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EDEXCEL A LEVEL FURTHER MATHEMATICS: QUESTIONS BY TOPIC

## Further Calculus (2)

### Further Pure Mathematics 1 · Topic 2

8 questions · 77 marks · Mark schemes included

*Pearson Edexcel A Level Further Mathematics (9FM0), June 2019 - June 2025*

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**HOW TO USE THIS BOOKLET**

Questions are grouped by specification sub-topic and run from the lowest mark tariff to the highest within each one.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme.

Diagrams, graphs, tables and data are printed exactly as they appeared in the original papers.

| QUESTION                                           | SOURCE                                             | MARKS           | PAGE      |
|----------------------------------------------------|----------------------------------------------------|-----------------|-----------|
| <b>2.1 TAYLOR SERIES</b>                           |                                                    | 4 Qs · 42 marks | <b>3</b>  |
| 1. <b>Q4</b> <i>also Topic 3</i>                   | Option 3A: Further Pure Mathematics 1 June2024     | 8               | <b>4</b>  |
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| <b>2.2 SERIES EXPANSIONS AND LIMITS</b>            |                                                    | 1 Qs · 8 marks  | <b>20</b> |
| 5. <b>Q4</b>                                       | Option 3A: Further Pure Mathematics 1 June2025     | 8               | <b>21</b> |
| <b>2.3 LEIBNITZ'S THEOREM</b>                      |                                                    | 2 Qs · 19 marks | <b>25</b> |
| 6. <b>Q4</b>                                       | Option 3A: Further Pure Mathematics 1 October2020  | 8               | <b>26</b> |
| 7. <b>Q6</b>                                       | Option 3A: Further Pure Mathematics 1 June2025     | 11              | <b>30</b> |
| <b>2.5 WEIERSTRASS SUBSTITUTION IN INTEGRATION</b> |                                                    | 1 Qs · 8 marks  | <b>34</b> |
| 8. <b>Q5</b> <i>also Topic 1</i>                   | Option 3A: Further Pure Mathematics 1 June2025     | 8               | <b>35</b> |



# QUESTIONS

8 questions · 77 marks · 4 sub-topics



SUB-TOPIC 2.1

# TAYLOR SERIES

Sub-topic 2.1 · 4 questions · 42 marks · lowest tariff first









**(Total for Question 4 is 8 marks)**



**8.**

$$\left[ \begin{array}{l} \text{The Taylor series expansion of } f(x) \text{ about } x = a \text{ is given by} \\ f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \dots \end{array} \right]$$

- (i) (a) Use differentiation to determine the Taylor series expansion of  $\ln x$ , in ascending powers of  $(x - 1)$ , up to and including the term in  $(x - 1)^2$

(4)

- (b) Hence prove that

$$\lim_{x \rightarrow 1} \left( \frac{\ln x}{x - 1} \right) = 1$$

(2)

- (ii) Use L'Hospital's rule to determine

$$\lim_{x \rightarrow 0} \left( \frac{1}{(x+3)\tan(6x)\operatorname{cosec}(2x)} \right)$$

(Solutions relying entirely on calculator technology are not acceptable.)

(4)







**(Total for Question 8 is 10 marks)**









**(Total for Question 6 is 12 marks)**









**(Total for Question 6 is 12 marks)**

SUB-TOPIC 2.2

# SERIES EXPANSIONS AND LIMITS

Sub-topic 2.2 · 1 question · 8 marks · lowest tariff first



4.

$$y = \sqrt{(15 + e^{2x})}$$

(a) Show that

$$\frac{d^2y}{dx^2} = \frac{e^{2x}(e^{2x} + 30)}{(15 + e^{2x})^{\frac{3}{2}}} \quad (3)$$

(b) Hence determine the Maclaurin series expansion for  $y$ , in ascending powers of  $x$ , up to and including the term in  $x^2$ , giving each term in simplest form.

(2)

(c) Use the series expansion for  $\sin x$  in ascending powers of  $x$  to determine, in simplest form, the first 2 non-zero terms in ascending powers of  $x$  of the series for  $\sin 3x$ .

(1)

(d) Use the answers to parts (b) and (c) to show that

$$\lim_{x \rightarrow 0} \frac{\sqrt{(15 + e^{2x})} - 4}{\sin 3x} = \frac{1}{12} \quad (2)$$





**Question 4 continued**





SUB-TOPIC 2.3

# LEIBNITZ'S THEOREM

Sub-topic 2.3 · 2 questions · 19 marks · lowest tariff first













**Question 6 continued**



**Question 6 continued**





## SUB-TOPIC 2.5

# WEIERSTRASS SUBSTITUTION IN INTEGRATION

Sub-topic 2.5 · 1 question · 8 marks · lowest tariff first



5. (a) Use the substitution  $t = \tan\left(\frac{x}{2}\right)$  to show that

$$\int \frac{1}{1 + \operatorname{cosec} x} dx = \int \frac{4t}{(1 + t^2)(1 + t)^2} dt \quad (3)$$

(b) Hence show that

$$\int \frac{1}{1 + \operatorname{cosec} x} dx = Ax + \frac{B}{1 + \tan\left(\frac{x}{2}\right)} + k$$

where  $A$  and  $B$  are constants to be determined and  $k$  is an arbitrary constant.

(5)

