



# GuideMyGrades

— YOUR GOALS. OUR EXPERTISE. —

EDEXCEL A LEVEL FURTHER MATHEMATICS: QUESTIONS BY TOPIC

## Further Matrix Algebra (3)

### Further Pure Mathematics 2 · Topic 3

7 questions · 72 marks · Mark schemes included

*Pearson Edexcel A Level Further Mathematics (9FM0), June 2019 - June 2025*

 [guidemygrades.com](https://www.guidemygrades.com)

 @Guidemygrades

 @Guidemygrades

 @Guidemygrades

 [info@guidemygrades.com](mailto:info@guidemygrades.com)

 Piccadilly Business Centre, Unit C, Aldow Enterprise Park  
Blackett Street, Manchester, M12 6AE

**HOW TO USE THIS BOOKLET**

Questions are grouped by specification sub-topic and run from the lowest mark tariff to the highest within each one.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme.

Diagrams, graphs, tables and data are printed exactly as they appeared in the original papers.

| QUESTION                                 | SOURCE                                             | MARKS           | PAGE      |
|------------------------------------------|----------------------------------------------------|-----------------|-----------|
| <b>1.3 ORDER AND SUBGROUPS</b>           |                                                    | 1 Qs · 10 marks | <b>3</b>  |
| 1. <b>Q7</b> <i>also Topic 1 &amp; 5</i> | Option 4A: Further Pure Mathematics 2 June2024     | 10              | <b>4</b>  |
| <b>3.1 EIGENVALUES AND EIGENVECTORS</b>  |                                                    | 1 Qs · 8 marks  | <b>8</b>  |
| 2. <b>Q2</b>                             | Option 4A: Further Pure Mathematics 2 June2022     | 8               | <b>9</b>  |
| <b>3.2 DIAGONALISATION</b>               |                                                    | 4 Qs · 48 marks | <b>13</b> |
| 3. <b>Q2</b>                             | Option 4A: Further Pure Mathematics 2 June2019     | 11              | <b>14</b> |
| 4. <b>Q4</b>                             | Option 4A: Further Pure Mathematics 2 June2024     | 12              | <b>18</b> |
| 5. <b>Q8</b> <i>also Topic 4</i>         | Option 4A: Further Pure Mathematics 2 November2021 | 17              | <b>22</b> |
| 6. <b>Q2</b>                             | Option 4A: Further Pure Mathematics 2 June2025     | 8               | <b>27</b> |
| <b>3.3 THE CAYLEY-HAMILTON THEOREM</b>   |                                                    | 1 Qs · 6 marks  | <b>31</b> |
| 7. <b>Q1</b>                             | Option 4A: Further Pure Mathematics 2 June2023     | 6               | <b>32</b> |



# QUESTIONS

7 questions · 72 marks · 4 sub-topics



SUB-TOPIC 1.3

# ORDER AND SUBGROUPS

Sub-topic 1.3 · 1 question · 10 marks · lowest tariff first









(continued)



SUB-TOPIC 3.1

# EIGENVALUES AND EIGENVECTORS

Sub-topic 3.1 · 1 question · 8 marks · lowest tariff first









**(Total for Question 2 is 8 marks)**

SUB-TOPIC 3.2

# DIAGONALISATION

Sub-topic 3.2 · 4 questions · 48 marks · lowest tariff first









**(Total for Question 2 is 11 marks)**









(Total for Question 4 is 12 marks)



8.

$$\mathbf{A} = \begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & p \\ -6 & 6 & -4 \end{pmatrix} \quad \text{where } p \text{ is a constant}$$

Given that  $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$  is an eigenvector for  $\mathbf{A}$

(a) (i) determine the eigenvalue corresponding to this eigenvector

(1)

(ii) hence show that  $p = 2$

(2)

(iii) determine the remaining eigenvalues and corresponding eigenvectors of  $\mathbf{A}$

(7)

(b) Write down a matrix  $\mathbf{P}$  and a diagonal matrix  $\mathbf{D}$  such that  $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$

(1)

(c) (i) Solve the differential equation  $\dot{u} = ku$ , where  $k$  is a constant.

(2)

With respect to a fixed origin  $O$ , the velocity of a particle moving through space is modelled by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

By considering  $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$  so that  $\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$

(ii) determine a general solution for the displacement of the particle.

(4)

---

---

---

---

---

---

---

---

---

---





**Question 8 continued**



2.

