$ or $n \leq 45$ or $n < 46$ - condone an argument which is mathematical in nature only e.g., $n < 30 + 12 + 5$ or $n < 5 + 23 + 18$

B1: Correct reasoning why $n \geq 44$ - condone 'largest bin filled is 44 so $n \geq 44$ ' or other similar mathematical argument. This mark can be awarded for an argument which is mathematical in nature only e.g., $n \geq 30 + 12 + 2$

B1: Completely correct reasoning for why n is either 44 or 45 only – this mark is dependent on the two previous **B** marks and must give sufficient detail (so an argument that contains no clear explanation of why certain inequalities hold cannot score this mark)

(c)

No MR or follow through in this part

M1: First five values placed correctly - with at least eight values placed (the squared values)

A1: First eight values placed correctly with no additional/repeated values (the squared and underlined values)

A1: cso - no additional/repeated values

Middle left pivot(s)

30	12	5	2	23	18	36	10	15	24
30	36	24	<u>23</u>	12	5	2	18	10	15
<u>36</u>	30	24	<u>23</u>	12	5	18	10	15	<u>2</u>
<u>36</u>	<u>30</u>	24	<u>23</u>	<u>18</u>	12	5	10	15	<u>2</u>
<u>36</u>	<u>30</u>	24	<u>23</u>	<u>18</u>	12	10	15	<u>5</u>	<u>2</u>
<u>36</u>	<u>30</u>	24	<u>23</u>	<u>18</u>	12	15	<u>10</u>	<u>5</u>	<u>2</u>
<u>36</u>	<u>30</u>	24	<u>23</u>	<u>18</u>	15	<u>12</u>	<u>10</u>	<u>5</u>	<u>2</u>

Therefore, the sort is complete



Ques tion	Scheme	Marks	AOs																																																																													
4. (a)	$132 + x > 120$ for all positive values of x therefore 3 bins is not possible	B1	2.4																																																																													
		(1)																																																																														
(b)	$23 < x \leq 28$	B1 B1	2.2a 2.2a																																																																													
		(2)																																																																														
(c)	Bin 1: <u>15</u> <u>22</u> <u>3</u> Bin 2: <u>9</u> <u>23</u> <u>5</u> Bin 3: <u>x</u> <u>4</u> Bin 4: 18 20 Bin 5: 13	M1 <u>A1</u> A1	1.1b 1.1b 1.1b																																																																													
		(3)																																																																														
(d)	<table border="1"><tr><td>15</td><td>22</td><td>3</td><td>9</td><td>23</td><td>x</td><td>5</td><td>4</td><td>18</td><td>20</td><td>13</td></tr><tr><td><u>x</u></td><td>15</td><td>22</td><td>3</td><td>9</td><td>23</td><td><u>5</u></td><td>4</td><td>18</td><td>20</td><td>13</td></tr><tr><td><u>x</u></td><td>15</td><td>22</td><td>9</td><td>23</td><td>18</td><td>20</td><td>13</td><td><u>5</u></td><td>3</td><td>4</td></tr><tr><td><u>x</u></td><td><u>23</u></td><td>15</td><td>22</td><td>9</td><td>18</td><td>20</td><td>13</td><td><u>5</u></td><td><u>4</u></td><td>3</td></tr><tr><td><u>x</u></td><td><u>23</u></td><td>22</td><td>20</td><td><u>18</u></td><td>15</td><td>9</td><td>13</td><td><u>5</u></td><td><u>4</u></td><td>3</td></tr><tr><td><u>x</u></td><td><u>23</u></td><td>22</td><td><u>20</u></td><td><u>18</u></td><td>15</td><td>13</td><td><u>9</u></td><td><u>5</u></td><td><u>4</u></td><td>3</td></tr><tr><td><u>x</u></td><td><u>23</u></td><td>22</td><td><u>20</u></td><td><u>18</u></td><td>15</td><td><u>13</u></td><td><u>9</u></td><td><u>5</u></td><td><u>4</u></td><td>3</td></tr></table>	15	22	3	9	23	x	5	4	18	20	13	<u>x</u>	15	22	3	9	23	<u>5</u>	4	18	20	13	<u>x</u>	15	22	9	23	18	20	13	<u>5</u>	3	4	<u>x</u>	<u>23</u>	15	22	9	18	20	13	<u>5</u>	<u>4</u>	3	<u>x</u>	<u>23</u>	22	20	<u>18</u>	15	9	13	<u>5</u>	<u>4</u>	3	<u>x</u>	<u>23</u>	22	<u>20</u>	<u>18</u>	15	13	<u>9</u>	<u>5</u>	<u>4</u>	3	<u>x</u>	<u>23</u>	22	<u>20</u>	<u>18</u>	15	<u>13</u>	<u>9</u>	<u>5</u>	<u>4</u>	3	M1 A1 A1ft A1	1.1b 1.1b 1.1b 1.1b
15	22	3	9	23	x	5	4	18	20	13																																																																						
<u>x</u>	15	22	3	9	23	<u>5</u>	4	18	20	13																																																																						
<u>x</u>	15	22	9	23	18	20	13	<u>5</u>	3	4																																																																						
<u>x</u>	<u>23</u>	15	22	9	18	20	13	<u>5</u>	<u>4</u>	3																																																																						
<u>x</u>	<u>23</u>	22	20	<u>18</u>	15	9	13	<u>5</u>	<u>4</u>	3																																																																						
<u>x</u>	<u>23</u>	22	<u>20</u>	<u>18</u>	15	13	<u>9</u>	<u>5</u>	<u>4</u>	3																																																																						
<u>x</u>	<u>23</u>	22	<u>20</u>	<u>18</u>	15	<u>13</u>	<u>9</u>	<u>5</u>	<u>4</u>	3																																																																						
		(4)																																																																														
(e)	e.g. attempt at first-fit decreasing (or clearly considering the cases in the correct order) Bin 1: x Bin 2: 23 15 If 15 cannot go in Bin 1 then $x > 25$ Bin 3: 22 18 Bin 4: 20 13 If 13 cannot go in Bin 1 then $x > 27$	B1	3.1a																																																																													
	Therefore, x must be 28	dB1	2.2a																																																																													
		(2)																																																																														
(12 marks)																																																																																

Notes for Question 4

(a)

B1: Correct reasoning for why 3 bins is not possible or at least 4 bins are required

Examples of B1:

- If $x > 23$ then $132 + x$ is at least 156 and $\frac{156}{40} = 3.9$ so 3 is not possible
- The lower bound for the number of bins is $\frac{156}{40} = 3.9$ so 3 is not possible
- If $x = 24$ then $\frac{156}{40} = 3.9$ so 3 is not possible (or comparing 156 with 120)
- If $x > 23$ then $132 + x > 155$ so you would need to have more than $\frac{155}{40} = 3.875$ bins so is not possible (condone 'at least' rather than 'more than' on the bod so condone comparing 155 with 120)
- 4 values are at least half the capacity of a bin (or at least 20) and they are 20, 22, 23 and x , so 3 is not possible (**must** explicitly state the four values but condone those candidates who imply that the 20, 22, 23 and the x are all greater than half the capacity of a bin)
- Comparing 132 with 120 and saying that 3 is not possible (no mention of x is required)
- x would have to be -12 for it to be possible (but not just that x would have to be negative)

(b)

B1: One correct inequality ($x > 23$, $x \geq 24$, $x \leq 28$ or $x < 29$)**B1:** CAO ($23 < x \leq 28$ or $24 \leq x \leq 28$ or 29 with a strict inequality)(c) **NO MISREADS MARK EXACTLY TO THE SCHEME – if x is given a value then M1 only****M1:** First five values placed correctly (underlined values) and the x not in Bin 1 or 2 – at least eight values placed (condone cumulative values for **M1** only)**A1:** First eight values placed correctly (underlined and boxed values with no repeated values)**A1:** CSO no additional/repeated values

(d)

M1: Quick sort, pivot, p , chosen (must be choosing middle left or right – choosing first/last item as the pivot is **M0**). After the first pass the list must read (values greater than the pivot), pivot, (values less than the pivot). **If only choosing one pivot per iteration then max of M1A1 only** – Bubble sort is not a MR and scores **M0****A1:** First two passes correct (pivots for third pass need not be chosen)**A1ft:** Third and fourth passes correct (follow through from their second pass and choice of consistent pivots). They do not need to be choosing a pivot for the fifth pass for this mark. After their second pass their list must contain either 10, 11 or 12 numbers (so allow one additional or one missing number or a slip in one number e.g. the 5 becoming a 6)**A1:** CSO (correct solution only – all previous marks in this part **must** have been awarded) – both middle left and middle right require six passes**SC:** If the list is sorted in ascending order, then award a maximum of **M1A1A0A0** (so 2 marks) as in the scheme above even if the list is re-ordered after the sort is complete

Ascending (below is middle-right for middle-left the pivot is the 23):

15	22	3	9	23	x	5	4	18	20	13
15	22	3	9	23	5	4	18	20	13	<u>x</u>
3	4	5	15	22	9	23	18	20	13	x

Middle left:

15	22	3	9	23	x	5	4	18	20	13
<u>x</u>	15	22	3	9	23	5	4	18	20	13
<u>x</u>	<u>23</u>	15	22	3	9	5	4	18	20	13
<u>x</u>	<u>23</u>	15	22	9	18	20	13	<u>5</u>	3	4
<u>x</u>	<u>23</u>	15	22	18	20	13	<u>9</u>	<u>5</u>	4	<u>3</u>
<u>x</u>	<u>23</u>	22	20	<u>18</u>	15	13	<u>9</u>	<u>5</u>	4	<u>3</u>
<u>x</u>	<u>23</u>	<u>22</u>	20	<u>18</u>	<u>15</u>	13	<u>9</u>	<u>5</u>	4	<u>3</u>

(e)

B1: any clear indication that as the 15 does not fit in Bin 1 then $x + 15 > 40$ (condone for this mark just stating $x > 25$) **or** any clear indication that as the 13 does not fit in Bin 1 then $x + 13 > 40$ (condone for this mark just stating that $x > 27$)

dB1: correct answer of $x = 28$ – this mark is dependent on the previous B mark **and the correct upper bound of 28 in (b)** (so if $x = 28$ stated with no justification then no marks). As a minimum accept $x + 13 > 40$ therefore $x > 27$ so $x = 28$ (just ' $x > 27$ so $x = 28$ ' scores **B1 B0** unless explicit mention of where $x > 27$ has come from)

SC in (e): if **B0B0** then award **B1B0** for the first 7 values placed correctly in bins just once (ignore the other values)