**Question 5 continued**



**Question 5 continued**



**Question 5 continued**

# MARK SCHEMES

official mark schemes



| Question         | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                   | Marks      | AOs  |
|------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|------|
| <b>4(a)</b>      | $\cos\left(\frac{\pi}{4}\right)\frac{d^2y}{dx^2} + (1)^2(0) + \sin\left(\frac{\pi}{4}\right) = 0 \Rightarrow \frac{d^2y}{dx^2} = \dots$                                                                                                                                                                                                                                                                                  | <b>M1</b>  | 3.1a |
|                  | $\frac{d^2y}{dx^2} = -1 < 0$ therefore a (local) maximum                                                                                                                                                                                                                                                                                                                                                                 | <b>A1</b>  | 2.4  |
|                  |                                                                                                                                                                                                                                                                                                                                                                                                                          | <b>(2)</b> |      |
| <b>(b)</b>       | Achieves $\pm \sin x \frac{d^2y}{dx^2} + \cos x \frac{d^3y}{dx^3} + \alpha y \left(\frac{dy}{dx}\right)^{1 \text{ or } 2} + y^2 \frac{d^2y}{dx^2} \pm \cos x = 0$ or<br>$\frac{d^2y}{dx^2} = -\tan x - y^2 \sec x \frac{dy}{dx} \rightarrow$<br>$\frac{d^3y}{dx^3} = \pm \sec^2 x \pm \alpha y \sec x \left(\frac{dy}{dx}\right)^{1 \text{ or } 2} \pm y^2 \sec x \tan x \frac{dy}{dx} \pm y^2 \sec x \frac{d^2y}{dx^2}$ | <b>M1</b>  | 1.1b |
|                  | $-\sin x \frac{d^2y}{dx^2} + \cos x \frac{d^3y}{dx^3} + 2y \left(\frac{dy}{dx}\right)^2 + y^2 \frac{d^2y}{dx^2} + \cos x = 0$ or<br>$\frac{d^3y}{dx^3} = -\sec^2 x - 2y \sec x \left(\frac{dy}{dx}\right)^2 - y^2 \sec x \tan x \frac{dy}{dx} - y^2 \sec x \frac{d^2y}{dx^2}$                                                                                                                                            | <b>A1</b>  | 1.1b |
|                  | $-\sin\left(\frac{\pi}{4}\right)(-1) + \cos\left(\frac{\pi}{4}\right)\frac{d^3y}{dx^3} + 2(1)(0)^2 + (1)^2(-1) + \cos\left(\frac{\pi}{4}\right) = 0 \rightarrow \frac{d^3y}{dx^3} = \dots$<br>Alt: $\frac{d^3y}{dx^3} = -2 - 2(1)\sqrt{2}(0) - 1^2\sqrt{2}(1)(0) - 1^2\sqrt{2}(-1)$                                                                                                                                      | <b>M1</b>  | 1.1b |
|                  | $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \frac{d^3y}{dx^3} - 1 + \frac{\sqrt{2}}{2} = 0 \Rightarrow \frac{\sqrt{2}}{2} \frac{d^3y}{dx^3} = 1 - \sqrt{2}\right)$<br>$\frac{d^3y}{dx^3} \left( = \frac{2 - 2\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2} - 4}{2} \right) = \sqrt{2} - 2^*$                                                                                                                                  | <b>A1*</b> | 2.1  |
|                  |                                                                                                                                                                                                                                                                                                                                                                                                                          | <b>(4)</b> |      |
| <b>(c)</b>       | $(y=)1 + \left(x - \frac{\pi}{4}\right)(0) + \frac{\left(x - \frac{\pi}{4}\right)^2}{2!}(\text{their ' -1' }) + \frac{\left(x - \frac{\pi}{4}\right)^3}{3!}(\sqrt{2} - 2) + \dots$                                                                                                                                                                                                                                       | <b>M1</b>  | 2.5  |
|                  | $(y=)1 - \frac{\left(x - \frac{\pi}{4}\right)^2}{2} + \frac{\left(x - \frac{\pi}{4}\right)^3}{6}(\sqrt{2} - 2) + \dots$                                                                                                                                                                                                                                                                                                  | <b>A1</b>  | 1.1b |
|                  |                                                                                                                                                                                                                                                                                                                                                                                                                          | <b>(2)</b> |      |
| <b>(8 marks)</b> |                                                                                                                                                                                                                                                                                                                                                                                                                          |            |      |



**Notes:**
**(a)**

**M1:** A complete method to find the value for  $\frac{d^2y}{dx^2}$ . Substitutes into the differential equation

$x = \frac{\pi}{4}$ ,  $y = 1$ ,  $\frac{dy}{dx} = 0$  and rearranges to find a value for  $\frac{d^2y}{dx^2}$ .

**A1:**  $\frac{d^2y}{dx^2} = -1 < 0$  therefore a maximum. Must achieve the correct value, or at least a correct expression and give reason “<0” (oe) and conclusion and no contradictory statements.

**(b)**

**M1:** Differentiates to the form  $\pm \sin x \frac{d^2y}{dx^2} + \cos x \frac{d^3y}{dx^3} + \alpha y \left( \frac{dy}{dx} \right)^{1 \text{ or } 2} + y^2 \frac{d^2y}{dx^2} \pm \cos x = 0$ .

Alternatively, divides through by  $\cos x$  first and differentiates to achieve the form shown in scheme.

**A1:** Correct differentiation, any alternative form is acceptable.

**M1:** Substitutes in  $x = \frac{\pi}{4}$ ,  $y = 1$ ,  $\frac{dy}{dx} = 0$ , their value of  $\frac{d^2y}{dx^2} = -1$  as appropriate for their expression,

and rearranges to find a value for  $\frac{d^3y}{dx^3}$  (if not already done so).

**A1\*:** Shows that  $\frac{d^3y}{dx^3} = \sqrt{2} - 2$  from correct working with at least one suitable intermediate line –

being either an unsimplified  $\frac{d^3y}{dx^3} = \dots$  or  $\alpha \frac{d^3y}{dx^3} = \dots$ , before the final answer. Must have come from a correct derivative expression. Note that in the Alt the correct form comes out much easier, so look for at least one unsimplified line before the final answer.

**(c)**

**M1:** Uses the Taylor’s series expansion with the correct values for  $y_{\frac{\pi}{4}} = 1$ ,  $\frac{dy}{dx} = 0$ ,  $\frac{d^3y}{dx^3} = \sqrt{2} - 2$  (or

their  $\frac{d^3y}{dx^3}$  if they had a different answer to (b)) and their value of  $\frac{d^2y}{dx^2}$ . Must include the 1 at the start.