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

- (a) Determine the eigenvalues of matrix **A**

(3)

- (b) Hence determine an orthogonal matrix  $\mathbf{P}$  and a diagonal matrix  $\mathbf{D}$  such that

$$\mathbf{D} = \mathbf{P}^T \mathbf{A} \mathbf{P}$$

(5)







SUB-TOPIC 3.3

# THE CAYLEY-HAMILTON THEOREM

Sub-topic 3.3 · 1 question · 6 marks · lowest tariff first



**1.**

$$\mathbf{A} = \begin{pmatrix} -1 & a \\ 3 & 8 \end{pmatrix}$$

where  $a$  is a constant.

- (a) Determine, in expanded form in terms of  $a$ , the characteristic equation for  $\mathbf{A}$ .

(2)

- (b) Hence use the Cayley-Hamilton theorem to determine values of  $a$  and  $b$  such that

$$\mathbf{A}^3 = \mathbf{A} + b\mathbf{I}$$

where  $\mathbf{I}$  is the  $2 \times 2$  identity matrix.

(4)

[illegible]



# MARK SCHEMES

official mark schemes



| Question    | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | Marks      | AOs  |
|-------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|------|
| <b>7(a)</b> | $\mathbf{B}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \{=\mathbf{E}\}$ <p>Accept <math>\begin{pmatrix} 2 &amp; 1 \\ 1 &amp; 1 \end{pmatrix}</math> as long as <math>2 \equiv 0 \pmod{2}</math> is used later.</p> <p>Or</p> $\mathbf{B}^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \{=\mathbf{I}\}$ <p>Accept <math>\begin{pmatrix} 3 &amp; 2 \\ 2 &amp; 1 \end{pmatrix}</math> as long as <math>2 \equiv 0 \pmod{2}</math> and <math>3 \equiv 1 \pmod{2}</math> is used later.</p>                                                                                                                                                                      | <b>M1</b>  | 2.5  |
|             | $\mathbf{B}^3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \{=\mathbf{I}\}$ (and $\mathbf{B}$ is not the identity) hence $\mathbf{B}$ has order 3 or $ \mathbf{B}  = 3$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            | <b>A1</b>  | 2.1  |
|             |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | <b>(2)</b> |      |
|             |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |            |      |
| <b>(b)</b>  | $\mathbf{I}$ {is the identity so} has order 1, and $\mathbf{E}$ {is $\mathbf{B}^{-1}$ so also} has order 3.                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      | <b>B1</b>  | 1.1b |
|             | <p>Finds at least one of the remaining orders e.g. <math>\mathbf{A}^2 = \begin{pmatrix} 1 &amp; 2 \\ 0 &amp; 1 \end{pmatrix} = \begin{pmatrix} 1 &amp; 0 \\ 0 &amp; 1 \end{pmatrix} \{=\mathbf{I}\}</math></p> <p>or <math>\mathbf{C}^2 = \begin{pmatrix} 1 &amp; 2 \\ 0 &amp; 1 \end{pmatrix} = \begin{pmatrix} 1 &amp; 0 \\ 0 &amp; 1 \end{pmatrix} \{=\mathbf{I}\}</math> or <math>\mathbf{D}^2 = \begin{pmatrix} 1 &amp; 2 \\ 0 &amp; 1 \end{pmatrix} = \begin{pmatrix} 1 &amp; 0 \\ 0 &amp; 1 \end{pmatrix} \{=\mathbf{I}\}</math></p>                                                                                                                                                                                                                                                                      | <b>M1</b>  | 1.1b |
|             | (So $\mathbf{I}$ has order 1, $\mathbf{B}$ and $\mathbf{E}$ order 3 and ) $\mathbf{A}$ , $\mathbf{C}$ and $\mathbf{D}$ each have order 2                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         | <b>A1</b>  | 1.1b |
|             |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | <b>(3)</b> |      |
| <b>(c)</b>  | (i) There is no element of <b>order 6</b> in $G$ {so cannot be isomorphic to $C_6$ }                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | <b>B1</b>  | 2.4  |
|             | <p>(ii) Either</p> <ul style="list-style-type: none"> <li>• There are 12 symmetries of a regular hexagon</li> <li>• Hexagon has more than 6 symmetries</li> <li>• Hexagon has 2 <b>rotations</b> of order 6, could give an example</li> <li>• Hexagon has 2 <b>rotations</b> of order 3</li> <li>• Hexagon has 1 <b>rotation</b> of order 2</li> <li>• Hexagon has 6 <b>reflections</b> of order 2</li> <li>• Group of symmetries of a hexagon has an order greater than 6</li> </ul> <p>B0 for</p> <ul style="list-style-type: none"> <li>• The group of symmetries of a hexagon is also cyclic</li> <li>• The groups are not of the same order</li> <li>• Any incorrect orders of element stated</li> <li>• Comments on the order of the whole group</li> <li>• The group of symmetries has order 6</li> </ul> | <b>B1</b>  | 2.4  |
|             |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | <b>(2)</b> |      |
|             |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |            |      |
| <b>(d)</b>  | Any two correct matchings                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | <b>M1</b>  | 2.2a |
|             | Completes the matching with at least 4 correct                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | <b>M1</b>  | 3.1a |



