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EDEXCEL A LEVEL FURTHER MATHEMATICS: QUESTIONS BY TOPIC

Further Trigonometry (1)

Further Pure Mathematics 1 · Topic 1

5 questions · 49 marks · Mark schemes included

Pearson Edexcel A Level Further Mathematics (9FM0), June 2019 - June 2025

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HOW TO USE THIS BOOKLET

Questions are grouped by specification sub-topic and run from the lowest mark tariff to the highest within each one.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme.

Diagrams, graphs, tables and data are printed exactly as they appeared in the original papers.

QUESTION	SOURCE	MARKS	PAGE
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QUESTIONS

5 questions · 49 marks · 3 sub-topics



SUB-TOPIC 1.1

THE T-FORMULAE

Sub-topic 1.1 · 3 questions · 23 marks · lowest tariff first



7.

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Use the substitution $t = \tan\left(\frac{\theta}{2}\right)$ to show that

$$\int \frac{1}{2\sin\theta + \cos\theta + 2} d\theta = \int \frac{a}{(t+b)^2 + c} dt$$

where a , b and c are constants to be determined.

(3)

(b) Hence show that

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{2\sin\theta + \cos\theta + 2} d\theta = \ln\left(\frac{2\sqrt{3}}{3}\right)$$

(4)



Question 7 continued

(Total for Question 7 is 7 marks)



5.

$$I = \int \frac{1}{4 \cos x - 3 \sin x} dx \quad 0 < x < \frac{\pi}{4}$$

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$I = \frac{1}{5} \ln \left(\frac{2 + \tan\left(\frac{x}{2}\right)}{1 - 2 \tan\left(\frac{x}{2}\right)} \right) + k$$

where k is an arbitrary constant.

(8)









SUB-TOPIC 1.2

TRIGONOMETRIC IDENTITIES USING T

Sub-topic 1.2 · 1 question · 10 marks · lowest tariff first



2. (i) Use the substitution $t = \tan \frac{x}{2}$ to prove the identity

$$\frac{\sin x - \cos x + 1}{\sin x + \cos x - 1} \equiv \sec x + \tan x \quad x \neq \frac{n\pi}{2} \quad n \in \mathbb{Z}$$

(5)

(ii) Use the substitution $t = \tan \frac{\theta}{2}$ to determine the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{5}{4 + 2 \cos \theta} \, d\theta$$

giving your answer in simplest form.

(5)

Question 2 continued



Question 2 continued

Question 2 continued

(Total for Question 2 is 10 marks)