Bin 1: x

Bin 2: 23 15

Bin 3: 22 18

Bin 4: 20 13



Question	Scheme	Marks	AOs
1(a)	Bin 1: <u>4.7</u> <u>2.9</u> <u>1.4</u> 2.8 Bin 2: <u>5.5</u> 5.8 Bin 3: 3.8 6.5 Bin 4: 5.1 6.3 Bin 5: 4.1	M1 A1 A1	1.1b 1.1b 1.1b
		(3)	
(b)	Bin 1: <u>6.5</u> <u>6.3</u> Bin 2: <u>5.8</u> <u>5.5</u> 1.4 Bin 3: <u>5.1</u> <u>4.7</u> 2.9 Bin 4: <u>4.1</u> <u>3.8</u> 2.8	M1 A1	1.1b 1.1b
		(2)	
(5 marks)			
Notes:			
a1M1: First four items placed correctly (the boxed values) and at least eight values placed in bins – condone cumulative totals for M1 only a1A1: First eight items placed correctly (the boxed and bold values) a2A1: CSO (no additional/repeated values) b1M1: First eight items placed correctly (the underlined values) and at least ten values placed in bins (no additional/repeated values) - condone cumulative totals for M1 only b1A1: CSO			

Question	Scheme										Marks	AOs
2(a)	e.g. Middle right										M1 A1 A1 ft A1	1.1b 1.1b 1.1b 1.1b
										Pivots		
	271	828	182	845	904	523	536	028	747	135	523	
	828	845	904	536	747	<u>523</u>	271	182	028	135	904, 028	
	<u>904</u>	828	845	536	747	<u>523</u>	271	182	135	<u>028</u>	536, 182	
	<u>904</u>	828	845	747	<u>536</u>	<u>523</u>	271	<u>182</u>	135	<u>028</u>	845	
	<u>904</u>	<u>845</u>	828	747	<u>536</u>	<u>523</u>	<u>271</u>	<u>182</u>	<u>135</u>	<u>028</u>	747	
	<u>904</u>	<u>845</u>	828	<u>747</u>	<u>536</u>	<u>523</u>	<u>271</u>	<u>182</u>	<u>135</u>	<u>028</u>		
	Middle left											
										Pivots		
	271	828	182	845	904	523	536	028	747	135	904	
	<u>904</u>	271	828	182	845	523	536	028	747	135	523	
	<u>904</u>	828	845	536	747	<u>523</u>	271	182	028	135	845, 182	
	<u>904</u>	<u>845</u>	828	536	747	<u>523</u>	271	<u>182</u>	028	135	536, 028	
	<u>904</u>	<u>845</u>	828	747	<u>536</u>	<u>523</u>	<u>271</u>	<u>182</u>	135	<u>028</u>	828	
	<u>904</u>	<u>845</u>	<u>828</u>	747	<u>536</u>	<u>523</u>	<u>271</u>	<u>182</u>	<u>135</u>	<u>028</u>		
											(4)	
(b)	In the worst case, the first pass requires $(n - 1)$ comparisons, the second $(n - 2)$, and so on										M1	3.1a
	$\sum_{r=1}^{n-1} r = \frac{1}{2}n(n-1)$										A1	2.2a
											(2)	

(c)	<table><tr><td><i>a</i></td><td><i>b</i></td><td><i>c</i></td><td><i>d</i></td><td>decimal</td><td>3 sig fig</td><td></td></tr><tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>Row 1</td></tr><tr><td></td><td></td><td>1</td><td>2</td><td>2</td><td>Row 2</td></tr><tr><td></td><td>2</td><td>2</td><td>$\frac{5}{2}$</td><td>2.5</td><td>2.5</td><td>Row 3</td></tr><tr><td></td><td>3</td><td>6</td><td>$\frac{8}{3}$</td><td>2.666...</td><td>2.67</td><td>Row 4</td></tr><tr><td></td><td>4</td><td>24</td><td>$\frac{65}{24}$</td><td>2.708333...</td><td>2.71</td><td>Row 5</td></tr><tr><td></td><td>5</td><td>120</td><td>$\frac{163}{60}$</td><td>2.71666...</td><td>2.72</td><td>Row 6</td></tr><tr><td></td><td>6</td><td>720</td><td>$\frac{1957}{720}$</td><td>2.7180555...</td><td>2.72</td><td>Row 7</td></tr></table>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	decimal	3 sig fig		1	1	1	1	1	Row 1			1	2	2	Row 2		2	2	$\frac{5}{2}$	2.5	2.5	Row 3		3	6	$\frac{8}{3}$	2.666...	2.67	Row 4		4	24	$\frac{65}{24}$	2.708333...	2.71	Row 5		5	120	$\frac{163}{60}$	2.71666...	2.72	Row 6		6	720	$\frac{1957}{720}$	2.7180555...	2.72	Row 7	M1 A1	1.1b 1.1b
	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	decimal	3 sig fig																																																			
	1	1	1	1	1	Row 1																																																			
			1	2	2	Row 2																																																			
		2	2	$\frac{5}{2}$	2.5	2.5	Row 3																																																		
		3	6	$\frac{8}{3}$	2.666...	2.67	Row 4																																																		
		4	24	$\frac{65}{24}$	2.708333...	2.71	Row 5																																																		
		5	120	$\frac{163}{60}$	2.71666...	2.72	Row 6																																																		
	6	720	$\frac{1957}{720}$	2.7180555...	2.72	Row 7																																																			
Output is $\frac{1957}{720}$ (accept 2.7180555... or 2.72 if working to 3 s.f.)	A1	2.2a																																																							
SC Accept																																																									
<table><tr><td><i>a</i></td><td><i>b</i></td><td><i>c</i></td><td><i>d</i></td><td>decimal</td><td>3 sig fig</td></tr><tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr><tr><td></td><td>2</td><td>1</td><td>2</td><td>2</td><td>2</td></tr><tr><td></td><td>3</td><td>2</td><td>$\frac{5}{2}$</td><td>2.5</td><td>2.5</td></tr><tr><td></td><td>4</td><td>6</td><td>$\frac{8}{3}$</td><td>2.666...</td><td>2.67</td></tr><tr><td></td><td>5</td><td>24</td><td>$\frac{65}{24}$</td><td>2.708333...</td><td>2.71</td></tr><tr><td></td><td>6</td><td>120</td><td>$\frac{163}{60}$</td><td>2.71666...</td><td>2.72</td></tr><tr><td></td><td></td><td>720</td><td>$\frac{1957}{720}$</td><td>2.7180555...</td><td>2.72</td></tr></table>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	decimal	3 sig fig	1	1	1	1	1	1		2	1	2	2	2		3	2	$\frac{5}{2}$	2.5	2.5		4	6	$\frac{8}{3}$	2.666...	2.67		5	24	$\frac{65}{24}$	2.708333...	2.71		6	120	$\frac{163}{60}$	2.71666...	2.72			720	$\frac{1957}{720}$	2.7180555...	2.72									
<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	decimal	3 sig fig																																																				
1	1	1	1	1	1																																																				
	2	1	2	2	2																																																				
	3	2	$\frac{5}{2}$	2.5	2.5																																																				
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		720	$\frac{1957}{720}$	2.7180555...	2.72																																																				
		(3)																																																							
(d)	$\frac{\left \frac{1957}{720} - e \right }{e} \times 100 = 0.00832(\%) \quad \text{If using 2.72 error} = 0.0632(\%)$	B1 ft	3.2b																																																						
		(1)																																																							
(10 marks)																																																									



Notes:

a1M1: Quick sort, pivot, p, chosen (must be choosing middle left or right – choosing any other item as the pivot is M0). After the first pass the list must read (values greater than the pivot), pivot, (values less than the pivot). Condone one error or omission in the list.

If only choosing one pivot per iteration then max of M1 only

Bubble sort is not a MR and scores M0.

If the list is sorted into ascending order, they can score M1 A1 A0 A0 for a fully correct sort even if reversed (if any error seen this can score M1 only).

If 028 is written as 28 at any stage this will lose the final CSO mark

The choice of pivots may only be seen once they have been used

a1A1: First pass correct **and** next pivots chosen correctly for the second pass (but the second pass does not need to be correct)

a2A1ft: Second and third passes correct (follow through from their first pass and choice of pivots). They do not need to be choosing a pivot for the fourth pass for this mark

If they alternate MR and ML pivots for the first and second passes, the pivots for the third pass must be consistent with their second pass

a3A1: CSO – so must have chosen and used 747 (MR) or 828 (ML) as a pivot for the fifth pass

b1M1: Considering the total number of comparisons in the worst case (possibly will see e.g. $(n-1) + (n-2) + \dots + 2 + 1$) which scores this mark (brackets not needed and sufficient terms to make sum clear. Accept a clear list without + signs for this mark)

b1A1: Using the standard series result that $\sum r = \frac{1}{2}n(n+1)$ with $n = (n-1)$ to get the correct quadratic expression for the total number of comparisons when sorting n values using bubble sort (allow any letter) Accept $(n-1) + (n-2) + \dots + 2 + 1 = \frac{1}{2}n(n-1)$ for both marks (sight of $\frac{1}{2}n(n-1)$ with no working scores both marks) (ISW if correct expression incorrectly expanded)

Note: Rows are shown above, and they may be spread across multiple rows in the table. Each row ends when the appropriate value in column d is seen

c1M1: At least four rows of cells completed with a correct first and second row – condone repeated values in all columns or a single value in each row (Note SC accept Column B with 1, 2, 3, 4, 5, 6 with no empty cell between 1 and 2)

c1A1: CAO – the values in the third, fourth and fifth rows correct (accept either full recurring decimals or decimals consistently rounded to 3 significant figures)

c2A1: CAO - correct output following a correct seventh row (their output must be clearly indicated e.g. on the Output line or identified in or near the table) If a value is written on the output line, this takes precedence over the table.

d1B1ft: Follow through their answer to (c) provided that % error is $< 0.1\%$ (must be given to at least 3 significant figures) (condone missing %) (accept - 0.00832%) (if a numerical approximation to e is used, it must be given to at least 3 d.p. – so 2.718 or better) (accept 8.32×10^{-3}). If incorrect working is seen, even if followed by a correct answer, this is B0.

$$\text{Note } \frac{\left| \frac{1957}{720} - e \right|}{\frac{1957}{720}} \times 100 = 0.00832\% \text{ scores B0}$$



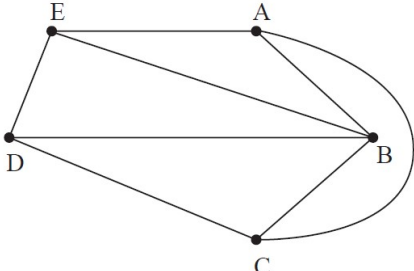
Sort Ascending Max M1 A1 A0 A0 e.g Middle right