**A1:** Correct simplified expansion. The “y=” may be missing. Accept with  $0\left(x - \frac{\pi}{4}\right)$  as a second term. ISW after a correct simplified expression if further attempts to simplify are made.



| Question                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | Scheme                                                                                                                                                                                                                            | Marks    | AOs          |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------|--------------|
| 8(i) (a)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | $f(x) = \ln x \Rightarrow f(1) = 0$ $f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$ $f'' = \frac{1}{x^2} f'''$                                                                                                                        | M1<br>A1 | 1.1b<br>1.1b |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          | $(\ln x) = (0 +)(x - 1) - \frac{1}{2}(x - 1)^2 + \dots$                                                                                                                                                                           | M1<br>A1 | 2.5<br>1.1b  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |                                                                                                                                                                                                                                   | (4)      |              |
| (i) (b)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | $\lim_{x \rightarrow 1} \left( \frac{\ln x}{x - 1} \right) = \lim_{x \rightarrow 1} \left( \frac{(x - 1) - \frac{1}{2}(x - 1)^2 + \dots}{x - 1} \right)$ $= \lim_{x \rightarrow 1} \left( 1 - \frac{1}{2}(x - 1) + \dots \right)$ | M1       | 2.1          |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          | $= \lim_{x \rightarrow 1} \left( 1 - \frac{1}{2}(x - 1) + \dots \right) = 1^* \text{ cso}$                                                                                                                                        | A1*      | 2.2a         |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |                                                                                                                                                                                                                                   | (2)      |              |
| (ii)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | Writes as an indeterminate form<br>For example $\frac{\sin(2x)}{(x+3)\tan(6x)}$ or $\frac{\sin(2x)\cos(6x)}{(x+3)\sin(6x)}$                                                                                                       | M1       | 3.1a         |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          | Differentiates numerator and denominator using appropriate rules<br>$\frac{2\cos(2x)}{\tan(6x)+6(x+3)\sec^2(6x)}$ or $\frac{2\cos(2x)\cos(6x)-6\sin(2x)\sin(6x)}{\sin(6x)+6(x+3)\cos(6x)}$                                        | M1<br>A1 | 1.1b<br>1.1b |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          | $\lim_{x \rightarrow 0} \left( \frac{1}{(x+3)\tan(6x)\operatorname{cosec}(2x)} \right) = \frac{2}{18} = \frac{1}{9} \text{ o.e.}$                                                                                                 | A1cso    | 2.2a         |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |                                                                                                                                                                                                                                   | (4)      |              |
| <b>(10 marks)</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |                                                                                                                                                                                                                                   |          |              |
| <b>Notes:</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |                                                                                                                                                                                                                                   |          |              |
| <p><b>(i) (a) Notes:</b> ignore extra terms throughout.</p> <p><b>M1:</b> Differentiates <math>f(x) = \ln x</math> twice and finds <math>f(1)</math>, <math>f'(1)</math> and <math>f''(1)</math></p> <p><b>A1:</b> Correct differentiation and values for <math>f(1)</math>, <math>f'(1)</math> and <math>f''(1)</math>.</p> <p><b>M1:</b> Uses correct mathematical notation to find the Taylor series for <math>\ln x</math> in powers of <math>(x - 1)</math> up to <math>(x - 1)^2</math></p> <p><b>A1:</b> Correct expansion with simplified coefficients. Do not be concerned with the left hand side.</p>                                                                                                                                                         |                                                                                                                                                                                                                                   |          |              |
| <p><b>(i) (b) Question says “hence” so the result of (a) must be used.</b> No marks for l’Hosptial’s rule on the original functions (send to review if attempted with their part (a)).</p> <p><b>M1:</b> Substitutes their Taylor series for <math>\ln x</math> in powers of <math>(x - 1)</math> up to <math>(x - 1)^2</math> into the limit and cancels a factor <math>(x - 1)</math> from each term. Allow for the cancelling seeing a relevant strikethrough in all <math>x - 1</math> terms.</p> <p><b>A1*:</b> <math>\lim_{x \rightarrow 1} \left( \frac{\ln x}{x - 1} \right) = \lim_{x \rightarrow 1} \left( 1 - \frac{1}{2}(x - 1) + \dots \right) = 1</math> cso Must have come from a correct expansion. Must see the <math>1 - \frac{1}{2}(x - 1)</math></p> |                                                                                                                                                                                                                                   |          |              |
| <b>(ii)</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |                                                                                                                                                                                                                                   |          |              |

**M1:** Writes the fraction in an indeterminate form  $\frac{f(x)}{g(x)}$  where  $\frac{f(0)}{g(0)} = \frac{0}{0}$  or  $\frac{f(0)}{g(0)} = \frac{\infty}{\infty}$

**M1:** Differentiates numerator and denominator using appropriate rules, ie product rule for a product etc. Allow slips in coefficients but the form should be correct. This mark is available as long as written as  $\frac{f'(x)}{g'(x)}$  even if not an indeterminate form. May be seen written as separate from the fraction, ie  $f'(x) = \dots$  and  $g'(x) = \dots$

**A1: Depend on both Ms.** Correct differentiation for derivatives that lead to a limit. Must have a derivative for which  $g'(0) \neq 0$  and is finite.