SUB-TOPIC 1.3

SOLVING EQUATIONS USING T

Sub-topic 1.3 · 1 question · 16 marks · lowest tariff first







MARK SCHEMES

official mark schemes



Question	Scheme	Marks	AOs
7(a)	$\int \frac{1}{2 \sin \theta + \cos \theta + 2} d\theta = \int \frac{1}{2\left(\frac{2t}{1+t^2}\right) + \left(\frac{1-t^2}{1+t^2}\right) + 2} \times \frac{2}{1+t^2} dt$	M1	2.1
	$\int \frac{2}{4t + (1-t^2) + 2(1+t^2)} dt = \int \frac{2}{t^2 + 4t + 3} dt$	M1	1.1b
	$\int \frac{2}{(t+2)^2 - 1} dt$	A1	2.2a
		(3)	
(b)	$\int \frac{a}{(t+b)^2 - c} dt = \frac{a}{2\sqrt{c}} \ln \left \frac{(t+b) - \sqrt{c}}{(t+b) + \sqrt{c}} \right , c > 0 \text{ or}$ $\int \frac{a}{(t+b)^2 + c} dt = \frac{a}{\sqrt{c}} \arctan \left(\frac{t+b}{\sqrt{c}} \right), c > 0$ Note correct answer is $\int \frac{2}{(t+2)^2 - 1} dt = \frac{2}{2(1)} \ln \left \frac{(t+2) - 1}{(t+2) + 1} \right$	M1	3.1a
	Correctly uses the limits $t=1$ and $t=\sqrt{3} \rightarrow \ln \left \frac{\sqrt{3}+1}{\sqrt{3}+3} \right - \ln \left \frac{2}{4} \right $ if correct.	M1	1.1b
	$= \ln \left(\frac{\sqrt{3}+1}{\sqrt{3}+3} \div \frac{1}{2} \right)$	M1	1.1b
	$= \ln \left(2 \times \frac{\sqrt{3}+1}{\sqrt{3}+3} \times \frac{\sqrt{3}-3}{\sqrt{3}-3} \right) = \ln \left(\frac{2\sqrt{3}}{3} \right)^*$	A1*	2.1
		(4)	
(7 marks)			
Notes:			
(a) M1: Applies the correct substitutions $\sin \theta = \frac{2t}{1+t^2}$, $\cos \theta = \frac{1-t^2}{1+t^2}$ and $d\theta = \frac{2}{1+t^2} dt$ The dt may be missing for this and the next M. M1: Simplifies the integrand to the form $\frac{a}{pt^2 + qt + r}$ allowing slips in coefficients but the “$1+t^2$” must be correctly dealt with. A1: Correct answer in the form required, including the $\int dt$			

(b)

M1: A correct method for the integration, by use of standard formula. Modulus signs are not necessary. Note that $-\frac{a}{\sqrt{c}} \operatorname{artanh}\left(\frac{t+b}{\sqrt{c}}\right) = -\frac{a}{2\sqrt{c}} \ln \left| \frac{t+b+\sqrt{c}}{t+b-\sqrt{c}} \right|$ is also a correct version. If by error they end up with a positive c then accept for the arctan integral shown in scheme.

M1: Correctly uses the limits $t=1$ and $t=\sqrt{3}$ in their integral to produce an exact expression. Use of the original limits is M0, must have changed to t (unless they reverse the substitution).

M1: Correctly combines the $\ln$ terms with at least one step showing the result of substituting limits before the final answer, and their final answer must follow this step if no further working is shown. May be seen before or after the rationalisation. (M0 if no $\ln$ terms.)

A1*cs0: Shows the method to rationalise the denominator and correctly completes to the given answer. Some suitable working must be seen. Note that $\sqrt{3}+3 = \sqrt{3}(1+\sqrt{3})$ may be used in the denominator, followed by cancelling.

(b) Alt	$\int \frac{2}{(t+2)^2 - 1} dt = \int \frac{2}{(t+1)(t+3)} dt$ $= \int \frac{1}{t+1} - \frac{1}{t+3} dt = \ln t+1 - \ln t+3 $	M1	3.1a
	Correctly uses the limits $t=1$ and $t=\sqrt{3}$ $\ln(\sqrt{3}+1) - \ln(\sqrt{3}+3) - \ln 2 + \ln 4$	M1	1.1b
	$= \ln \left(2 \times \frac{\sqrt{3}+1}{\sqrt{3}+3} \right)$	M1	2.1
	$= \ln \left(2 \times \frac{\sqrt{3}+1}{\sqrt{3}(1+\sqrt{3})} \right) = \ln \left(\frac{2\sqrt{3}}{3} \right)^*$	A1*	1.1b
		(4)	

(7 marks)**Notes:****(b)**

M1: A correct method for the integration, applying partial fractions and integrating to $\ln$ terms.

M1: Correctly uses the limits $t=1$ and $t=\sqrt{3}$ (or correct θ limits if substitution is reversed) to produce an exact expression.

M1: Correctly combines the $\ln$ terms with at least one step showing the result of substituting limits before the final answer, and their final answer must follow this step if no further working is shown. May be seen before or after the rationalisation. (M0 if no $\ln$ terms.)