										Pivots
271	828	182	845	904	523	536	028	747	135	523
271	182	028	135	<u>523</u>	828	845	904	536	747	028, 904
<u>028</u>	271	182	135	<u>523</u>	828	845	536	747	<u>904</u>	182, 536
<u>028</u>	135	<u>182</u>	271	<u>523</u>	<u>536</u>	828	845	747	<u>904</u>	845
<u>028</u>	<u>135</u>	<u>182</u>	<u>271</u>	<u>523</u>	<u>536</u>	828	747	<u>845</u>	<u>904</u>	747
<u>028</u>	<u>135</u>	<u>182</u>	<u>271</u>	<u>523</u>	<u>536</u>	<u>747</u>	828	<u>845</u>	<u>904</u>	

Middle left

										Pivots
271	828	182	845	904	523	536	028	747	135	904
271	828	182	845	523	536	028	747	135	<u>904</u>	523
271	182	028	135	<u>523</u>	828	845	536	747	<u>904</u>	182, 845
028	135	<u>182</u>	271	<u>523</u>	828	536	747	<u>845</u>	<u>904</u>	028, 536
<u>028</u>	135	<u>182</u>	<u>271</u>	<u>523</u>	<u>536</u>	828	747	<u>845</u>	<u>904</u>	828
<u>028</u>	<u>135</u>	<u>182</u>	<u>271</u>	<u>523</u>	<u>536</u>	747	828	<u>845</u>	<u>904</u>	



Question	Scheme	Marks	AOs																																				
1.(a)	Graph G is neither as there are more than two vertices of odd degree	B1	2.4																																				
		(1)																																					
(b)	e.g. A – B – C – D – E – A	B1	1.1b																																				
		(1)																																					
(c)	G <u>is</u> planar as it can be drawn with no arcs intersecting/crossing each other e.g. 	B1	2.4																																				
		(1)																																					
(d)	2	B1	1.1b																																				
		(1)																																					
(e)	<table border="1" data-bbox="389 1117 844 1330"><tr><td></td><td>A</td><td>B</td><td>C</td><td>D</td><td>E</td></tr><tr><td>A</td><td>-</td><td>10</td><td>15</td><td>∞</td><td>5</td></tr><tr><td>B</td><td>10</td><td>-</td><td>3</td><td>8</td><td>4</td></tr><tr><td>C</td><td>15</td><td>3</td><td>-</td><td>2</td><td>∞</td></tr><tr><td>D</td><td>∞</td><td>8</td><td>2</td><td>-</td><td>7</td></tr><tr><td>E</td><td>5</td><td>4</td><td>∞</td><td>7</td><td>-</td></tr></table>		A	B	C	D	E	A	-	10	15	∞	5	B	10	-	3	8	4	C	15	3	-	2	∞	D	∞	8	2	-	7	E	5	4	∞	7	-	B1	1.1b
	A	B	C	D	E																																		
A	-	10	15	∞	5																																		
B	10	-	3	8	4																																		
C	15	3	-	2	∞																																		
D	∞	8	2	-	7																																		
E	5	4	∞	7	-																																		
		(1)																																					
(f)	<table border="1" data-bbox="389 1420 844 1632"><tr><td></td><td>A</td><td>B</td><td>C</td><td>D</td><td>E</td></tr><tr><td>A</td><td>-</td><td>[9]</td><td>[12]</td><td>[12]</td><td>5</td></tr><tr><td>B</td><td>[9]</td><td>-</td><td>3</td><td>5</td><td>4</td></tr><tr><td>C</td><td>[12]</td><td>3</td><td>-</td><td>2</td><td>7</td></tr><tr><td>D</td><td>[12]</td><td>5</td><td>2</td><td>-</td><td>7</td></tr><tr><td>E</td><td>5</td><td>4</td><td>7</td><td>7</td><td>-</td></tr></table>		A	B	C	D	E	A	-	[9]	[12]	[12]	5	B	[9]	-	3	5	4	C	[12]	3	-	2	7	D	[12]	5	2	-	7	E	5	4	7	7	-	M1 A1	1.1b 1.1b
	A	B	C	D	E																																		
A	-	[9]	[12]	[12]	5																																		
B	[9]	-	3	5	4																																		
C	[12]	3	-	2	7																																		
D	[12]	5	2	-	7																																		
E	5	4	7	7	-																																		
		(2)																																					
(7 marks)																																							

Notes for Question 1

(a)

B1: ‘Neither’ together with a correct reason (ignore irrelevant statements but do not include incorrect statements)

Examples of B1:

- G contains more than 2 odd nodes
- G contains 4 odd nodes (**or** stating that A, C, D and E are odd)
- G contains (only) 1 even node
- The number of odd degree nodes in G is not 0 or 2

Examples of B0

- G contains odd nodes
- G contains at least 2 odd nodes
- G contains 4 nodes of degree 3 (not linking this to ‘odd’)

(b)

B1: CAO - must begin and end at the same node and include every other node in the graph exactly once. Accept if given in terms of arcs e.g. AC, CB, BD, DE, EA

(c)

B1: Correct answer of planar + justification

Examples of correct justification:

- AC can be moved outside **or** EB, DB can be moved outside
- AC(O), BE(I), BD(I) (or vice-versa) accept just AC(O) **or** BE(O) and BD(O)
- A correct drawing of the graph as planar (condone nodes not being labelled)

Examples of insufficient justification:

- Move arc BE (or BD) outside (need to mention both)
- Move arcs outside so that no arcs cross each other (must give specific examples of which arc(s) are being moved outside)
- Comments about moving nodes

(d)

B1: CAO (2 only)

(e)

B1: CAO - no blank entries (apart from the lead diagonal) and must include ∞ in cells AD, CE, DA, EC)

(f)

M1: No change in the fifth row and fifth column with at least **two** values reduced correctly (no blank entries – apart from the lead diagonal)**A1:** CAO for the final iteration

Question	Scheme	Marks	AOs
1(a)	..., D, Y, B, U, C	B1	1.1b
		(1)	
(b)	Or list of arcs: AU, AV, BV, CW, CX, DX, EY	M1	2.1
	e.g., select (and label) AU (as I) and the arcs that intersect AU are BV, EY, CW and DX (so label them O so AU(I), AV, BV(O), CW(O), CX, DX(O), EY(O))	A1	1.1b
	Edges BV and CW intersect and so the graph is not planar (oe)	A1	2.2a
		(3)	
(4 marks)			
Notes: (a) B1: CAO (CVEXAWDYBUC) – must return to C (b) M1: Either draws their Hamiltonian cycle from part (a) as the edges of a polygon and shows the remaining arcs intersecting inside OR lists the arcs that are not part of the Hamiltonian cycle A1: Selects any arc (that is not part of the Hamiltonian cycle) and lists/references the correct arcs that intersect with this selected arc – dependent on any correct Hamiltonian cycle and correct arcs that are not part of this cycle A1: cao – based on their initial arc selection, states the two arcs that are unlabelled (or that are labelled with the same label) which intersect each other (e.g., if CX chosen as the initial selection then this arc intersects with BV, EY and AV but EY and AV intersect) and concludes that the graph is not planar. This mark is dependent on the correct Hamiltonian cycle stated in either (a) or (b)			

Question	Scheme	Marks	AOs
1(a)		M1 A1	1.1b 1.1b
		(2)	
(b)	Kruskal's algorithm: AD, AB, BC, CG, reject BD, EG, reject CE, reject CD, reject AE, CF (reject EF, reject FG, reject DF)	M1 A1 A1	1.1b 1.1b 1.1b
		(3)	
(c)	Weight of MST: 162 (km)	B1	1.1b
		(1)	
(6 marks)			
Notes for Question 1			
(a) M1: Either all arcs correct (ignore weights) or two arcs correct (including correct weights) A1: CAO (b) M1: Kruskal's algorithm – first three arcs correctly chosen and at least one rejection seen at some point A1: All six arcs selected correctly AD, AB, BC, CG, EG, CF only A1: CSO – all selections and rejections correct (in correct order and at the correct time) (c) B1: CAO (condone lack of units)			

Question	Scheme	Marks	AOs
3(a)	(i)	M1 A1 (ABCD) A1 (GEH) A1ft (FJ)	1.1b 1.1b 1.1b 1.1b
	Shortest path from A to J is ABCEHJ	A1	2.2a
	Length of shortest path is 100	A1ft	2.2a
		(6)	
(b)	$0.0312 \times \left(\frac{9000}{9} \right)^2$	M1	1.1a
	= 31 200 seconds therefore 520 minutes	A1	2.2a
		(2)	
(8 marks)			

Notes for Question 3

In (a) it is important that all values at each node are checked very carefully – the order of the working values must be correct for the corresponding A mark to be awarded e.g. at D the working values must be 41 38 in that order (so 38 41 is incorrect)

It is also important that the order of labelling is checked carefully – some candidates start with a label of 0 at A (rather than 1) – which is fine. Also the order of labelling must be a strictly increasing sequence – so 1, 2, 3, 3, 4, ... will be penalised once (see notes below) but 1, 2, 3, 5, 6, ... is fine. Errors in the final values and working values are penalised before errors in the order of labelling

(a) M1: A larger value replaced by a smaller value in the working values at **two** of C or D or E or F or G or H or J

A1: All values in A, B, C and D correct. Condone lack of 0 in A's working value

A1: All values G, E and H correct and the working values in the correct order. Penalise order of labelling only once per question (G, E and H must be labelled in that order and G must be labelled after A, B, C and D). Note that an additional working value of 80 at E after the 61 is not an error so 65 61 80 is fine, however, any other number or e.g. 80 65 61 in this order is incorrect and scores A0 in this part

A1ft: All values in F and J correct on the follow through and the working values in the correct order. Penalise order of labelling only once per question.