**A1cso:** Deduces the correct limit from fully correct work.



| Question                                                                                                                                                                                                                                                                                                     | Scheme                                                                                                                                                                                                                                 | Marks    | AOs         |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------|-------------|
| 6(a)                                                                                                                                                                                                                                                                                                         | $y = \ln(e^{2x} \cos 3x) \Rightarrow \frac{dy}{dx} = \frac{2e^{2x} \cos 3x - 3e^{2x} \sin 3x}{e^{2x} \cos 3x} \text{ or}$ $y = \ln e^{2x} + \ln \cos 3x = 2x + \ln \cos 3x \Rightarrow \frac{dy}{dx} = 2 + \frac{-3 \sin 3x}{\cos 3x}$ | M1       | 2.1         |
|                                                                                                                                                                                                                                                                                                              | $= 2 - 3 \tan 3x^*$                                                                                                                                                                                                                    | A1*      | 1.1b        |
|                                                                                                                                                                                                                                                                                                              |                                                                                                                                                                                                                                        | (2)      |             |
| (b)                                                                                                                                                                                                                                                                                                          | $\frac{d^2 y}{dx^2} = -9 \sec^2 3x$                                                                                                                                                                                                    | B1       | 1.1b        |
|                                                                                                                                                                                                                                                                                                              | $\frac{d^3 y}{dx^3} = -54 \sec^2 3x \tan 3x$ $\Rightarrow \frac{d^4 y}{dx^4} = -162 \sec^4 3x - 324 \sec^2 3x \tan^2 3x \text{ (oe)}$                                                                                                  | M1<br>A1 | 2.1<br>1.1b |
|                                                                                                                                                                                                                                                                                                              |                                                                                                                                                                                                                                        | (3)      |             |
| (c)                                                                                                                                                                                                                                                                                                          | $(y)_0 = 0, \left(\frac{dy}{dx}\right)_0 = 2, \left(\frac{d^2 y}{dx^2}\right)_0 = -9, \left(\frac{d^3 y}{dx^3}\right)_0 = 0, \left(\frac{d^4 y}{dx^4}\right)_0 = -162$                                                                 | M1       | 1.1b        |
|                                                                                                                                                                                                                                                                                                              | $y = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$                                                                                                                                                           | M1       | 2.1         |
|                                                                                                                                                                                                                                                                                                              | $(y)0 + 2x - \frac{9x^2}{2} - \frac{162x^4}{24} + \dots = 2x - \frac{9x^2}{2} - \frac{27x^4}{4}$                                                                                                                                       | A1cso    | 1.1b        |
|                                                                                                                                                                                                                                                                                                              |                                                                                                                                                                                                                                        | (3)      |             |
| (d)                                                                                                                                                                                                                                                                                                          | $\ln(1+kx) = kx - \frac{k^2 x^2}{2} + \frac{k^3 x^3}{3} - \frac{k^4 x^4}{4} + \dots$                                                                                                                                                   | B1       | 2.2a        |
|                                                                                                                                                                                                                                                                                                              |                                                                                                                                                                                                                                        | (1)      |             |
| (e)                                                                                                                                                                                                                                                                                                          | $\ln \frac{e^{2x} \cos 3x}{1+kx} = \ln(e^{2x} \cos 3x) - \ln(1+kx)$ $= \ln(e^{2x} \cos 3x) - \ln(1+kx)$ $= 2x - \frac{9x^2}{2} - \frac{27x^4}{4} + \dots - \left(kx - \frac{k^2 x^2}{2} + \dots\right)$                                | M1       | 3.1a        |
|                                                                                                                                                                                                                                                                                                              | $= \frac{1}{x^2} \left( (2-k)x - \frac{(9-k^2)x^2}{2} - \frac{k^3 x^3}{3} - \frac{(27-k^4)x^4}{4} + \dots \right)$ $\text{For the limit to exist } 2-k=0 \Rightarrow k=...$                                                            | M1       | 3.1a        |
|                                                                                                                                                                                                                                                                                                              | $k=2$                                                                                                                                                                                                                                  | A1       | 2.2a        |
|                                                                                                                                                                                                                                                                                                              |                                                                                                                                                                                                                                        | (3)      |             |
|                                                                                                                                                                                                                                                                                                              |                                                                                                                                                                                                                                        |          |             |
| (12 marks)                                                                                                                                                                                                                                                                                                   |                                                                                                                                                                                                                                        |          |             |
| Notes                                                                                                                                                                                                                                                                                                        |                                                                                                                                                                                                                                        |          |             |
| (a)<br>M1: Attempts differentiation of the given function by applying both the chain rule and the product rule. Look for the correct form but they may make slips with signs or coefficients.<br>Alternatively, applies the sum law for logs and differentiates using the chain rule.<br>A1*: Correct proof. |                                                                                                                                                                                                                                        |          |             |

(b)

B1: Correct second derivative.

M1: Continues the differentiation using the chain rule and product rule to reach the 4<sup>th</sup> derivative. Look for correct forms, allowing for slips in signs or coefficients only when differentiating.

May be given in terms of  $\frac{dy}{dx}$ ,  $\frac{d^2y}{dx^2}$  etc for this mark.

A1: Correct 4<sup>th</sup> derivative in terms of  $x$ . Accept alternatives. Like terms should be gathered and coefficients simplified but isw after a correct suitable answer.

(c)

M1: Attempts the values of the derivatives at  $x = 0$  up to at least the third derivative. If substitution not seen at least two values should be correct for their derivatives, with values for others.

M1: Substitutes the values of their derivatives into the correct Maclaurin expansion formula up to at least the third non-zero value for their derivatives.

A1cso: Must have obtained a correct 4<sup>th</sup> derivative in (b) (though need not have been in terms of  $x$ ). Correct expansion with simplified coefficients. May be missing the “ $y =$ ” and allow “ $f(x) =$ ”. Note the correct series will be obtained from various incorrect 4<sup>th</sup> derivatives due to some terms evaluating to 0, but use of a clearly incorrect 4<sup>th</sup> derivative score A0 here. However, an initially correct 4<sup>th</sup> derivative, with incorrect simplification, can score A1 bod if no incorrect work is shown (ie correct values stated with no substitution shown, as they may have used the initially correct answer).

(d)

B1: Deduces the correct expansion in any form.

(e)

M1: Applies the subtraction law of logarithms and then substitutes their expansions.

M1: Realises that the  $x$  terms will determine the required value of  $k$  and so collects the  $x$  terms and sets the coefficient to zero and solves for  $k$  (may be implied).