|                                                                                                                                                                       |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------|---|---|---|---|---|---|---|---|---|---|---|-----|-----|
|                                                                                                                                                                       | Hence an isomorphism is |   |   |   |   |   |   |   |   |   |   |   | A1  | 2.1 |
|                                                                                                                                                                       | I                       | A | B | C | D | E | I | A | B | C | D | E |     |     |
|                                                                                                                                                                       | e                       | c | a | d | f | b | e | c | b | f | d | a |     |     |
|                                                                                                                                                                       | e                       | d | a | f | c | b | e | d | b | c | f | a |     |     |
|                                                                                                                                                                       | e                       | f | a | c | d | b | e | f | b | d | c | a |     |     |
|                                                                                                                                                                       |                         |   |   |   |   |   |   |   |   |   |   |   | (3) |     |
| (10 marks)                                                                                                                                                            |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| Notes:                                                                                                                                                                |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| (a)                                                                                                                                                                   |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| M1: Squares B to begin the process of finding the order, or uses a calculator to find B <sup>3</sup>                                                                  |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| A1: Shows that B <sup>3</sup> = I and concludes that order must be 3.                                                                                                 |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| (b)                                                                                                                                                                   |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| B1: States the order for I and gives the order for E (which is the inverse of B).                                                                                     |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| M1: Finds the order of at least one of the other elements to get an element of order 2. May use a calculator                                                          |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| A1: Correct orders for the three elements of order 2.                                                                                                                 |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| (c)                                                                                                                                                                   |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| B1: Correct reason given, identifies there is no element of order 6. It is not sufficient to say that it does not have a group generator, they need to give a reason. |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| B1: Any correct reason given.                                                                                                                                         |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| Considers the total number of elements of the group, order of the group is more than 6                                                                                |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| Considers the orders of either reflections or rotations                                                                                                               |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| (d)                                                                                                                                                                   |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| M1: Any two correct matchings                                                                                                                                         |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| M1: Completes the matching with at least 4 correct                                                                                                                    |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |
| A1: A fully specified correct isomorphism.                                                                                                                            |                         |   |   |   |   |   |   |   |   |   |   |   |     |     |