To follow through F check that the working values at F follow from the candidate's final values for the nodes that are directly attached to F (which are C, D, G and H). For example, if correct then the order of labelling of nodes C, D, G and H are 3, 4, 5 and 7 respectively so the working values at F should come from C, D, G and H in that order. The first working value at F should be their 31 (the Final value at C) + 43 (the weight of the arc CF), the second working value at F should be their 38 (the Final value at D) + 34 (the weight of the arc DF), the third working value at F should be their 59 (the Final value at G) + 11 (the weight of the arc GF), the fourth working value need not be there (as $68 + 22 > 70$) but do not penalise if seen. Repeat the process for J (which will have working values from E and H with the order of these nodes determined by the candidate's order of labelling at E and H)

A1: CAO - correct path from A to J (ABCEHJ) and not the path from J to A

A1ft: ft their final value at J only (if 100 stated and 100 is not the final value at J then **A0**)

(b)

M1: Complete, correct method – allow reciprocal e.g. $0.0312 \times \left(\frac{9}{9000}\right)^2$ – allow slips in values

only e.g. 0.312 for 0.0312. If using anything other than squared e.g. n^3 then **M0**. The correct answer in seconds, minutes or hours (8 hours 40 minutes) implies this mark (and if in minutes implies the A mark too)

A1: CAO – the exact value of 520 must be stated at some point (as question specifically asked for the answer in minutes) – condone lack of units (but if present must be correct)



Question	Scheme										Marks	AOs		
3(a)	Time matrix						Route matrix						B1 B1	1.1b 1.1b
		A	B	C	D	E		A	B	C	D	E		
	A	-	8	4	7	∞	A	A	B	C	D	E		
	B	8	-	3	∞	10	B	A	B	C	D	E		
	C	4	3	-	∞	6	C	A	B	C	D	E		
	D	7	∞	1	-	1	D	A	B	C	D	E		
	E	∞	10	6	1	-	E	A	B	C	D	E		
											(2)			
(b)	Time matrix						Route matrix						M1 A1 M1 A1	1.1b 1.1b 1.1b 1.1b
		A	B	C	D	E		A	B	C	D	E		
	A	-	7	4	7	10	A	A	C	C	D	C		
	B	7	-	3	14	9	B	C	B	C	C	C		
	C	4	3	-	11	6	C	A	B	C	A	E		
	D	5	4	1	-	1	D	C	C	C	D	E		
	E	10	9	6	1	-	E	C	C	C	D	E		
	Time matrix						Route matrix							
		A	B	C	D	E		A	B	C	D	E		
	A	-	7	4	7	8	A	A	C	C	D	D		
	B	7	-	3	14	9	B	C	B	C	C	C		
	C	4	3	-	11	6	C	A	B	C	A	E		
	D	5	4	1	-	1	D	C	C	C	D	E		
	E	6	5	2	1	-	E	D	D	D	D	E		
											(4)			
(c)(i)	NNA: A – C – B – E – D – A										B1	1.1b		
	(ii) 4 + 3 + 9 + 1 + 5 = 22 minutes										dB1	1.1b		
	(iii) A – C – B – C – E – D – C – A										B1	3.2a		
											(3)			
(9 marks)														
Notes for Question 3														
(a)														
B1: Correct time matrix														
B1: Correct route matrix														

(b)

M1: No change in the third row and third column of both matrices with at least one value in the time matrix reduced correctly and one value in the route matrix changed to C

A1: CAO

M1: No change in the fourth row and fourth column of both matrices with at least one value in the time matrix reduced correctly (follow through their first iteration) and one value in the route matrix changed to D

A1: CAO

(c)(i)

B1: CAO

(c)(ii)

dB1: CAO – not from A – C – D – E – B – A

(c)(iii)

B1: CAO

Question	Scheme	Marks	AOs																																																																																																		
3(a)		M1 A1	1.1b 1.1b																																																																																																		
		(2)																																																																																																			
(b)	Third iteration: <table><tr><td></td><td>A</td><td>B</td><td>C</td><td>D</td><td>E</td><td>F</td></tr><tr><td>A</td><td>-</td><td>12</td><td>29</td><td>20</td><td>29</td><td>11</td></tr><tr><td>B</td><td>12</td><td>-</td><td>17</td><td>8</td><td>29</td><td>23</td></tr><tr><td>C</td><td>29</td><td>17</td><td>-</td><td>4</td><td>12</td><td>40</td></tr><tr><td>D</td><td>24</td><td>21</td><td>4</td><td>-</td><td>16</td><td>13</td></tr><tr><td>E</td><td>41</td><td>29</td><td>12</td><td>16</td><td>-</td><td>12</td></tr><tr><td>F</td><td>11</td><td>23</td><td>40</td><td>13</td><td>12</td><td>-</td></tr></table> Fourth iteration: <table><tr><td></td><td>A</td><td>B</td><td>C</td><td>D</td><td>E</td><td>F</td></tr><tr><td>A</td><td>-</td><td>12</td><td>24</td><td>20</td><td>29</td><td>11</td></tr><tr><td>B</td><td>12</td><td>-</td><td>12</td><td>8</td><td>24</td><td>21</td></tr><tr><td>C</td><td>28</td><td>17</td><td>-</td><td>4</td><td>12</td><td>17</td></tr><tr><td>D</td><td>24</td><td>21</td><td>4</td><td>-</td><td>16</td><td>13</td></tr><tr><td>E</td><td>40</td><td>29</td><td>12</td><td>16</td><td>-</td><td>12</td></tr><tr><td>F</td><td>11</td><td>23</td><td>17</td><td>13</td><td>12</td><td>-</td></tr></table>		A	B	C	D	E	F	A	-	12	29	20	29	11	B	12	-	17	8	29	23	C	29	17	-	4	12	40	D	24	21	4	-	16	13	E	41	29	12	16	-	12	F	11	23	40	13	12	-		A	B	C	D	E	F	A	-	12	24	20	29	11	B	12	-	12	8	24	21	C	28	17	-	4	12	17	D	24	21	4	-	16	13	E	40	29	12	16	-	12	F	11	23	17	13	12	-	M1 A1 M1 A1	1.1b 1.1b 1.1b 1.1b
	A	B	C	D	E	F																																																																																															
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F	11	23	17	13	12	-																																																																																															
		(4)																																																																																																			
(c)	Nearest neighbour routes: E – C – D – F – A – B – E Length = 76 E – F – A – B – D – C – E Length = 59 The cycle that begins E – F – ... is the better upper bound as the value is the smaller of the two	M1 A1 A1 A1	3.4 1.1b 1.1b 2.4																																																																																																		
		(4)																																																																																																			
(10 marks)																																																																																																					

Notes for Question 3

a1M1: Correct five arcs from A

a1A1: cao no additional arcs or arrows, correct weights and arc AE directed from A to E

In the distance matrix for part (b) ignore whatever is written in the lead diagonal (top left to bottom right) and just consider the values (ignore shading etc.) – check bottom of page 9 for replaced matrices

b1M1: No change in the third row and third column with at least two values reduced correctly (no blank entries – apart from the lead diagonal)

b1A1: cao for the third iteration

b2M1: No change in their fourth row and fourth column (following their third iteration) with at least two values correctly reduced follow through from their previous iteration (no blank entries – apart from the lead diagonal)

b2A1: cso for both iterations (no follow through from an incorrect third iteration even if the fourth iteration is ‘correct’ so M1A0M1A1 is not possible in this part)

c1M1: Either cycle correct (must include returning to E) – nodes must be in the correct order and therefore not reversed but allow if stated in terms of arcs. If only one cycle correct then M1 only

c1A1: Both cycles correct – nodes must be in the correct order but allow if stated in terms of arcs

c2A1: Both cycles and corresponding lengths correct

c3A1: Correct reasoning that the smallest (oe) value is the better upper bound – dependent on all previous marks in this part (cso)



Question	Scheme	Marks	AOs
6(a)(i)		M1 A1 A1 A1ft	1.1b 1.1b 1.1b 1.1b
	Shortest path from A to H: ABEFGH	A1	2.2a
(a)(ii)	Length of shortest path from A to H is 112	A1ft	2.2a
		(6)	
(b)	Applying Dijkstra repeatedly to n nodes implies that the order is $n(n^2) = n^3$	B1	3.1b
	$t = 0.082 \left(\frac{200}{10} \right)^3$	M1	3.4
	= 656 (seconds)	A1	2.2a
		(3)	
(c)	e.g. order of n^3 does not mean that the order is proportional to n^3 (which is the assumption behind the answer in (b)) but merely means that the dominant term is of order n^3	B1	3.2b
		(1)	
(10 marks)			
Notes: In (a) it is important that all values at each node are checked very carefully – the order of the working values must be correct for the corresponding A mark to be awarded e.g. at H the working values must be 129 118 112 in that order (so 129 112 118 is incorrect) It is also important that the order of labelling is checked carefully – some candidates start with a label of 0 at A (rather than 1) – which is fine. Also the order of labelling must be a strictly increasing sequence – so 1, 2, 3, 3, 4, ... will be penalised once (see notes below) but 1, 2, 3, 5, 6, ... is fine. Errors in the final values and working values are penalised before errors in the order of labelling			

(a)

M1: A larger value replaced by a smaller value in at least two of the working boxes at either D or F or G or H

A1: All values in A, B, C and E correct. Condone lack of 0 in A's working value

A1: All values D and F correct and the working values in the correct order. Penalise order of labelling only once per question (D and F must be labelled in that order and D must be labelled after A, B, C and E)

A1ft: All values in G and H correct on the follow through and the working values in the correct order. Penalise order of labelling only once per question. To follow through G check that the working value at G follows from the candidate's final values from their feeds into G (which will most likely come from nodes C, D and/or F (in the order in which the candidate has labelled them)) and that the final value, and order of labelling, follows through correctly. Repeat this process for H (which will possibly have working values from C, F and/or G with the order of these values determined by the candidate's order of labelling at C, F and G)

A1: cao - correct path from A to H (ABEFGH)

A1ft: Follow through their final value at H only (if 112 stated and 112 is not the final value at H then **A0**)

(b)

B1: Any indication that repeated application of Dijkstra has cubic order

M1: Complete method – allow 10/200 – allow slips in values e.g. 0.82 for 0.082 and accept 200/10 (or 10/200) either squared or cubed **only**

A1: cao

(c)

B1: Any indication that the run-time is not exactly proportional to n^3 e.g., may suggest that there are other terms ($n^3 + \dots$), or that n^3 is the dominant term, or that order does not imply proportionality. Do not accept only that ' n^3 is not exact'. Condone use of n^2 (oe) for n^3



Qu	Scheme										Marks	AOs		
3(a)	Distance table					Route table					B1	1.1b		
		A	B	C	D	E		A	B	C			D	E
	A	-	15	7	18	3	A	A	B	C			D	E
	B	15	-	∞	∞	∞	B	A	B	C			D	E
	C	7	5	-	4	9	C	A	B	C			D	E
	D	18	∞	4	-	3	D	A	B	C			D	E
	E	∞	∞	9	3	-	E	A	B	C	D	E		
											(2)			
(b)	1 st iteration:										M1 A1	1.1b 1.1b		
	Distance table					Route table								
		A	B	C	D	E		A	B	C			D	E
	A	-	15	7	18	3	A	A	B	C			D	E
	B	15	-	22	33	18	B	A	B	A			A	A
	C	7	5	-	4	9	C	A	B	C			D	E
	D	18	33	4	-	3	D	A	A	C			D	E
E	∞	∞	9	3	-	E	A	B	C	D	E			
	2 nd iteration:										A1ft	1.1b		
Distance table					Route table									
	A	B	C	D	E		A	B	C	D			E	
A	-	15	7	18	3	A	A	B	C	D			E	
B	15	-	22	33	18	B	A	B	A	A			A	
C	7	5	-	4	9	C	A	B	C	D			E	
D	18	33	4	-	3	D	A	A	C	D			E	
E	∞	∞	9	3	-	E	A	B	C	D	E			
	3 rd iteration (SEE NOTES FOR VALID ALTERNATIVE):										M1 A1	1.1b 1.1b		
Distance table					Route table									
	A	B	C	D	E		A	B	C	D			E	
A	-	12	7	11	3	A	A	C	C	C			E	
B	15	-	22	26	18	B	A	B	A	C			A	
C	7	5	-	4	9	C	A	B	C	D			E	
D	11	9	4	-	3	D	C	C	C	D			E	
E	16	14	9	3	-	E	C	C	C	D	E			
											(5)			
(c)(i)	Start at E(5 th row) and read across to the A (1 st column), there is a D there so the route from E to A is via D										B1	2.4		
	Now consider both E to D and D to A – for E reading across to the D (4 th column), there is a D indicating that the shortest path from E to D is ED. For D reading across to the A (1 st column), there is a C indicating that the shortest path from D to A is via C										dB1	2.4		
(ii)	EDCA										B1	2.2a		
											(3)			