A1: For  $k = 2$

| Question    | Scheme                                                                                                                                                                                                                                        | Marks      | AOs  |
|-------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|------|
| <b>6(a)</b> | $y = (1 + \ln x)^2 \Rightarrow \frac{dy}{dx} = k(1 + \ln x) \times \frac{1}{x}$ or<br>$y = 1 + 2 \ln x + (\ln x)^2 \Rightarrow \frac{dy}{dx} = \frac{A}{x} + B \ln x \times \frac{1}{x}$                                                      | <b>M1</b>  | 1.1b |
|             | $y = (1 + \ln x)^2 \Rightarrow \frac{dy}{dx} = 2(1 + \ln x) \times \frac{1}{x}$ or<br>$y = 1 + 2 \ln x + (\ln x)^2 \Rightarrow \frac{dy}{dx} = \frac{2}{x} + 2 \ln x \times \frac{1}{x}$                                                      | <b>A1</b>  | 1.1b |
|             | $\frac{d^2y}{dx^2} = \frac{k\left(\frac{1}{x}\right) \times x - k(1 + \ln x) \times 1}{x^2}$ or $\frac{k}{x} x^{-1} + k(1 + \ln x)(-x^{-2})$<br>or $\frac{d^2y}{dx^2} = -\frac{A}{x^2} + \frac{\frac{B}{x} \times x - 2 \ln x \times 1}{x^2}$ | <b>M1</b>  | 1.1b |
|             | $\frac{d^2y}{dx^2} = \frac{\left(\frac{2}{x}\right) \times x - 2(1 + \ln x)}{x^2}$ or $\frac{2}{x} x^{-1} + 2(1 + \ln x)(-x^{-2})$<br>Leading to<br>$\frac{d^2y}{dx^2} = -\frac{2 \ln x}{x^2}$ * achieved from correct work.                  | <b>A1*</b> | 2.1  |
|             |                                                                                                                                                                                                                                               | <b>(4)</b> |      |
| <b>(b)</b>  | $\frac{d^3y}{dx^3} = \frac{\pm \frac{C}{x} \times x^2 \pm Dx \ln x}{x^4}$ or $\frac{d^3y}{dx^3} = (-2 \ln x)(-Cx^{-3}) + \left(-\frac{D}{x}\right)(x^{-2})$                                                                                   | <b>M1</b>  | 1.1b |
|             | $\frac{d^3y}{dx^3} = -\frac{\frac{2}{x} \times x^2 - 4x \ln x}{x^4}$ or $\frac{-\frac{2}{x} \times x^2 - (-2 \ln x)(2x)}{x^4}$ or $-\frac{2}{x^3} + \frac{4 \ln x}{x^3}$                                                                      | <b>A1</b>  | 1.1b |
|             |                                                                                                                                                                                                                                               | <b>(2)</b> |      |
| <b>(c)</b>  | $y(1) = 1, y'(1) = 2, y''(1) = 0, y'''(1) = -2$                                                                                                                                                                                               | <b>M1</b>  | 1.1b |
|             | $[y] = 1 + 2(x-1) + \frac{0}{2!}(x-1)^2 + \frac{-2}{3!}(x-1)^3 + \dots$                                                                                                                                                                       | <b>M1</b>  | 2.5  |
|             | $[y] = 1 + 2(x-1) - \frac{1}{3}(x-1)^3 + \dots$ or $[y] = -1 + 2x - \frac{1}{3}(x-1)^3 + \dots$                                                                                                                                               | <b>A1</b>  | 1.1b |
|             |                                                                                                                                                                                                                                               | <b>(3)</b> |      |
| <b>(d)</b>  | $\frac{2x-1-(1+\ln x)^2}{(x-1)^3} = \frac{2x-1-1-2(x-1)+\frac{1}{3}(x-1)^3+\dots}{(x-1)^3} = \frac{\frac{1}{3}(x-1)^3+\dots}{(x-1)^3}$                                                                                                        | <b>M1</b>  | 1.1b |

|                                                                                                                                                                                                                                                                                                                                                                                                                          |                                                                                                                                                                                                                       |            |      |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|------|
|                                                                                                                                                                                                                                                                                                                                                                                                                          | <p>Simplifies and realises that terms cancel to leave a constant term</p> $\frac{\frac{1}{3} \cancel{(x-1)^3} + \dots}{\cancel{(x-1)^3}} = \frac{1}{3}$                                                               | <b>M1</b>  | 3.1a |
|                                                                                                                                                                                                                                                                                                                                                                                                                          | <p>Hence <math>\lim_{x \rightarrow 1} \frac{2x-1-(1+\ln x)^2}{(x-1)^3} = \frac{1}{3}</math> as all remaining terms will become zero in the limit as they are multiples of <math>(x-1)^k</math>, which tends to 0.</p> | <b>A1</b>  | 2.4  |
|                                                                                                                                                                                                                                                                                                                                                                                                                          |                                                                                                                                                                                                                       | <b>(3)</b> |      |
| <b>(12 marks)</b>                                                                                                                                                                                                                                                                                                                                                                                                        |                                                                                                                                                                                                                       |            |      |
| <b>Notes:</b>                                                                                                                                                                                                                                                                                                                                                                                                            |                                                                                                                                                                                                                       |            |      |
| <p><b>(a)</b><br/> <b>M1:</b> Attempts the first derivative, including use of the chain rule. May expand first. E.g. accept forms as shown.<br/> <b>A1:</b> Correct first derivative, need not be simplified.<br/> <b>M1:</b> Attempts second derivative using quotient rule or product rule – examples as shown, or equivalents accepted.<br/> <b>A1*:</b> Correct result achieved from correct work.</p>               |                                                                                                                                                                                                                       |            |      |
| <p><b>(b)</b><br/> <b>M1:</b> Applies quotient rule or product rule to achieve third derivative. If formula is quoted it must be correct, if not accept derivatives of the form shown as there may be confusion with the minus sign.<br/> <b>A1:</b> Correct third derivative, any form.</p>                                                                                                                             |                                                                                                                                                                                                                       |            |      |
| <p><b>(c)</b><br/> <b>M1:</b> Find value of derivatives at <math>x = 1</math>.<br/> <b>M1:</b> Applies Taylor series expansion<br/> <b>A1:</b> Correct series, may be unsimplified, isw once correct series seen. Must be using a correct third derivative.</p>                                                                                                                                                          |                                                                                                                                                                                                                       |            |      |
| <p><b>(d)</b><br/> <b>M1:</b> Applies the series to the limit and cancels terms in numerator to leave term in <math>(x-1)^3</math> and above only (may not see <math>+\dots</math> for this mark)<br/> <b>M1:</b> Simplifies and realises that the <math>(x-1)^3</math> cancels and achieves a constant <math>A</math><br/> <b>A1:</b> Correct limit deduced with reasoning given why the remaining terms disappear.</p> |                                                                                                                                                                                                                       |            |      |