A1*cs0: Shows the method to rationalise the denominator and correctly completes to the given answer. Some suitable working must be seen. Note that $\sqrt{3}+3 = \sqrt{3}(1+\sqrt{3})$ may be used in the denominator, followed by cancelling.



Question	Scheme	Marks	AOs
5	$4\cos x - 3\sin x = 4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)$	B1	1.1a
	$\frac{dt}{dx} = \frac{1+t^2}{2}$ or $\frac{dx}{dt} = \frac{2}{1+t^2}$ or $dx = \frac{2dt}{1+t^2}$ or $dt = \frac{1+t^2}{2}dx$ oe	B1 M1 on ePEN	2.1
	$\int \frac{1}{4\cos x - 3\sin x} dx = \int \frac{1}{4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$	M1	2.1
	$= \int \frac{2}{4-4t^2-6t} (dt)$ or $\int \frac{1}{2-2t^2-3t} (dt)$ or $\int \frac{-1}{2t^2+3t-2} (dt)$ etc.	A1	1.1b
	$\frac{-2}{4t^2+6t-4} = \frac{-1}{(t+2)(2t-1)} = \frac{A}{(t+2)} + \frac{B}{(2t-1)}$ $\frac{-1}{(t+2)(2t-1)} = \frac{1}{5(t+2)} + \frac{2}{5(1-2t)}$	M1	3.1a
	$\Rightarrow I = \frac{1}{5} \int \frac{1}{(t+2)} - \frac{2}{(2t-1)} (dt)$ or equivalent	A1	1.1b
	$= \frac{1}{5} \int \frac{1}{(t+2)} - \frac{2}{(2t-1)} dt = \frac{1}{5} \ln(t+2) - \frac{1}{5} \ln(1-2t) (+k)$	A1	1.1b
	$= \frac{1}{5} \ln\left(\frac{2+t}{1-2t}\right) (+k) = \frac{1}{5} \ln\left(\frac{2+\tan(\frac{x}{2})}{1-2\tan(\frac{x}{2})}\right) + k^*$	A1*	2.1
		(8)	
Alternative for final 4 marks:			
	$= \int \frac{2}{4-4t^2-6t} (dt) = -\frac{1}{2} \int \frac{1}{t^2 + \frac{3}{2}t - 1} (dt) = -\frac{1}{2} \int \frac{1}{(t+\frac{3}{4})^2 - \frac{25}{16}} (dt)$ or e.g. $\int \frac{1}{\frac{25}{8} - 2(t+\frac{3}{4})^2} (dt)$	M1 A1	3.1a 1.1b
	$-\frac{1}{2} \times \frac{1}{2} \times \frac{4}{5} \ln\left(\frac{t+\frac{3}{4}-\frac{5}{4}}{t+\frac{3}{4}+\frac{5}{4}}\right) (+c)$	A1	1.1b
	$-\frac{1}{5} \ln\left \frac{\tan(\frac{x}{2})-\frac{1}{2}}{\tan(\frac{x}{2})+2}\right + c = \frac{1}{5} \ln\left \frac{\tan(\frac{x}{2})+2}{\tan(\frac{x}{2})-\frac{1}{2}}\right + c = \frac{1}{5} \ln\left(\frac{\tan(\frac{x}{2})+2}{\frac{1}{2}-\tan(\frac{x}{2})}\right) + c$ $= \frac{1}{5} \ln\left(\frac{2(\tan(\frac{x}{2})+2)}{1-2\tan(\frac{x}{2})}\right) + c = \frac{1}{5} \ln\left(\frac{(\tan(\frac{x}{2})+2)}{1-2\tan(\frac{x}{2})}\right) + \frac{1}{5} \ln 2 + c$ $= \frac{1}{5} \ln\left(\frac{(\tan(\frac{x}{2})+2)}{1-2\tan(\frac{x}{2})}\right) + k$	A1*	2.1
(8 marks)			

Notes

B1: Uses the **correct** formulae to express $4\cos x - 3\sin x$ in terms of t

B1(M1 on ePEN): Correct equation in terms of dx , dt and t – can be implied if seen as part of their substitution.