(d)(i)	NNA: D – E – C – B – A – D	B1	1.1b
(ii)	$3 + 7 + 5 + 15 + 6 = 36$ miles	B1	1.1b
(iii)	D – E – D – C – B – A – E – D	B1	3.2a
(iv)	e.g. the cycle A – E – D – C – B – A has a length of 30 miles < 36 miles so Mabintou's route is not optimal	B1	2.4
		(4)	

(14 marks)

Notes for Question 3

IN THE DISTANCE AND ROUTE TABLE FOR PARTS (a) and (b) IGNORE WHATEVER IS WRITTEN IN THE LEAD DIAGONAL (TOP LEFT TO BOTTOM RIGHT)

(a) B1: Correct distance table (condone dashes, crosses, etc. for infinity but do not condone a 'large' number in these cells)

B1: Correct route table

(b) M1: No change in the first row and first column of both tables with at least two values in the distance table correctly reduced and two letters in the route table correctly changed – all cells complete

A1: CAO (condone dashes, etc. in cells EA and EB)

A1ft: No change from candidate's first iteration to second iteration for either table or ft from candidate's first iteration

M1: No change in the third row and third column of both tables with at least two values in the distance table correctly reduced from their second iteration and two values in the route table correctly changed

A1: CAO for third iteration (**note that the entry in row B column D for the route table could be an A**)

(c) B1: Row E column A is D so the route is E to A via D (or implies that the order of the nodes in the route is EDA) **or** D implied from general argument **or** Row E column A is D therefore the route begins ED (in all cases must clearly imply **row E** and **column A**)

dB1: Row D column A is C therefore the route goes via C (before A) **or** complete general argument that allows the route from D to A to be found **or** allow those who say that row D column A is C so the route is EDC and then row C column A is A

B1: CAO (EDCA)

(d)(i) B1: CAO (D – E – C – B – A – D)

(ii) B1: CAO (36 – no units required)

(iii) B1: CAO (D – E – D – C – B – A – E – D) **or** mentions that the cycle would visit E twice and D three times (or visit D before the end of the cycle – if D visited once stated and it is not clear that this isn't the start or finish then B0) **or** mention of E to C via D and A to D via E

(iv) B1: A correct cycle stated (e.g. a cyclic permutation of A – E – D – C – B – A) with corresponding correct length – **dependent on second B mark in this part** (so must have had 36 in (d)(ii))



Question	Scheme	Marks	AOs																																																																																																		
4(a)	Initial time matrix <table><tr><td></td><td>A</td><td>B</td><td>C</td><td>D</td><td>E</td><td>F</td></tr><tr><td>A</td><td>-</td><td>57</td><td>95</td><td>150</td><td>63</td><td>230</td></tr><tr><td>B</td><td>57</td><td>-</td><td>72</td><td>∞</td><td>132</td><td>∞</td></tr><tr><td>C</td><td>95</td><td>72</td><td>-</td><td>289</td><td>160</td><td>125</td></tr><tr><td>D</td><td>150</td><td>∞</td><td>289</td><td>-</td><td>84</td><td>∞</td></tr><tr><td>E</td><td>63</td><td>132</td><td>160</td><td>84</td><td>-</td><td>191</td></tr><tr><td>F</td><td>230</td><td>∞</td><td>125</td><td>∞</td><td>191</td><td>-</td></tr></table>		A	B	C	D	E	F	A	-	57	95	150	63	230	B	57	-	72	∞	132	∞	C	95	72	-	289	160	125	D	150	∞	289	-	84	∞	E	63	132	160	84	-	191	F	230	∞	125	∞	191	-	B1	1.1b																																																	
	A	B	C	D	E	F																																																																																															
A	-	57	95	150	63	230																																																																																															
B	57	-	72	∞	132	∞																																																																																															
C	95	72	-	289	160	125																																																																																															
D	150	∞	289	-	84	∞																																																																																															
E	63	132	160	84	-	191																																																																																															
F	230	∞	125	∞	191	-																																																																																															
		(1)																																																																																																			
(b)	1st iteration: <table><tr><td></td><td>A</td><td>B</td><td>C</td><td>D</td><td>E</td><td>F</td></tr><tr><td>A</td><td>-</td><td>57</td><td>95</td><td>150</td><td>63</td><td>230</td></tr><tr><td>B</td><td>57</td><td>-</td><td>72</td><td>207</td><td>120</td><td>287</td></tr><tr><td>C</td><td>95</td><td>72</td><td>-</td><td>245</td><td>158</td><td>125</td></tr><tr><td>D</td><td>150</td><td>207</td><td>245</td><td>-</td><td>84</td><td>380</td></tr><tr><td>E</td><td>63</td><td>120</td><td>158</td><td>84</td><td>-</td><td>191</td></tr><tr><td>F</td><td>230</td><td>287</td><td>125</td><td>380</td><td>191</td><td>-</td></tr></table> <table><tr><td></td><td>A</td><td>B</td><td>C</td><td>D</td><td>E</td><td>F</td></tr><tr><td>A</td><td>A</td><td>B</td><td>C</td><td>D</td><td>E</td><td>F</td></tr><tr><td>B</td><td>A</td><td>B</td><td>C</td><td>A</td><td>A</td><td>A</td></tr><tr><td>C</td><td>A</td><td>B</td><td>C</td><td>A</td><td>A</td><td>F</td></tr><tr><td>D</td><td>A</td><td>A</td><td>A</td><td>D</td><td>E</td><td>A</td></tr><tr><td>E</td><td>A</td><td>A</td><td>A</td><td>D</td><td>E</td><td>F</td></tr><tr><td>F</td><td>A</td><td>A</td><td>C</td><td>A</td><td>E</td><td>F</td></tr></table>		A	B	C	D	E	F	A	-	57	95	150	63	230	B	57	-	72	207	120	287	C	95	72	-	245	158	125	D	150	207	245	-	84	380	E	63	120	158	84	-	191	F	230	287	125	380	191	-		A	B	C	D	E	F	A	A	B	C	D	E	F	B	A	B	C	A	A	A	C	A	B	C	A	A	F	D	A	A	A	D	E	A	E	A	A	A	D	E	F	F	A	A	C	A	E	F	M1 A1	1.1b 1.1b
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C	A	B	C	A	A	F																																																																																															
D	A	A	A	D	E	A																																																																																															
E	A	A	A	D	E	F																																																																																															
F	A	A	C	A	E	F																																																																																															
		(2)																																																																																																			
(c)	Route must start and finish at B, E therefore need to consider pairings of the other four odd nodes (A, C, D and F)	M1	3.1b																																																																																																		
	AC + DF = 95 + 275 = 370 AD + CF = 147 + 125 = 272* AF + CD = 220 + 242 = 462	A1 A1	1.1b 1.1b																																																																																																		
	Repeat arcs: AE, DE and CF	A1	2.2a																																																																																																		
		(4)																																																																																																			
(d)	Length: 1648 + 272 = 1920 (minutes)	B1	2.2a																																																																																																		
		(1)																																																																																																			
(8 marks)																																																																																																					

Notes:**(a)****B1:** Correct distance table (condone dashes, crosses, etc. for infinity but do not condone a ‘large’ number in these cells or these cells left blank)**(b)****M1:** No change in the first row and first column of both tables with at least two values in the distance table correctly reduced and two letters in the route table correctly changed – all cells complete**A1:** cao**(c)****M1:** Either the correct three pairings of the correct four nodes A, C, D and F **or** recognises that as the route begins at B and finishes at E that only the nodes A, C, D and F need to be considered**A1:** Two rows correct including pairings **and** totals**A1:** All three rows correct including pairings **and** totals**A1:** selecting the shortest pairing, and stating that these arcs (AE, DE and CF) should be repeated. Must be these arcs and not e.g., AED or AD via E, etc.**(d)****B1:** cao

Question	Scheme	Marks	AOs
6(a)	The graph has exactly two odd nodes and so the graph is semi-Eulerian	B1 dB1	2.4 2.2a
		(2)	
(b)	Shortest path from A to F is $58 + x$ and shortest path from A to G is $60 + y$ $58 + x + 60 + y = 140$ The only odd nodes in the network are A and G Route inspection algorithm: Shortest route between A and G is $60 + y$ $\Rightarrow 320 + x + y + 60 + y = 409$ $x = 15$ and $y = 7$	M1 A1 A1 A1 A1ft M1 B1 M1 A1	1.1b 1.1b 1.1b 1.1b 2.2a 2.1 2.2a 3.1b 2.2a
		(9)	

(11 marks)

Notes for Question 6

(a)

B1: Explanation which consists of the graph having two odd nodes or stating graph is semi-Eulerian
dB1: Exactly two odd nodes (or two odd nodes and five even nodes or the rest even) together with the deduction that therefore the graph is semi-Eulerian

(b)

M1: For a larger number replaced by a smaller one in two working value boxes at C, D, G or F

A1: For all values correct (and in correct order) at A, B and C

A1: For all values correct (and in correct order) at E and D

A1: For all values correct (and in correct order) at G and F

A1ft: Length of shortest path from A to F or A to G stated (may be seen in an equation(s))

M1: (length of shortest path from A to F) + (length of shortest path from A to G) = 140 – linear equation in x and y

B1: Correctly stating the two odd nodes (A and G) – could be implied by subsequent working

M1: For an equation based on the route from A to G ($320 + x + y + \text{final value at G (in } y) = 409$)

A1: CAO for x and y



Question	Scheme	Marks	AOs
2(a)	(i) (ii) Path from A to K: ABEH GK (iii) Length of shortest path from A to K is 59 (km)	M1 A1 A1 A1ft A1 A1ft	1.1b 1.1b 1.1b 1.1b 2.2a 2.2a
		(6)	
(b)(i)	Repeated edges: AB, BE, EH	B1	2.2a
(b)(ii)	Length of route = $299 + 28 = 327$ (km)	B1ft	2.2a
		(2)	
(c)	Route must start at A and finish at K therefore need to consider pairings of the nodes D, E, H and K	M1	3.1b
	$D(AB)E + H(G)K = 26 + 31 = 57$ $D(FG)H + E(HG)K = 30 + 47 = 77$ $D(FG)K + EH = 49 + 16 = 65$	A1 A1 A1	1.1b 1.1b 1.1b
	Length of new route = $299 + 57 = 356$ (km)	A1	2.2a
		(5)	
(13 marks)			