| Question                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | Marks | AOs  |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------|------|
| 2(a)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | $\begin{vmatrix} 1-\lambda & 0 & a \\ -3 & b-\lambda & 1 \\ 0 & 1 & a-\lambda \end{vmatrix}$ $= (1-\lambda)[(b-\lambda)(a-\lambda) - 1] + a(-3)(= 0)$                                                                                                                                                                                                                                                                                                                                                                                        | M1    | 1.1b |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | $\lambda^3 - (a+b+1)\lambda^2 + (a+b+ab-1)\lambda + (3a+1-ab)(= 0)$ o.e.<br>$-\lambda^3 + (a+b+1)\lambda^2 - (a+b+ab-1)\lambda + (ab-3a-1)(= 0)$ o.e.                                                                                                                                                                                                                                                                                                                                                                                        | A1    | 1.1b |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | $\lambda^2 \Rightarrow a+b+1 = 7 \text{ and } \lambda \Rightarrow a+b+ab-1 = 13$ Solves simultaneously e.g. $a+b=6, ab=8$<br>For example: leading to $a^2 - 6a + 8 = 0 \Rightarrow a = \dots$                                                                                                                                                                                                                                                                                                                                                | M1    | 3.1a |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | $a = 2, b = 4$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               | A1    | 1.1b |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | $c = -1$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | A1    | 2.2a |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | (5)   |      |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |       |      |
| (b)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               | $\mathbf{M}^3 - 7\mathbf{M}^2 + 13\mathbf{M} + \text{'their c'} \mathbf{I} = 0$                                                                                                                                                                                                                                                                                                                                                                                                                                                              | B1ft  | 1.1b |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | $I = M^3 - 7M^2 + 13M \Rightarrow M^{-1} = M^2 - 7M + 13I$ $\Rightarrow M^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}^2 - 7 \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix} = \dots$ $= \begin{pmatrix} 1 & 2 & 6 \\ -15 & 17 & 0 \\ -3 & 6 & 5 \end{pmatrix} - 7 \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix} = \dots$ | M1    | 3.1a |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | $M^{-1} = \begin{pmatrix} 7 & 2 & -8 \\ 6 & 2 & -7 \\ -3 & -1 & 4 \end{pmatrix}$                                                                                                                                                                                                                                                                                                                                                                                                                                                             | A1    | 1.1b |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | (3)   |      |
| (8 marks)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |       |      |
| <b>Notes:</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |       |      |
| <p>(a)</p> <p><b>M1:</b> Correct method to find the characteristic equation for <b>M</b>, condone missing = 0, and one slip as long as the intention is clear</p> <p><b>A1:</b> Multiplies out to achieve a correct characteristic equation, condone missing = 0</p> <p><b>M1:</b> A complete method to find the values of the constants <math>a</math> or <math>b</math>. Equates their coefficients for <math>\lambda^2</math> and <math>\lambda</math> and solves simultaneously to find values for <math>a</math> or <math>b</math>.</p> <p><b>A1:</b> Deduces the correct values for <math>a</math> and <math>b</math>. (<math>a &lt; b</math>) following correct simultaneous equations</p> <p><b>A1:</b> Deduces the correct value for <math>c</math>.</p> |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |       |      |
| <p>(b)</p> <p><b>B1ft:</b> Uses Cayley-Hamilton theorem to produce equation replacing <math>\lambda</math> with <b>M</b> and constant term with constant multiple of the identity matrix <b>I</b>. Follow through on their value for <math>c</math>. This mark may be implied by the M mark.</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |       |      |
| <p><b>M1:</b> A complete method to find <math>M^{-1}</math> using the Cayley-Hamilton theorem. The minimum is for writing an expression for <math>M^{-1}</math> from their characteristic equation, for example <math>M^{-1} = M^2 - 7M + 13I</math> and then stating an answer for <math>M^{-1}</math>, they may have used their calculator, there is no need to check.</p> <p><b>A1:</b> Correct <math>M^{-1}</math></p>                                                                                                                                                                                                                                                                                                                                        |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |       |      |

| Question   | Scheme                                                                                                                                                                                                                                                                                                     | Marks | AOs  |
|------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------|------|
| 2(a)       | $ \mathbf{A} - \lambda \mathbf{I}  = \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = (6-\lambda)[...] - (-2)[...] + 2[...] = \dots$                                                                                                                        | M1    | 1.1b |
|            | $(6-\lambda)((3-\lambda)^2 - 1) + 2(2(\lambda-3) + 2) + 2(2 - 2(3-\lambda)) (=0)$ $(\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0)$                                                                                                                                                                         | A1    | 1.1b |
|            | $= (\lambda - 2)(\lambda^2 + \dots\lambda + \dots)$                                                                                                                                                                                                                                                        | M1    | 2.1  |
|            | $= (\lambda - 2)(\lambda^2 - 10\lambda + 16) = (\lambda - 2)^2(\lambda - 8) \Rightarrow \lambda = 2 \text{ is a repeated eigenvalue } *$                                                                                                                                                                   | A1*   | 2.2a |
|            | $\lambda = 8$                                                                                                                                                                                                                                                                                              | B1    | 1.1b |
|            |                                                                                                                                                                                                                                                                                                            | (5)   |      |
| (b)        | $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \mathbf{v} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ or } \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \mathbf{v} = 8 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \mathbf{v} = \dots$ | M1    | 1.1b |
|            | Obtains any multiple of $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ for $\lambda = 8$                                                                                                                                                                                                                     | A1    | 1.1b |
|            | Obtains any (non-zero) multiple or linear combination of $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ for $\lambda = 2$                                                                                      | A1    | 1.1b |
|            | Obtains a <b>different</b> linear combination or (non-zero) multiple of different vector from $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ for $\lambda = 2$                                                 | A1    | 3.1a |
|            |                                                                                                                                                                                                                                                                                                            | (4)   |      |
| (c)        | Forms a matrix with their eigenvectors as columns                                                                                                                                                                                                                                                          | M1    | 1.2  |
|            | E.g. $\begin{pmatrix} -1 & 1 & 2 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{pmatrix}$                                                                                                                                                                                                                                 | A1ft  | 1.1b |
|            |                                                                                                                                                                                                                                                                                                            | (2)   |      |
| (11 marks) |                                                                                                                                                                                                                                                                                                            |       |      |



## Notes

(a)

M1: Attempts to expand the determinant to find the characteristic polynomial.