| Question | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | Marks    | AOs          |
|----------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------|--------------|
| 4(a)     | $y = (15 + e^{2x})^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(15 + e^{2x})^{-\frac{1}{2}} \times 2e^{2x}$ <p style="text-align: center;">or</p> $y^2 = 15 + e^{2x} \Rightarrow 2y \frac{dy}{dx} = 2e^{2x}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | B1       | 1.1b         |
|          | $\frac{d^2y}{dx^2} = \frac{(15 + e^{2x})^{\frac{1}{2}} \times Ae^{2x} + B(15 + e^{2x})^{-\frac{1}{2}} \times e^{2x} \times e^{2x}}{15 + e^{2x}}$ $\left\{ \frac{d^2y}{dx^2} = \frac{(15 + e^{2x})^{\frac{1}{2}} \times 2e^{2x} - \frac{1}{2}(15 + e^{2x})^{-\frac{1}{2}} \times 2e^{2x} \times e^{2x}}{15 + e^{2x}} \right\}$ <p style="text-align: center;">or</p> $\frac{d^2y}{dx^2} = Ae^{2x}(15 + e^{2x})^{-\frac{1}{2}} + Be^{2x} \times e^{2x}(15 + e^{2x})^{-\frac{3}{2}}$ $\left\{ \frac{d^2y}{dx^2} = 2e^{2x}(15 + e^{2x})^{-\frac{1}{2}} - \frac{1}{2} \times 2e^{2x} \times e^{2x}(15 + e^{2x})^{-\frac{3}{2}} \right\}$ <p style="text-align: center;">Or</p> $A\left(\frac{dy}{dx}\right)^2 + By \frac{d^2y}{dx^2} = Ce^{2x}$ $\left\{ 2\left(\frac{dy}{dx}\right)^2 + 2y \frac{d^2y}{dx^2} = 4e^{2x} \right\}$ | M1       | 1.1b         |
|          | $\frac{d^2y}{dx^2} = \frac{(15 + e^{2x}) \times 2e^{2x} - e^{4x}}{(15 + e^{2x})^{\frac{3}{2}}} = \frac{e^{2x}(e^{2x} + 30)}{(15 + e^{2x})^{\frac{3}{2}}} *$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | A1*      | 2.1          |
|          |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | (3)      |              |
| (b)      | $x = 0 \Rightarrow y = 4, \frac{dy}{dx} = \frac{1}{4}, \frac{d^2y}{dx^2} = \frac{31}{64} \text{ leading to}$ $\Rightarrow y = 4 + \frac{x}{4} + \frac{31}{64} \frac{x^2}{2!} + \dots = 4 + \frac{x}{4} + \frac{31}{128} x^2$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | M1<br>A1 | 1.1b<br>1.1b |
|          |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | (2)      |              |
| (c)      | $\sin 3x = 3x - \frac{9}{2}x^3 + \dots$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      | B1       | 2.2a         |
|          |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | (1)      |              |
| (d)      | $\left\{ \lim_{x \rightarrow 0} \right\} \frac{\sqrt{(15 + e^{2x})} - 4}{\sin 3x} = \left\{ \lim_{x \rightarrow 0} \right\} \frac{4 + \frac{x}{4} + \frac{31}{128}x^2 - 4}{3x - \frac{9}{2}x^3} = \left\{ \lim_{x \rightarrow 0} \right\} \frac{\frac{1}{4} + \frac{31}{128}x}{3 - \frac{9}{2}x^2}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          | M1       | 2.1          |

|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |                                                                                                                                                           |     |      |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------|-----|------|
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      | $\lim_{x \rightarrow 0} \frac{\frac{1}{4} + \frac{31}{128}x}{3 - \frac{9}{2}x^2} \left\{ = \frac{1}{4} \div 3 \right\} = \frac{1}{12} \quad \text{* cso}$ | A1* | 1.1b |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |                                                                                                                                                           | (2) |      |
| <b>(8 marks)</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |                                                                                                                                                           |     |      |
| <b>Notes</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         |                                                                                                                                                           |     |      |
| <p>(a)</p> <p>B1: Correct first derivative in any form, may square first and use implicit differentiation</p> <p>M1: Differentiates again applying the quotient or product rule correctly to achieve the correct form or uses implicit differentiation to achieve a correct form</p> <p>A1*: Correct proof with no errors. Must show sufficient working.</p> <p>If uses the product rule they must show a common denominator</p> <p>If uses the quotient rule there must be evidence to show the multiplication of <math>(15 + e^{2x})^{\frac{1}{2}}</math> top and bottom</p> <p>If uses implicit differentiation we need to see the substitution for <math>\frac{dy}{dx}</math> and <math>y</math> and rearranging.</p> <p>(b)</p> <p>M1: A full method to obtain the required expansion. I.e. finds the values up to the second derivative and uses a correct formula</p> <p>A1: Correct simplified expansion with <math>y =</math> or <math>f(x) =</math> seen somewhere in their solution</p> <p>(c)</p> <p>B1: Deduces the correct simplified expansion, ignore any extra terms, whether correct or not</p> <p>(d)</p> <p>M1: A rigorous argument using their answers to (b) and (c), showing the cancelling constant term and the division by <math>x</math> in the numerator and denominator to establish the limiting behaviour.</p> <p>A1*: Correct proof with no errors including correct limiting notation seen in the final stage cso</p> <p>Note: Using L'Hospital's Rule with the functions <math>\sqrt{(15 + e^{2x})} - 4</math> and <math>\sin 3x</math> is M0A0</p> <p>Using L'Hospital's Rule with their expansions <math>\frac{4 + \frac{x}{4} + \frac{31}{128}x^2 - 4}{3x - \frac{9}{2}x^3}</math> leading to <math>\frac{\frac{1}{4} + \frac{31}{64}x}{3 - \frac{27}{2}x^2}</math> could</p> <p>score M1A1</p> |                                                                                                                                                           |     |      |