Notes for Question 2

In (a) it is important that all values at each node are checked very carefully – the order of the working values must be correct for the corresponding A mark to be awarded, for example at F the working values must be 29 28 25 in that order (so 29 25 28 is incorrect)

It is also important that the order of labelling is checked carefully – some candidates start with a 0 label at A (rather than 1) – which is fine. Also the order of labelling must be a strictly increasing sequence – so 1, 2, 3, 3, 4,... will be penalised once (see notes below) but 1, 2, 3, 5, 6,... is fine. Errors in the final values and working values are penalised before errors in the order of labelling. Condone crossed out working values if they are still legible

a1M1: A larger value replaced by a smaller value in at least **two** different Working Value boxes at either C or F or G or H or J or K

a1A1: All values at A, B, E, D and C correct. Condone lack of 0 in A's working value

a2A1: All values at F, H and G correct and the working values in the correct order. Penalise order of labelling only once per question (F, H and G must be labelled in that order and F must be labelled after A, B, E, D and C)

a3A1ft: All values in J and K correct on the follow through and the working values in the correct order. To follow through J check that the working values at J follow from the candidate's final values for the nodes that are directly attached to J (which are G and H). For example, **if** correct then the order of labelling of nodes G and H are 8 and 7 respectively so the working values at J should come from H and G in that order. The first working value at J should be their 28 (the Final value at H) + 23 (the weight of the arc HJ), the second working value at J should be their 34 (the Final value at G) + 10 (the weight of the arc GJ). Repeat the process for K (which will have working values from G and H with the order of these nodes determined by the candidate's order of labelling at G and H)

a4A1: cao (correct path from A to K – ABEHGK). Do not accept KGHEBA. Condone the length of the shortest path and the path interchanged on the answer lines in the answer book. Allow the paths written in terms of arcs, e.g. AB, BE, EH, HG, GK

a5A1ft: ft their final value at K only (if 59 stated and 59 is not the final value at K then A0)

bi1B1: cao (AB, BE, EH only but in any order) – must be stated as edges so B0 for ABEH

bii1B1ft: 327 or follow through their final value at H from **(a)** + 299

c1M1: The correct three pairings of the correct four nodes (D, E, H and K)

c1A1: One row correct including pairing **and** total

c2A1: Two rows correct including pairings **and** totals

c3A1: All three rows correct including pairings **and** totals

c4A1: cao (356) – dependent on all previous marks in this part

Condone lack of units throughout



Question	Scheme	Marks	AOs
5.(a)	(i) Route must start at F and finish at J therefore need to consider pairings of the nodes F, C, D and G	M1	3.1b
	$CD + FG = 82 + 28 = 110$	A1	1.1b
	$CF + DG = 71 + 23 = 94$	A1	1.1b
	$CG + DF = 59 + 22 = 81$		
	Repeated roads: CB, BA, AG, and DF	A1	2.2a
	(ii) Length of route = $423 + 81 = 504$ (miles)	A1ft	2.2a
		(5)	
(b)	Prim's starting at A: AG, AF, DF, AB, BC, CJ, HJ, EJ	M1	1.1b
		A1	1.1b
		A1	1.1b
		(3)	
(c)	NNA starting at G: $G - A - F - D - B - C - J - H - E - G$ $8 + 20 + 22 + 65 + 17 + 10 + 12 + 30 + 87 = 271$	M1	1.1b
		A1	1.1b
		(2)	
(d)	$(141 - 8) + 8 + 23 = 164$ (miles)	M1	3.1b
		A1	2.2a
		(2)	
(e)	GAFD GA BCJH J E JCBA G	B1	3.2a
		(1)	
(13 marks)			

Notes for Question 5

(a)**M1:** The correct three pairings of the correct four nodes (F, C, D and G)**A1:** Two rows correct including pairings **and** totals**A1:** All three rows correct including pairings **and** totals**A1:** CAO (CB, BA, AG, DF only but in any order) – must be stated as edges so **A0** for any of CG or CBAG or CG via B and A**A1ft:** 423 + weight of their shortest pairing**(b)****M1:** First three arcs correctly chosen in order {AG, AF, DF, ...} or first four nodes correctly chosen in order {A, G, F, D, ...}. If any explicit rejections seen at any point then **M1** (max) only. Order of nodes may be seen at the top of one of the tables {1, -, -, 4, -, 3, 2, -, -} so please check the top of the tables carefully**A1:** First five arcs correctly chosen in order {AG, AF, DF, AB, BC, ...} or all nine nodes correctly chosen in order {A, G, F, D, B, C, J, H, E}. Order of nodes may be seen at the top of one of the tables so for the first two marks accept {1, 5, 6, 4, 9, 3, 2, 8, 7} (do not condone any missing numbers e.g. the number 9 must be above E)**A1:** CSO – all arcs correct stated and chosen in the correct order. Candidates must be considering arcs for this final mark (do not accept a list of nodes or numbers across the top of one of the tables unless the correct list of arcs (in the correct order) is also seen)**(c)****M1:** Nearest neighbour route starting at G – must have at least G – A – F – D – B – C – ... allow if stated in terms of arcs**A1:** CAO on length (271) **and** route (must return to G and can be stated in terms of arcs)**(d)****M1:** For one of the following

- RMST arcs: AF, DF, AB, BC, CJ, HJ, EJ (stated in any order) – condone AG added too but must be removed at some later stage (e.g. subtracting 8 at some point)
- RMST weight calculation: $20 + 22 + 34 + 17 + 10 + 12 + 18$ (not just 20, 22, etc.)
- RMST weight stated: 133 or $141 - 8$
- Lower bound stated: 164

A1: CAO (164)**(e)****B1:** CAO (GAFD **GA** BCJH **J E** JCBA G or in terms of arcs)

Qu	Scheme	Marks	AOs
2(a)	Path from A to D is AFGJD Length of path from A to D is 78 metres	M1 A1 A1 A1ft A1 A1ft	1.1b 1.1b 1.1b 1.1b 2.2a 2.2a
		(6)	
(b)	$A(FG)C + E(J)H = 61 + 23 = 84$ $A(B)E + C(GJ)H = 53 + 27 = 80^*$ $A(FGJ)H + C(G)E = 74 + 17 = 91$ Repeat arcs: AB, BE, CG, GJ and JH	M1 A1ft A1 A1	3.1b 1.1b 1.1b 2.2a
		(4)	
(c)	Length of the route is $370 + 80 = 450$ metres	B1ft	2.2a
		(1)	
(d)(i)	If node B is removed this makes D, C, G and H odd The shortest path between any two odd nodes is CG (so repeat CG) so the route should start at D and finish at H (or vice-versa)	M1 A1	3.1b 2.2a
(ii)	Length of new route is $370 - 38 - 42 - 15 + 7 = 282$ metres	B1	2.2a
		(3)	
(14 marks)			

Notes for Question 2

In (a) it is important that all values at each node are checked very carefully – the order of the working values must be correct for the corresponding A mark to be awarded e.g. at H the working values must be 81 77 74 in that order (so 81 74 77 is incorrect)

It is also important that the order of labelling is checked carefully – some candidates start with a label of 0 at A (rather than 1) – which is fine. Also the order of labelling must be a strictly increasing sequence – so 1, 2, 3, 3, 4, ... will be penalised once (see notes below) but 1, 2, 3, 5, 6, ... is fine. Errors in the final values and working values are penalised before errors in the order of labelling

(a) M1: A larger value replaced by a smaller value in at least two of the working boxes at either C or D or E or G or J

A1: All values in A, B, F and E correct. Condone lack of 0 in A's working value

A1: All values G, C and J correct and the working values in the correct order. Penalise order of labelling only once per question (G, C and J must be labelled in that order and G must be labelled after A, B, F and E). Note that an additional working value of 63 at G after the 54 is not an error so 54 63 is fine, however, any other number or 63 54 in this order is incorrect and scores A0 in this part

A1ft: All values in H and D correct on the follow through and the working values in the correct order. Penalise order of labelling only once per question. To follow through H check that the working value at H follows from the candidate's final values from their feeds into H (which will come from nodes F, G and/or J (in the order in which the candidate has labelled them)) and that the final value, and order of labelling, follows through correctly. Repeat this process for D (which will possibly have working values from B and J with the order of these values determined by the candidate's order of labelling at B and J)

A1: CAO - correct path from A to D (AFGJD)

A1ft: ft their final value at D only (if 78 stated and 78 is not the final value at D then A0)

(b) M1: correct three pairings of the correct four odd nodes (A, C, E and H)

A1ft: any row correct including pairing **and** total (ft the final values from **(a)** for their shortest paths from A to the three other nodes C, E or H only (so the pairing that does not include A must be correct))

A1: all three rows correct including pairings **and** totals

A1: selecting the shortest pairing, and stating that these arcs (AB, BE, CG, GJ and JH) should be repeated. Must be these arcs and not e.g. ABE, CGJH or AE via B, etc.