M1: Makes a **complete** substitution to obtain an integral in terms of t only. Allow slips with the substitution of “ dx ” but must be $dx = f(t)dt$ where $f(t) \neq 1$. This mark is also available if the

candidate makes errors when attempting to simplify $4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)$ before attempting the substitution.

A1: For obtaining a fully correct simplified integral with a constant in the numerator and a 3 term quadratic expression in the denominator. (“ dt ” not required)

M1: Realises the need to express the integrand in terms of partial fractions in order to attempt the integration. **Must have a 3 term quadratic expression in the denominator and a constant in the numerator.**

A1: Correct integral in terms of partial fractions – allow any equivalent **correct** integral. (“ dt ” not required)

A1: Fully correct integration in terms of t

A1*: Correct solution with no errors including “ $+k$ ” (allow “ $+c$ ”) and with the constant dealt with correctly if necessary. The denominator must also be dealt with correctly. E.g. if it appears as $2t - 1$ initially and becomes $1 - 2t$ without justification, this final mark should be withheld.

Alternative for final 4 marks:

M1: Realises the need to express the integrand in completed square form in order to attempt the integration. **Must have a 3 term quadratic expression in the denominator and a constant in the numerator.**

A1: Correct integral with the square completed – allow any equivalent **correct** integral (“ dt ” not required)

A1: Fully correct integration in terms of t

A1*: Correct solution with no errors including “ $+k$ ” (allow “ $+c$ ”) and with the constant dealt with correctly if necessary as shown in the scheme and with the denominator dealt with correctly if necessary.

Note that it is acceptable for the “ dt ” to appear and disappear throughout the proof as long as the intention is clear.



Question	Scheme	Marks	AOs
5(a)	$1 + \operatorname{cosec} x = 1 + \frac{1}{\sin x} = 1 + \frac{1}{\frac{2t}{1+t^2}} \text{ or } 1 + \frac{1+t^2}{2t} \left(= \frac{t^2 + 2t + 1}{2t} \right)$ or $1 + \operatorname{cosec} x = \frac{\sin x + 1}{\sin x} = \frac{\frac{2t}{1+t^2} + 1}{\frac{2t}{1+t^2}}$	B1	1.1a
	$\int \frac{1}{1 + \operatorname{cosec} x} dx = \int \frac{2t}{t^2 + 2t + 1} \times \frac{2}{1+t^2} dt$	M1	2.1
	$\int \frac{1}{1 + \operatorname{cosec} x} dx = \int \frac{4t}{(1+t^2)(1+t)^2} dt^*$	A1*	1.1b
		(3)	
(b)	$\frac{4t}{(1+t^2)(1+t)^2} \equiv \frac{At+B}{1+t^2} + \frac{C}{1+t} + \frac{D}{(1+t)^2}$ or $\frac{4t}{(1+t^2)(1+t)^2} \equiv \frac{At+B}{1+t^2} + \frac{Ct+D}{(1+t)^2}$ or $\frac{4t}{(1+t^2)(1+t)^2} \equiv \frac{At+B}{1+t^2} + \frac{C}{1+t} + \frac{Dt+E}{(1+t)^2}$ Leading to finding the value for at least one constant	M1	3.1a
	$\frac{4t}{(1+t^2)(1+t)^2} \equiv \frac{2}{1+t^2} - \frac{2}{(1+t)^2}$	A1	1.1b
	$\int \frac{2}{1+t^2} - \frac{2}{(1+t)^2} dt = 2 \arctan t + \frac{2}{1+t}$	A1	2.2a
	$= 2 \arctan\left(\tan \frac{x}{2}\right) + \frac{2}{1 + \tan \frac{x}{2}} + k = x + \frac{2}{1 + \tan \frac{x}{2}} + k \text{ cso}$	M1 A1	1.1b 2.1
		(5)	
(8 marks)			