Note: other methods of expanding the determinant are possible. If unsure send to review.

A1: Correct expansion need not be simplified. (Need not set equal to zero) Allow recovery of missing brackets if indicated by later working.

M1: Attempts to take out a factor of  $(\lambda - 2)$  of their equation (may first expand to cubic or may spot the factor and take out without full expansion). E.g

$$(6-\lambda)((3-\lambda)^2-1)+2(2(\lambda-3)+2)+2(2-2(3-\lambda))=(6-\lambda)(4-\lambda)(2-\lambda)+4(\lambda-2)+4(\lambda-2)$$

$$=(\lambda-2)((6-\lambda)(4-\lambda)+4+4)$$

This is for a method that will allow  $\lambda$  to be shown as a repeated eigenvalue, so just stating two solutions is not sufficient, factorisation must be seen.A1\*: Obtains a correct factor of  $(\lambda - 2)^2$  and deduces that 2 is a repeated eigenvalue. Must see **statement** about 2 being repeated. (Just listing 2 twice is not sufficient.)

B1: (Note this is A1 on ePEN) Obtains and identifies 8 as the other eigenvalue (B0 if not identified in (a) but full marks can be scored in (b) and (c) for use of 8 as eigenvalue)

(b)

M1: Uses a correct method to find at least one eigenvector

A1: Obtains one correct eigenvector for  $\lambda = 8$ A1: Obtains one correct eigenvector for  $\lambda = 2$ A1: Obtains two correct linearly independent eigenvectors for  $\lambda = 2$ 

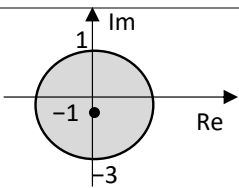
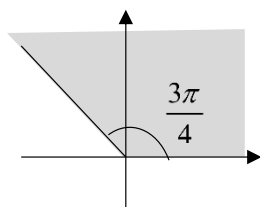
Note some other common eigenvectors for  $\lambda = 2$  are  $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

(c)

M1: Forms a matrix with their three **different non-zero** eigenvectors as columns or with their normalised (or any scaled version) of their eigenvectors.A1ft: Correct matrix with the eigenvectors (normalised/scaled) as columns in any order (follow through their three **different** vectors which are **not multiples** of any other)