(c) B1ft: For 370 + their smallest repeat out of a choice of at least **two** totals seen in **(b)** – this mark is dependent on M1 in (b)

(d) M1: Mention of the fact that these four nodes D, C, G and H only are now odd **or** clear consideration of these four nodes only

A1: CAO D and H **and** must have clearly indicated that the shortest path is from C to G (or vice-versa) – but A0 if clearly selecting the shortest pairing first before selecting the shortest path (as the shortest path is embedded in the shortest pairing)

B1: CAO (282)



Question	Scheme	Marks	AOs
3(a)	Prim's starting at A: AH, AB, DH; CD, CE; DG, EF	M1 A1 A1	1.1b 1.1b 1.1b
		(3)	
(b)	Weight of MST is 183 (miles)	B1	2.2a
		(1)	
(c)	NNA: J – F – E – C – D – H – A – B – G – J	B1	1.1b
	Upper bound is 267 (miles)	B1	2.2a
		(2)	
(d)	$183 + 27 + 25 = \dots$	M1	3.1b
	$\dots = 235$ (miles)	A1	2.2a
		(2)	
(8 marks)			
Notes: (a) M1: First three arcs correctly chosen in order {AH, AB, DH, ...} or first four nodes correctly chosen in order {A, H, B, D, ...}. If any rejections seen at any point then M1 (max) only. Order of nodes may be seen at the top of the matrix {1, 3, -, 4, -, -, -, 2} so please check the top of the matrix carefully A1: First five arcs correctly chosen in order {AH, AB, DH, CD, CE, ...} or all eight nodes correctly chosen in order {A, H, B, D, C, E, G, F}. Order of nodes may be seen at the top of the matrix so for the first two marks accept {1, 3, 5, 4, 6, 8, 7, 2} (do not condone any missing numbers e.g. the number 8 must be above F) A1: cso – all arcs correct stated and chosen in the correct order. Candidates must be considering arcs for this final mark (do not accept a list of nodes or numbers across the top of the matrix unless the correct list of arcs (in the correct order) is also seen) (b) B1: cao (183) (c) B1: cao (for route – must return to J) B1: cao (for upper bound of 267) (d) M1: Their answer to (b) + 27 + 25 (the two smallest arcs incident to J) A1: cao (235)			

Qu	Scheme	Marks	AOs																																																						
7 (a)	$x + y, 8 \Rightarrow x + y + s_1 = 8$ $x + 2y, 12 \Rightarrow x + 2y + s_2 = 12$ $7x + 2y, 46 \Rightarrow 7x + 2y + s_3 = 46$ $y, 2x + 1 \Rightarrow -2x + y + s_4 = 1$ $P = x + ky \Rightarrow P - x - ky = 0$	M1 A1 B1	3.4 1.1b 1.1b																																																						
	e.g. <table><tr><th>b.v.</th><th>x</th><th>y</th><th>s_1</th><th>s_2</th><th>s_3</th><th>s_4</th><th>Value</th></tr><tr><td>s_1</td><td>1</td><td>1</td><td>1</td><td>0</td><td>0</td><td>0</td><td>8</td></tr><tr><td>s_2</td><td>1</td><td>2</td><td>0</td><td>1</td><td>0</td><td>0</td><td>12</td></tr><tr><td>s_3</td><td>7</td><td>2</td><td>0</td><td>0</td><td>1</td><td>0</td><td>46</td></tr><tr><td>s_4</td><td>-2</td><td>1</td><td>0</td><td>0</td><td>0</td><td>1</td><td>1</td></tr><tr><td>P</td><td>-1</td><td>-k</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr></table>	b.v.	x	y	s_1	s_2	s_3	s_4	Value	s_1	1	1	1	0	0	0	8	s_2	1	2	0	1	0	0	12	s_3	7	2	0	0	1	0	46	s_4	-2	1	0	0	0	1	1	P	-1	-k	0	0	0	0	0	M1 A1	3.3 2.2a						
b.v.	x	y	s_1	s_2	s_3	s_4	Value																																																		
s_1	1	1	1	0	0	0	8																																																		
s_2	1	2	0	1	0	0	12																																																		
s_3	7	2	0	0	1	0	46																																																		
s_4	-2	1	0	0	0	1	1																																																		
P	-1	-k	0	0	0	0	0																																																		
		(5)																																																							
(b)	$x = 2, y = 5, s_1 = 1, s_2 = 0, s_3 = 22, s_4 = 0$ ($P = 5k + 2$)	B1	3.4																																																						
		(1)																																																							
(c)	<table><tr><th>b.v.</th><th>x</th><th>y</th><th>s_1</th><th>s_2</th><th>s_3</th><th>s_4</th><th>Value</th><th>Row Ops</th></tr><tr><td>s_4</td><td>0</td><td>0</td><td>5</td><td>-3</td><td>0</td><td>1</td><td>5</td><td>5r1</td></tr><tr><td>x</td><td>1</td><td>0</td><td>2</td><td>-1</td><td>0</td><td>0</td><td>4</td><td>r2 + 0.4R1</td></tr><tr><td>s_3</td><td>0</td><td>0</td><td>-12</td><td>5</td><td>1</td><td>0</td><td>10</td><td>r3 - 2.4R1</td></tr><tr><td>y</td><td>0</td><td>1</td><td>-1</td><td>1</td><td>0</td><td>0</td><td>4</td><td>r4 - 0.2R1</td></tr><tr><td>P</td><td>0</td><td>0</td><td>2 - k</td><td>k - 1</td><td>0</td><td>0</td><td>4k + 4</td><td>r5 - (0.2k - 0.4)R1</td></tr></table>	b.v.	x	y	s_1	s_2	s_3	s_4	Value	Row Ops	s_4	0	0	5	-3	0	1	5	5r1	x	1	0	2	-1	0	0	4	r2 + 0.4R1	s_3	0	0	-12	5	1	0	10	r3 - 2.4R1	y	0	1	-1	1	0	0	4	r4 - 0.2R1	P	0	0	2 - k	k - 1	0	0	4k + 4	r5 - (0.2k - 0.4)R1	B1 M1 A1 A1 B1	1.1b 2.1 1.1b 1.1b 2.4
b.v.	x	y	s_1	s_2	s_3	s_4	Value	Row Ops																																																	
s_4	0	0	5	-3	0	1	5	5r1																																																	
x	1	0	2	-1	0	0	4	r2 + 0.4R1																																																	
s_3	0	0	-12	5	1	0	10	r3 - 2.4R1																																																	
y	0	1	-1	1	0	0	4	r4 - 0.2R1																																																	
P	0	0	2 - k	k - 1	0	0	4k + 4	r5 - (0.2k - 0.4)R1																																																	
	Optimal value of P is $4k + 4$ (at $x = y = 4$)	M1	3.4																																																						
	Second iteration not optimal $\Rightarrow -\frac{2}{5} + \frac{1}{5}k < 0 \therefore k < 2$	B1	3.1a																																																						
	Third iteration optimal $\Rightarrow 2 - k \dots 0$ and $k - 1 \dots 0$ ($\therefore k \dots 1$)	dM1	3.4																																																						
	1,, $k < 2 \Rightarrow 8$,, $P < 12$	A1	2.2a																																																						
		(9)																																																							
(15 marks)																																																									



Notes for Question 7

a1M1: Correctly re-writing any two inequalities as equations with slack variables (can be implied by two correct rows in the Simplex tableau ignoring b.v. column). Or correctly stating all four constraints as inequalities

a1A1: Correctly re-writing all inequalities as equations with slack variables (can be implied by the four correct constraint rows in the Simplex tableau ignoring b.v. column)

a1B1: Correctly re-writes objective function (can be implied by correct row in tableau)

a2M1: Any two rows correct including consistent b.v. column entries **or** any three rows correct (ignoring b.v. column)

a2A1: cao (including consistent b.v. column) – **note that the candidate's order in which the rows appear in the tableau (and choice of letter to represent the slack variable) may be different. A correct tableau implies full marks in this part**

b1B1: cao for x, y, s_1, s_2, s_3 and s_4 only (ignore any mention of P)

c1B1: Pivot row completely correct including change of b.v.

c1M1: All **values** in one of the non-pivot rows correct (so ignore b.v. column and 'Row Ops' column) **or** one of the 'non zero and one' columns (which are s_1, s_2 or Value) correct (must have pivoted on the correct value)

c1A1: Row operations used correctly at least twice, i.e. two of the 'non zero and one' columns (s_1, s_2 or Value) correct

c2A1: cao **all** values including b.v. column – ignore 'Row Ops' column for this mark

c2B1: Correct row operations stated

(alternatives row operations are $5r_1; r_2 + 2r_1; r_3 - 12r_1; r_4 - r_1; r_5 - (k - 2)r_1$)

c2M1: Their optimal value (as a linear expression in k) stated correctly following their third iteration (must have pivoted on a positive value from the s_4 column and completed the bottom row of the tableau). Condone this expression ($4k + 4$ **if** correct) being stated as part of an equation/inequality (or as part of their final answer) – sight of this expression (but must be seen **outside** of the tableau) scores this mark

c3B1: Correctly inferring that $k < 2$ either from $-\frac{2}{5} + \frac{1}{5}k < 0$ or from $4k + 4 > 5k + 2$ – just stating $k < 2$ without it being clear where this comes from is B0

c3dM1: Considering (at least) two of their linear expressions in k from their objective row (not including the Value column) after the third iteration ...0 (**dependent on the previous M mark**) – note that working may be minimal here so please follow through their expressions in k from their objective row (so **if** correct, stating $k \geq 2$ and $k \leq 1$ implies this mark)

c3A1: cao for the range of values for P - this mark is dependent on a correct objective row in the tableau and the previous three marks in this part



Qu	Scheme	Marks	AOs																																																		
6(a)	Simplex can only be applied when the non-negativity constraints are $\leq$	B1	3.5b																																																		
		(1)																																																			
(b)	$3x + y + 2z \leq 30 \Rightarrow 3x + y + 2z + s_1 = 30$	B1	1.1b																																																		
	$x - y + z \geq 8 \Rightarrow x - y + z - s_2 + a_1 = 8$	B1	2.5																																																		
	$4y + 2z \geq 15 \Rightarrow 4y + 2z - s_3 + a_2 = 15$	B1	1.1b																																																		
	$P = 2x + 2y - z \Rightarrow P = 2x + 2y - z - M(a_1 + a_2)$ together with $a_1 + a_2 = 23 - x - 3y - 3z + s_2 + s_3$	M1	2.1																																																		
	$P - (2 + M)x - (2 + 3M)y - (-1 + 3M)z + Ms_2 + Ms_3 = -23M$	A1	1.1b																																																		
	<table><tr><td>b.v</td><td>x</td><td>y</td><td>z</td><td>s₁</td><td>s₂</td><td>s₃</td><td>a₁</td><td>a₂</td><td>Value</td></tr><tr><td>s₁</td><td>3</td><td>1</td><td>2</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>30</td></tr><tr><td>a₁</td><td>1</td><td>-1</td><td>1</td><td>0</td><td>-1</td><td>0</td><td>1</td><td>0</td><td>8</td></tr><tr><td>a₂</td><td>0</td><td>4</td><td>2</td><td>0</td><td>0</td><td>-1</td><td>0</td><td>1</td><td>15</td></tr><tr><td>P</td><td>-(2 + M)</td><td>-(2 + 3M)</td><td>-(3M - 1)</td><td>0</td><td>M</td><td>M</td><td>0</td><td>0</td><td>-23M</td></tr></table>	b.v	x	y	z	s ₁	s ₂	s ₃	a ₁	a ₂	Value	s ₁	3	1	2	1	0	0	0	0	30	a ₁	1	-1	1	0	-1	0	1	0	8	a ₂	0	4	2	0	0	-1	0	1	15	P	-(2 + M)	-(2 + 3M)	-(3M - 1)	0	M	M	0	0	-23M	M1 A1	3.3 2.2a
b.v	x	y	z	s ₁	s ₂	s ₃	a ₁	a ₂	Value																																												
s ₁	3	1	2	1	0	0	0	0	30																																												
a ₁	1	-1	1	0	-1	0	1	0	8																																												
a ₂	0	4	2	0	0	-1	0	1	15																																												
P	-(2 + M)	-(2 + 3M)	-(3M - 1)	0	M	M	0	0	-23M																																												
		(7)																																																			
(c)	$s_1 = 26.25, a_1 = 11.75, y = 3.75, x = z = s_2 = s_3 = a_2 = 0$	B1	3.4																																																		
		(1)																																																			
(d)	The solution after the 1 st iteration is not feasible because $a_1 = 11.75$ is an artificial variable which must be zero in a feasible solution	B1	2.4																																																		
		(1)																																																			
(e)	The most negative value in the objective row is $2 - 1.5M$ so the pivot is a value from the z-column	B1	2.4																																																		
	The 0.5 in the y row is the pivot because $\frac{3.75}{0.5}$ is less than both $\frac{26.25}{1.5}$ and $\frac{11.75}{1.5}$	dB1	2.2a																																																		
		(2)																																																			
(12 marks)																																																					