Notes

(a)

B1: Selects the correct formulae to express $1 + \operatorname{cosec} x$ in terms of t . May be seen within the integral

M1: Makes a complete substitution to obtain an integral in terms of t only, including

$$dx = \frac{2}{1+t^2} dt$$

A1*: Correct proof with no errors or omissions, must see the factorisation if the denominator is expanded first. Long division or equivalent with $t^2 + 1$ or $(t+1)^2$ or $t+1$

(b)

M1: Realises the need to express the integrand in terms of partial fractions and attempts the correct form, condone the use of x instead of t on their numerator with t 's on the denominator. Must attempt to find at least one constant term

A1: Correct partial fractions.

A1: Correct integration, must have scored the previous marks, no need for the constant of integration

Note an incorrect form such as $\frac{A}{1+t^2} + \frac{B}{1+t} + \frac{C}{(1+t)^2}$ can lead to a correct answer but is

M0A0A0

M1: Reverses the substitution to express in terms of x

A1: Correct answer with no errors including “+ k ” must have scored all the previous marks in part (b)

Note Verification method using $\frac{A}{1+t^2} + \frac{B}{(1+t)^2}$ to find the values for A and B , please send to review



Question	Scheme	Marks	AOs
2(i)	$\frac{\sin x - \cos x + 1}{\sin x + \cos x - 1} = \frac{\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} + 1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} - 1} = \dots$	M1	1.1b
	$= \frac{2t - (1-t^2) + 1 + t^2}{2t + 1 - t^2 - (1+t^2)} \text{ or } = \frac{2t - 1 + t^2 + 1 + t^2}{2t + 1 - t^2 - 1 - t^2}$ $\text{numerator} = \frac{2t - (1-t^2) + 1 + t^2}{1+t^2} \quad \text{denominator} = \frac{2t + 1 - t^2 - (1+t^2)}{1+t^2}$ and divides	M1	2.1
	$= \frac{2t^2 + 2t}{2t - 2t^2} \left(= \frac{1+t}{1-t} \right)$	A1	1.1b
	$= \frac{t+1}{1-t} \times \frac{1+t}{1+t} = \frac{t^2 + 2t + 1}{1-t^2} = \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}$ Alt: $\sec x + \tan x = \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2} = \frac{1+2t+t^2}{1-t^2}$	M1	3.1a
	$= \frac{1}{\cos x} + \tan x = \sec x + \tan x^*$ Alt: $= \frac{(t+1)^2}{(1-t)(1+t)} = \frac{1+t}{1-t} = LHS \text{ hence result proved.}^*$	A1*	2.1
		(5)	
(ii)	$\int_{(0)}^{\left(\frac{\pi}{2}\right)} \frac{5}{4+2\cos\theta} d\theta = \int_{(0)}^{(1)} \frac{5}{4+2\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2} dt$ Alternatively $\frac{dt}{dq} = \frac{1}{2} \sec^2 \frac{q}{2} = \frac{1}{\cos q + 1} \text{ leading to}$ $\int_{(0)}^{\left(\frac{\pi}{2}\right)} \frac{5}{4+2\cos\theta} d\theta = \int_{(0)}^{(1)} \frac{5\cos\theta + 5}{4+2\cos\theta} dt = \int_{(0)}^{(1)} \frac{5\left(\frac{1-t^2}{1+t^2}\right) + 5}{4+2\left(\frac{1-t^2}{1+t^2}\right)} dt$	M1	2.1
	$= \int_{(0)}^{(1)} \frac{10}{4(1+t^2) + 2(1-t^2)} dt = \int_{(0)}^{(1)} \frac{5}{3+t^2} dt \text{ o.e.}$	A1	1.1b
	$= \left[5 \times \frac{1}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right) \right]_{(0)}^{(1)}$	M1	1.1b