| Question                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | Marks                               | AOs                  |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------|----------------------|
| 4                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | $u' = 4x^3, u'' = 12x^2, u''' = 24x, u^{(4)} = 24$ (and $u^{(n)} = 0$ for $n > 4$ )                                                                                                                                                                                                                                                                                                                                                                                          | <b>M1</b>                           | 1.1b                 |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                | $v' = 2 \cos(2x), v'' = -4 \sin(2x), v''' = -8 \cos(2x), v^{(4)} = 16 \sin(2x),$<br>$v^{(5)} = 32 \cos(2x), v^{(6)} = -64 \sin(2x), v^{(7)} = -128 \cos(2x),$<br>$v^{(8)} = 256 \sin(2x),$                                                                                                                                                                                                                                                                                   | <b>M1</b><br><b>A1</b><br><b>A1</b> | 3.1a<br>1.1b<br>1.1b |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                | $f^{(8)}(x) = x^4 \times 256 \sin(2x) + 8 \times 4x^3 \times -128 \cos(2x)$ Thus $+ \frac{8 \times 7}{2} \times 12x^2 \times -64 \sin(2x) + \frac{8 \times 7 \times 6}{6} \times 24x \times 32 \cos(2x)$ $+ \frac{8 \times 7 \times 6 \times 5}{24} \times 24 \times 16 \sin(2x)$ $f^{(8)}(x) = x^4 \times 256 \sin(2x) + 8 \times 4x^3 \times -128 \cos(2x)$ $+ 28 \times 12x^2 \times -64 \sin(2x) + 56 \times 24x \times 32 \cos(2x)$ $+ 70 \times 24 \times 16 \sin(2x)$ | <b>M1</b>                           | 2.1                  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                | $f^{(8)}(\pi) = 0 - 4096\pi^3 - 0 + 1344 \times 2^5 \pi + 0 \quad (= -4096\pi^3 + 43008\pi)$                                                                                                                                                                                                                                                                                                                                                                                 | <b>M1</b>                           | 1.1b                 |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                | Coefficient is $\frac{f^{(8)}(\pi)}{8!} = \frac{1344 \times 2^5 \pi - 4096\pi^3}{8! \text{ or } \{40320\}}$                                                                                                                                                                                                                                                                                                                                                                  | <b>M1</b>                           | 2.2a                 |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                | $= \frac{336\pi - 32\pi^3}{315}$<br>(So $a = 336$ and $b = -32$ )                                                                                                                                                                                                                                                                                                                                                                                                            | <b>A1</b>                           | 2.1                  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | <b>(8)</b>                          |                      |
| <b>(8 marks)</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |                                     |                      |
| <b>Notes:</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |                                     |                      |
| <p><b>M1:</b> Establishes the non-disappearing derivatives of <math>x^4</math>. Allow slips in coefficients, but powers must decrease.</p> <p><b>M1:</b> Identifies the relevant derivatives for <math>\sin(2x)</math>, up to the 8<sup>th</sup> derivative or establishes the correct pattern. Look for alternating between sin and cos. Condone use of <math>x</math>.</p> <p><b>A1:</b> Correct sizes for the coefficients, allow sign errors for this mark (may be due to incorrect signs when differentiating sin and cos) Must have angle <math>2x</math>.</p> <p><b>A1:</b> All derivatives correctly established. (Note the sin terms may be omitted if the student has made clear they will disappear, but if present they must be correct).</p> <p><b>M1:</b> Applies Leibnitz's theorem to get the 8<sup>th</sup> derivative with their expressions. Binomial coefficients must be present.</p> <p><b>M1:</b> Evaluates their 8<sup>th</sup> derivative at <math>\pi</math></p> <p><b>M1:</b> Uses Taylor series – divides their value for <math>f^{(8)}(\pi)</math> by 8!</p> <p><b>A1:</b> Simplifies to the correct answer.</p> <p><b>Note:</b> If do not use Leibnitz's theorem then maximum <b>M0 M0 A0 A0 M0 M1 M1 A0</b></p> |                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |                                     |                      |

| Question          | Scheme                                                                                                                                                                                         | Marks      | AOs          |
|-------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|--------------|
| <b>6(i)</b>       | $\left\{ \lim_{x \rightarrow 0} \frac{1 - \cos 7x}{x \sin 9x} = \left\{ \lim_{x \rightarrow 0} \frac{A \sin 7x}{\sin 9x + Bx \cos 9x} \right. \right.$                                         | M1         | 1.1b         |
|                   | $\left. \left\{ \lim_{x \rightarrow 0} \frac{1 - \cos 7x}{x \sin 9x} = \left\{ \lim_{x \rightarrow 0} \frac{7 \sin 7x}{\sin 9x + 9x \cos 9x} \right. \right. \right.$                          | A1         | 1.1b         |
|                   | $\left. \left\{ \lim_{x \rightarrow 0} \frac{7 \sin 7x}{\sin 9x + 9x \cos 9x} = \left\{ \lim_{x \rightarrow 0} \frac{49 \cos 7x}{9 \cos 9x + 9 \cos 9x - 81x \sin 9x} \right. \right. \right.$ | M1         | 3.1a         |
|                   | $\lim_{x \rightarrow 0} \frac{49 \cos 7x}{9 \cos 9x + 9 \cos 9x - 81x \sin 9x} = \frac{49}{9+9} = \frac{49}{18}^*$                                                                             | A1*        | 1.1b         |
|                   |                                                                                                                                                                                                | <b>(4)</b> |              |
| <b>(ii)(a)</b>    | $u = x^2 \Rightarrow \frac{du}{dx} = 2x, \frac{d^2u}{dx^2} = 2, \left\{ \frac{d^3u}{dx^3} = 0 \right\}$                                                                                        | M1         | 1.1b         |
|                   | $v = e^{3x} \Rightarrow \frac{dv}{dx} = 3e^{3x}, \frac{d^2v}{dx^2} = 9e^{3x}, \frac{d^3v}{dx^3} = 27e^{3x}, \dots, \left\{ \frac{d^k v}{dx^k} = 3^k e^{3x} \right\}$                           | M1         | 2.1          |
|                   | $\frac{d^k y}{dx^k} = 3^k e^{3x} x^2 + k \times 2x \times 3^{k-1} e^{3x} + \frac{k(k-1)}{2} \times 2 \times 3^{k-2} e^{3x}$                                                                    | M1<br>A1   | 3.1a<br>1.1b |
|                   | $= 3^{k-2} e^{3x} (9x^2 + 6kx + k(k-1))$                                                                                                                                                       | A1         | 2.2a         |
|                   |                                                                                                                                                                                                | <b>(5)</b> |              |
| <b>(b)</b>        | $\frac{d^9 y}{dx^9} = 3^{9-2} e^{3x} (9x^2 + 6 \times 9x + 9 \times 8) = 0$<br>$\Rightarrow 9x^2 + 54x + 72 = 0 \Rightarrow x = \dots$                                                         | M1         | 1.1b         |
|                   | $x = -4, -2$ cso                                                                                                                                                                               | A1         | 2.2a         |
|                   |                                                                                                                                                                                                | <b>(2)</b> |              |
| <b>(11 marks)</b> |                                                                                                                                                                                                |            |              |



## Notes

(i)

M1: Differentiates numerator and denominator to the correct form, **may be done separately**