Notes for Question 6

(a)

B1: CAO – e.g. not all of the constraints are $\leq$, the origin is not a (basic feasible) solution of the LP

(b)

B1: CAO $3x + y + 2z + s_1 = 30$ (may be seen in the simplex tableau – allow any s_i (or s) for s_1)**B1:** CAO $x - y + z - s_2 + a_1 = 8$ (may be seen in the simplex tableau – allow any consistent s_i for s_2 (or t say) but not the same s_i as in the previous mark and allow any a_i for a_1)**B1:** CAO $4y + 2z - s_3 + a_2 = 15$ (may be seen in the simplex tableau – same conditions as above)**M1:** setting up the new objective which must be $P = 2x + 2y - z - M(a_1 + a_2)$ and substituting for their a_1 and a_2 (if no working then the **correct** objective line in the tableau implies this mark)**A1:** CAO $P - (2 + M)x - (2 + 3M)y - (-1 + 3M)z + Ms_2 + Ms_3 = -23M$ (any equivalent form – need not be factorised and does not need to be re-arranged into this form - if no working then the **correct** objective line in the tableau implies this mark)**M1:** setting up initial tableau – all four rows complete with two correct rows (but ignore b.v. column for this mark)**A1:** CAO (any equivalent correct form)

(c)

B1: CAO $s_1 = 26.25, a_1 = 11.75, y = 3.75, x = z = s_2 = s_3 = a_2 = 0$ (ignore expression for P if given)

(d)

B1: correct reasoning of why the solution is not feasible e.g. a_1 is not zero but B0 for just stating that the artificial variable is non-zero (so must see either a_1 or 11.75 being stated as non-zero)

(e)

B1: correct reasoning of why the pivot comes from a value from the z -column so must say that the most negative value (in the objective row) is $2 - 1.5M$ (or this expression clearly implied)**dB1:** correct justification of why the 0.5 in the third row is the next pivot (dependent on previous B mark) – so must compare or state that $\frac{3.75}{0.5}$ or 7.5 is less than both $\frac{26.25}{1.5}$ or 17.5 and $\frac{11.75}{1.5}$ or 7.8(3333....) – just stating that the 0.5 in the third row is the next pivot without reasoning is no marks in this part

Ques tion	Scheme	Marks	AOs																																								
7.(a)	The pivot for this first iteration came from the x -column	B1	1.1b																																								
	as it is now a basic variable	dB1	2.5																																								
		(2)																																									
(b)	x -row: $4x + y - z \leq 40$ or s_3 - row: $y + z \leq 26$	B1	3.4																																								
	Eliminating s_2 from the s_1 row using the x row: $-\frac{1}{2}y + \frac{3}{2}z + s_1 - 2\left(10 - x - \frac{1}{4}y + \frac{1}{4}z\right) = 30$	M1	2.1																																								
	Eliminating s_2 from the P row using the x row: $P - \frac{1}{4}y - \frac{11}{4}z + 3\left(10 - x - \frac{1}{4}y + \frac{1}{4}z\right) = 30$	M1	1.1b																																								
	For either $P - 3x - y - 2z = 0$ or $2x + z + s_1 = 50$	A1	1.1b																																								
	Alternatively <table border="1"><thead><tr><th>b.v.</th><th>x</th><th>y</th><th>z</th><th>s_1</th><th>s_2</th><th>s_3</th><th>Value</th></tr></thead><tbody><tr><td>s_1</td><td>2</td><td>0</td><td>1</td><td>1</td><td>0</td><td>0</td><td>50</td></tr><tr><td>s_2</td><td>4</td><td>1</td><td>-1</td><td>0</td><td>1</td><td>0</td><td>40</td></tr><tr><td>s_3</td><td>0</td><td>1</td><td>1</td><td>0</td><td>0</td><td>1</td><td>26</td></tr><tr><td>P</td><td>-3</td><td>-1</td><td>-2</td><td>0</td><td>0</td><td>0</td><td>0</td></tr></tbody></table>	b.v.	x	y	z	s_1	s_2	s_3	Value	s_1	2	0	1	1	0	0	50	s_2	4	1	-1	0	1	0	40	s_3	0	1	1	0	0	1	26	P	-3	-1	-2	0	0	0	0		
b.v.	x	y	z	s_1	s_2	s_3	Value																																				
s_1	2	0	1	1	0	0	50																																				
s_2	4	1	-1	0	1	0	40																																				
s_3	0	1	1	0	0	1	26																																				
P	-3	-1	-2	0	0	0	0																																				
	LP: (Maximise) $P = 3x + y + 2z$ Subject to: $2x + z \leq 50$ $4x + y - z \leq 40$ $y + z \leq 26$ ($x, y, z \geq 0$)	A1	2.2a																																								
		(5)																																									

(c)	<table><tr><td>b.v.</td><td>x</td><td>y</td><td>z</td><td>s₁</td><td>s₂</td><td>s₃</td><td>Value</td><td>Row Ops</td></tr><tr><td>z</td><td>0</td><td>$-\frac{1}{3}$</td><td>1</td><td>$\frac{2}{3}$</td><td>$-\frac{1}{3}$</td><td>20</td><td>$\frac{2}{3}$r₁</td></tr><tr><td>x</td><td>1</td><td>$\frac{1}{6}$</td><td>0</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td>15</td><td>r₂ + 0.25R₁</td></tr><tr><td>s₃</td><td>0</td><td>$\frac{4}{3}$</td><td>0</td><td>$-\frac{2}{3}$</td><td>$\frac{1}{3}$</td><td>6</td><td>r₃– R₁</td></tr><tr><td>P</td><td>0</td><td>$-\frac{7}{6}$</td><td>0</td><td>$\frac{11}{6}$</td><td>$-\frac{1}{6}$</td><td>85</td><td>r₄ + 2.75R₁</td></tr></table>	b.v.	x	y	z	s ₁	s ₂	s ₃	Value	Row Ops	z	0	$-\frac{1}{3}$	1	$\frac{2}{3}$	$-\frac{1}{3}$	20	$\frac{2}{3}$ r ₁	x	1	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	15	r ₂ + 0.25R ₁	s ₃	0	$\frac{4}{3}$	0	$-\frac{2}{3}$	$\frac{1}{3}$	6	r ₃ – R ₁	P	0	$-\frac{7}{6}$	0	$\frac{11}{6}$	$-\frac{1}{6}$	85	r ₄ + 2.75R ₁	B1	1.1b
	b.v.	x	y	z	s ₁	s ₂	s ₃	Value	Row Ops																																			
	z	0	$-\frac{1}{3}$	1	$\frac{2}{3}$	$-\frac{1}{3}$	20	$\frac{2}{3}$ r ₁																																				
	x	1	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	15	r ₂ + 0.25R ₁																																				
	s ₃	0	$\frac{4}{3}$	0	$-\frac{2}{3}$	$\frac{1}{3}$	6	r ₃ – R ₁																																				
P	0	$-\frac{7}{6}$	0	$\frac{11}{6}$	$-\frac{1}{6}$	85	r ₄ + 2.75R ₁																																					
M1	2.1																																											
A1	1.1b																																											
B1	2.4																																											
		(4)																																										
(d)(i)	After the second iteration an optimal solution has not been found as the profit row still contains negative values	dB1	2.4																																									
(ii)	z = 20, x = 15, s ₃ = 6	dB1ft	2.2a																																									
		(2)																																										
(e)	e.g. Their attempt is not correct as, if y is now a basic variable, then the y column should contain only one value of 1 (in the third row) and therefore the entry of 1 in the profit row is incorrect (and should be 0)	B1	2.3																																									
		(1)																																										
(14 marks)																																												

Notes for Question 7

(a)

B1: CAO (correct column of x stated)**dB1:** **Dependent on first B1** Correct statement that x is now a basic variable (oe) Accept that x now appears in the first (bv) column or states x column has one 1 and rest 0 or s_2 row has been replaced with x

(b)

B1: Either the constraint for the x -row **or** s_3 - row correct (allow any equivalent form including non-integer coefficients but must be inequalities) Do not accept strict inequalities**M1:** Eliminating s_2 from the equation from the s_1 row using the equation from the x row Correct constraint implies this mark**M1:** Eliminating s_2 from the equation from the P row using the equation from the x row Correct objective implies this mark**A1:** $P - 3x - y - 2z = 0$ **or** $2x + z + s_1 = 50$ (allow any equivalent form including non-integer coefficients but must have been simplified to a single term in each variable) Correct constraint or objective implies the corresponding M mark**A1:** Correct LP formulation (condone lack of 'maximise' and condone lack of the non-negative trivial constraints)**Note: It is possible to score M1 M0 A1 A0 or M0 M1 A1 A0**

Alternatively – reproduces original tableau

B1: s_2 row correct**M1:** s_1 row correct - Correct constraint implies this mark**M1:** P row correct - Correct objective implies this mark**A1:** fully correct tableau**A1:** Correct LP formulation (condone lack of 'maximise' and condone lack of the non-negative trivial constraints)(c) **Note – accept correct recurring decimals in place of fractions****B1:** Pivot row completely correct including change of b.v.**M1:** All **values** in one of the non-pivot rows correct (so ignore b.v. column and 'Row Ops' column) **or** one of the 'non zero and one' columns (which are y , s_1 , s_2 or Value) correct (must have pivoted on the correct value)**A1:** **cao all** values including b.v. column – ignore 'Row Ops' column for this mark**B1:** Correct row operations stated (allow alternative numbering of rows as long as this is clear. Condone use of R_1 throughout)

(d)(i)

dB1: CAO (profit row still contains negative values (but not negative variables) – dependent on the M mark in (c) and a completed profit row in (c) (accept P row or objective row but not operations row or bottom row) Do not accept that not all values are positive**dB1ft:** Follow through their values of z , x , and s_3 – dependent on the M mark in (c) and a completed tableau in (c) ignore mention of any other variables including P **All values must be positive**

(e)

B1: CAO (e.g. correct indication that the (basic variable) column for y is not correct – see bold statement for the minimum acceptable) Accept there should not be a 1 in the objective row in the y column or that there should not be two 1s in the y column. If s_2 column also mentioned then B0