	$= \frac{5}{\sqrt{3}} \left(\arctan\left(\frac{1}{\sqrt{3}}\right) - 0 \right) \text{ or } \frac{5}{\sqrt{3}} \left(\arctan\left(\frac{\tan\left(\frac{\pi}{4}\right)}{\sqrt{3}}\right) - \arctan\left(\frac{\tan(0)}{\sqrt{3}}\right) \right)$	M1	2.2a
	$= \frac{5\pi\sqrt{3}}{18} \text{ oe in a surd form e.g. } \frac{5\pi}{6\sqrt{3}}$	A1	1.1b
		(5)	
(10 marks)			
Notes:			
(i) M1: Applies the t-formulae to the left-hand side of expression. Allow slips in signs of the terms. M1: Multiplies numerator and denominator through by $1+t^2$ (allow if they forget to multiply the 1's). Alternative works separately on the numerator and denominator to combine terms and then divides. A1: Correct $\frac{\text{quadratic}}{\text{quadratic}}$ with terms gathered, award where first seen, need not have cancelled $2t$ for this mark. M1: Cancels $2t$, multiplies numerator and denominator by $1+t$ and splits to sum of two terms. If working from both sides, this mark is for substituting the t-formulae into the right-hand side and combining to single fraction. A1*: Correct completion to given result. No errors in proof. If working from both sides, a suitable conclusion is needed, e.g. "hence proven".			
(ii) M1: Applies the substitution including the use of $d\theta = \frac{2}{1+t^2} dt$ (Limits not needed for first three marks). A1: Simplifies correctly to a recognisable integrable form. M1: Integrates to the form $K \arctan\left(\frac{t}{a}\right)$ where a^2 is their constant term. M1: Deduces correct limits and applies them the correct way round OR deduces integral in terms of θ from their integration and applies original limits the correct way. A1: $\frac{5\pi\sqrt{3}}{18}$ or equivalent in surd form. Note use of calculator does not lead to the exact value required in the question = 1.51149947 This can score M1 A1 M0 M1 A0			

Question	Scheme	Marks	AOs	
8(a)	$f(x) = \frac{3}{13+6 \times \frac{2t}{1+t^2} - 5 \times \frac{1-t^2}{1+t^2}}$	M1	1.1b	
	$= \frac{3(1+t^2)}{13(1+t^2)+12t-5(1-t^2)}$	M1	1.1b	
	$= \frac{3(1+t^2)}{18t^2+12t+8} \Rightarrow \text{for example } \frac{3(1+t^2)}{2(9t^2+6t+1)+6} \text{ or } \frac{3(1+t^2)}{2[(3t+1)^2-1]+8}$ $\Rightarrow \frac{3(1+t^2)}{2(3t+1)^2+6}^*$	A1*	2.1	
		(3)		
(b)	$f(x) = \frac{3}{7} \Rightarrow \frac{3(1+t^2)}{2(3t+1)^2+6} = \frac{3}{7} \Rightarrow 21+21t^2 = 54t^2+36t+24$ $\Rightarrow 11t^2+12t+1=0$	M1	1.1b	
	$\Rightarrow (11t+1)(t+1)=0 \Rightarrow t=...$	M1	1.1b	
	$t=-1, t=-\frac{1}{11}$	A1	1.1b	
	$\Rightarrow x=2 \arctan ("their t") + 2\pi \text{ for a negative } t$	dM1	3.1a	
	$x = \frac{3\pi}{2} \text{ or awrt } 4.71 \text{ and awrt } x = 6.10$	A1	1.1b	
		(5)		
(c)	$\int f(x) = \int \frac{3(1+t^2)}{2(3t+1)^2+6} \times \frac{2}{1+t^2} dt = \int \frac{3}{(3t+1)^2+3} dt$	B1	2.1	
	$= K \arctan (M(3t+1))$	$u = (3t+1) \Rightarrow K \arctan (Mu)$	M1	1.1b
	$= \frac{1}{\sqrt{3}} \arctan \left(\frac{3t+1}{\sqrt{3}} \right)$	$= \frac{1}{\sqrt{3}} \arctan \left(\frac{u}{\sqrt{3}} \right)$	A1	1.1b
	$\int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} f(x) dx = \int_{\frac{\pi}{3}}^{\pi} f(x) dx + \int_{\pi}^{\frac{4\pi}{3}} f(x) dx$ $= \int_{\frac{\sqrt{3}}{3}}^{\infty} ... dt + \int_{-\infty}^{-\sqrt{3}} ... dt \text{ or } \int_{\sqrt{3}+1}^{\infty} ... du + \int_{-\infty}^{1-3\sqrt{3}} ... du$		B1	3.1a
	$= \frac{1}{\sqrt{3}} \arctan \left(\frac{3(-\sqrt{3})+1}{\sqrt{3}} \right) - \frac{1}{\sqrt{3}} \arctan \left(\frac{3\left(\frac{\sqrt{3}}{3}\right)+1}{\sqrt{3}} \right) + ...$		M1	1.1b