A1: Correct derivatives

M1: Recognises the requirement to differentiate numerator and denominator again and achieves the correct form of the derivatives may be seen **separately** or as a fraction

$$\frac{A \cos 7x}{B \cos 9x + C \cos 9x + Dx \sin 9x}$$

A1\*: Completes the proof, including correct limit notation seen when  $x = 0$  is substituted with as

$$\text{a minimum } \frac{49}{9+9} = \frac{49}{18} \text{ or } \lim_{x \rightarrow 0} \frac{49 \cos 7x}{18 \cos 9x - 81x \sin 9x} = \frac{49}{18}$$

(ii)(a)

M1: Differentiates  $u = x^2$  twice, may be seen as part of the expression for the  $k^{\text{th}}$  derivativeM1: Uses  $v = e^{3x}$  to establish the form of the derivatives  $\frac{d^k v}{dx^k} = 3^k e^{3x}$ . Look for multiples of  $e^{3x}$ increasing by a factor of 3 each time, need at least three. This may be seen as part of the expression for the  $k^{\text{th}}$  derivativeM1: A correct strategy for the  $k^{\text{th}}$  derivative. This requires the correct derivative of  $x^2$  combined with the correct derivative of  $e^{3x}$  in terms of  $k$  together with the correct binomial coefficient.Condone using  $k(k-1)$  for the third term

A1: A correct unsimplified expression

A1: Correct expression in the required form with correct values of  $A$ ,  $B$  and  $C$ . Condone missing trailing bracket(NB  $A = 9$ ,  $B = 6$ ,  $C = 1$ )

Repeated differentiation to find at least the third derivative

$$y = x^2 e^{3x} \Rightarrow \frac{dy}{dx} = 2x e^{3x} + 3x^2 e^{3x} \Rightarrow \frac{d^2 y}{dx^2} = 2e^{3x} + 6x e^{3x} + 6x e^{3x} + 9x^2 e^{3x}$$

$$\Rightarrow \frac{d^3 y}{dx^3} = 6e^{3x} + 12e^{3x} + 36x e^{3x} + 18x e^{3x} + 27x^2 e^{3x}$$

they can score the first M1M1 for differentiating  $u$  twice and  $v$  three times. Then will score M0A0A0.

(b)

M1: Substitutes  $k = 9$  into their result from part (a) and solves the resulting 3TQ which leads to real roots

A1: These values only following full marks in (a)

| Question  | Scheme                                                                                                                                                                                                                                                                                                                                                | Marks    | AOs         |
|-----------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------|-------------|
| 5(a)      | $1 + \operatorname{cosec} x = 1 + \frac{1}{\sin x} = 1 + \frac{1}{\frac{2t}{1+t^2}} \text{ or } 1 + \frac{1+t^2}{2t} \left( = \frac{t^2 + 2t + 1}{2t} \right)$ <p>or</p> $1 + \operatorname{cosec} x = \frac{\sin x + 1}{\sin x} = \frac{\frac{2t}{1+t^2} + 1}{\frac{2t}{1+t^2}}$                                                                     | B1       | 1.1a        |
|           | $\int \frac{1}{1 + \operatorname{cosec} x} dx = \int \frac{2t}{t^2 + 2t + 1} \times \frac{2}{1+t^2} dt$                                                                                                                                                                                                                                               | M1       | 2.1         |
|           | $\int \frac{1}{1 + \operatorname{cosec} x} dx = \int \frac{4t}{(1+t^2)(1+t)^2} dt^*$                                                                                                                                                                                                                                                                  | A1*      | 1.1b        |
|           |                                                                                                                                                                                                                                                                                                                                                       | (3)      |             |
| (b)       | $\frac{4t}{(1+t^2)(1+t)^2} \equiv \frac{At+B}{1+t^2} + \frac{C}{1+t} + \frac{D}{(1+t)^2}$ <p>or</p> $\frac{4t}{(1+t^2)(1+t)^2} \equiv \frac{At+B}{1+t^2} + \frac{Ct+D}{(1+t)^2}$ <p>or</p> $\frac{4t}{(1+t^2)(1+t)^2} \equiv \frac{At+B}{1+t^2} + \frac{C}{1+t} + \frac{Dt+E}{(1+t)^2}$ <p>Leading to finding the value for at least one constant</p> | M1       | 3.1a        |
|           | $\frac{4t}{(1+t^2)(1+t)^2} \equiv \frac{2}{1+t^2} - \frac{2}{(1+t)^2}$                                                                                                                                                                                                                                                                                | A1       | 1.1b        |
|           | $\int \frac{2}{1+t^2} - \frac{2}{(1+t)^2} dt = 2 \arctan t + \frac{2}{1+t}$                                                                                                                                                                                                                                                                           | A1       | 2.2a        |
|           | $= 2 \arctan\left(\tan \frac{x}{2}\right) + \frac{2}{1 + \tan \frac{x}{2}} + k = x + \frac{2}{1 + \tan \frac{x}{2}} + k \text{ cso}$                                                                                                                                                                                                                  | M1<br>A1 | 1.1b<br>2.1 |
|           |                                                                                                                                                                                                                                                                                                                                                       | (5)      |             |
| (8 marks) |                                                                                                                                                                                                                                                                                                                                                       |          |             |

## Notes

(a)

B1: Selects the correct formulae to express  $1 + \operatorname{cosec} x$  in terms of  $t$ . May be seen within the integral

M1: Makes a complete substitution to obtain an integral in terms of  $t$  only, including

$$dx = \frac{2}{1+t^2} dt$$

A1\*: Correct proof with no errors or omissions, must see the factorisation if the denominator is expanded first. Long division or equivalent with  $t^2 + 1$  or  $(t+1)^2$  or  $t+1$

(b)

M1: Realises the need to express the integrand in terms of partial fractions and attempts the correct form, condone the use of  $x$  instead of  $t$  on their numerator with  $t$ 's on the denominator.

Must attempt to find at least one constant term

A1: Correct partial fractions.

A1: Correct integration, must have scored the previous marks, no need for the constant of integration

Note an incorrect form such as  $\frac{A}{1+t^2} + \frac{B}{1+t} + \frac{C}{(1+t)^2}$  can lead to a correct answer but is

M0A0A0

M1: Reverses the substitution to express in terms of  $x$

A1: Correct answer with no errors including “+  $k$ ” must have scored all the previous marks in part (b)

**Note Verification method using  $\frac{A}{1+t^2} + \frac{B}{(1+t)^2}$  to find the values for  $A$  and  $B$ , please send to review**