$= \frac{\sqrt{3}}{3} \left(\arctan \left(\frac{\sqrt{3}-9}{3} \right) - \arctan \left(\frac{\sqrt{3}+3}{3} \right) \right) + \dots$	A1	1.1b
$= \dots + \lim_{t \rightarrow \infty} \frac{1}{\sqrt{3}} \arctan \left(\frac{3t+1}{\sqrt{3}} \right) - \lim_{t \rightarrow -\infty} \frac{1}{\sqrt{3}} \arctan \left(\frac{3t+1}{\sqrt{3}} \right) = \dots + \frac{\pi}{2\sqrt{3}} - \left(-\frac{\pi}{2\sqrt{3}} \right)$	M1	3.1a
$= \frac{\sqrt{3}}{3} \left(\arctan \left(\frac{\sqrt{3}-9}{3} \right) - \arctan \left(\frac{\sqrt{3}+3}{3} \right) + \pi \right)$	A1	2.1
	(8)	

(16 marks)

Notes:
(a)
M1: Uses one correct substitution

M1: Both substitutions correct and attempts to multiply through numerator and denominator by $1+t^2$.

A1*: Completes to the correct expression with no errors seen. Must see an intermediate step simplifying the denominator – most likely one of the ones seen in the scheme.

(b)
M1: Equates the result in (a) to $\frac{3}{7}$ and simplifies to a 3TQ

M1: Solves their equation by any valid means.

A1: Correct values for t
dm1: Dependent on first method mark. Applies the correct process to find at least one value for x from a negative value for t . (If two positive values are found in error, this mark cannot be scored.)

A1: Both answers correct and no others in range.

(c)
B1: Applies the substitution including the use of $dx = \frac{2}{1+t^2} dt$
M1: Attempts the integration to achieve $K \arctan(M(1+3t))$ or $K \arctan(Mu)$ if using a substitution of $u = (3t+1)$.

 May use substitution $3t+1 = \sqrt{3} \tan q \Rightarrow \frac{dt}{dq} = \frac{\sqrt{3}}{3} \sec^2 q \Rightarrow \frac{2}{3} \frac{\sqrt{3}}{3} dq = \frac{\sqrt{3}}{3} dq = \frac{\sqrt{3}}{3} \arctan \frac{3t+1}{\sqrt{3}}$
A1: Correct integral.

B1: Changes the limits and splits the integral around π
M1: Applies their limits ' $\frac{1}{\sqrt{3}}$ ' and ' $-\sqrt{3}$ ' to their integrand.

A1: Correct "arctan" expressions.

M1: Correct work to evaluate the $\pm\infty$ limits

A1: Fully correct solution.