| Question      | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                             | Marks      | AOs  |
|---------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|------|
| <b>4(a)</b>   | $2 \times 4 - 1 \times 2 + 2 \times 0 = \lambda \times 2 \Rightarrow \lambda = \dots$ or<br>$2 \times 0 - 1 \times -2 + 2 \times 2 = \lambda \times 2 \Rightarrow \lambda = \dots$<br><b>Alternative 1</b><br>$\begin{pmatrix} 4 & 2 & 0 \\ 2 & p & -2 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -p \\ 6 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and deduces a value for $\lambda$ | <b>M1</b>  | 1.1b |
|               | <b>Alternative 2</b><br>$\begin{pmatrix} 4-\lambda & 2 & 0 \\ 2 & p-\lambda & -2 \\ 0 & -2 & 2-\lambda \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 0 \Rightarrow \begin{matrix} 2(4-\lambda) - 2 = 0 \\ 2 + 2(2-\lambda) = 0 \end{matrix} \Rightarrow \lambda = \dots$                                                                                                                                                                              |            |      |
|               | $\lambda = 3$                                                                                                                                                                                                                                                                                                                                                                                                                                                      | <b>A1</b>  | 2.2a |
|               |                                                                                                                                                                                                                                                                                                                                                                                                                                                                    | <b>(2)</b> |      |
| <b>(b)</b>    | $2 \times 2 - p - 2 \times 2 = 3 \times -1 \Rightarrow 4 - p - 4 = -3 \Rightarrow p = 3^*$<br><b>Alternative 1</b><br>$(4-3)((p-3)(2-3)-4) - 2(2(2-3)-0) + 0 = 0 \Rightarrow p = 3^*$<br><b>Using Alternative 2 from (a)</b><br>$4 - (p - \lambda) - 4 = 0$ uses $\lambda = 3$ to show $p = 3^*$                                                                                                                                                                   | <b>B1*</b> | 1.1b |
|               |                                                                                                                                                                                                                                                                                                                                                                                                                                                                    | <b>(1)</b> |      |
| <b>(c)(i)</b> | $(4-\lambda)((3-\lambda)(2-\lambda)-4) - 2(2(2-\lambda)-0) + 0 = 0$                                                                                                                                                                                                                                                                                                                                                                                                | <b>M1</b>  | 1.1b |
|               | $\Rightarrow (4-\lambda)(\lambda^2 - 5\lambda + 2) - 8 + 4\lambda = 0 \Rightarrow \lambda^3 - 9\lambda^2 + 18\lambda = 0$<br>$\Rightarrow \lambda(\lambda-3)(\lambda-6) = 0 \Rightarrow \lambda = \dots$                                                                                                                                                                                                                                                           | <b>M1</b>  | 1.1b |
|               | Remaining eigenvalues are 0 and 6.                                                                                                                                                                                                                                                                                                                                                                                                                                 | <b>A1</b>  | 1.1b |
|               |                                                                                                                                                                                                                                                                                                                                                                                                                                                                    | <b>(3)</b> |      |
| <b>(ii)</b>   | $\left. \begin{matrix} 4x + 2y = 0 \\ 2x + 3y - 2z = 0 \\ -2y + 2z = 0 \end{matrix} \right\} \text{ or } \left. \begin{matrix} -2x + 2y = 0 \\ 2x - 3y - 2z = 0 \\ -2y - 4z = 0 \end{matrix} \right\} \Rightarrow x = \dots, y = \dots, z = \dots$                                                                                                                                                                                                                 | <b>M1</b>  | 2.1  |
|               | $c \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \text{ or } d \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \text{ o.e.}$                                                                                                                                                                                                                                                                                                                                               | <b>A1</b>  | 1.1b |

|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |                                                                                                                                                                                                                                                          |             |      |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------|------|
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | $c \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \text{ and } d \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \text{ o.e.}$                                                                                                                                    | <b>A1</b>   | 1.1b |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |                                                                                                                                                                                                                                                          | <b>(3)</b>  |      |
| <b>(d)</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | $\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix} \text{ or equivalent}$                                                        | <b>B1ft</b> | 2.2a |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | $\sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3 \Rightarrow \mathbf{v} = \dots$                                                                                                                                                                                | <b>M1</b>   | 3.1a |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | $\mathbf{P} = \frac{1}{3} \begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix} \text{ or } \mathbf{P} = \frac{1}{3} \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{pmatrix}$<br>The order must correspond to their <b>D</b> | <b>A1ft</b> | 1.1b |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |                                                                                                                                                                                                                                                          | <b>(3)</b>  |      |
| <b>(12 marks)</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |                                                                                                                                                                                                                                                          |             |      |
| <b>Notes:</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |                                                                                                                                                                                                                                                          |             |      |
| <p><b>(a)</b></p> <p><b>M1:</b> Uses either the first or third row of matrix with the eigenvector to form and solve an equation in <math>\lambda</math>. May see the full matrix equation used, but this is not necessary.</p> <p>Alternative 1: multiplies the matrix by the eigenvector and compares to a multiple of the eigenvector to deduce a value for <math>\lambda</math></p> <p>Alternative 2: Uses <math>\begin{pmatrix} 4-\lambda &amp; 2 &amp; 0 \\ 2 &amp; p-\lambda &amp; -2 \\ 0 &amp; -2 &amp; 2-\lambda \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 0</math> to find an equation and find a value for <math>\lambda</math></p> <p><b>A1:</b> For the correct eigenvalue of 3.</p> <p><b>(b)</b></p> <p><b>B1*:</b> Uses the second row with eigenvector 3 to set up a correct equation in <math>p</math> and then solves correctly.</p> <p>Alternative 1: finds the characteristic equations, use <math>\lambda = 3</math> and correctly shows that <math>p = 3</math></p> <p>Alternative 2: Uses Alternative 2 from (a) and <math>\lambda = 3</math> to correctly show that <math>p = 3</math></p> <p><b>(c)(i)</b></p> <p><b>M1:</b> Attempts the characteristic equation. Allow sign slips.</p> <p><b>M1:</b> Expands, simplifies and factorises to find the values, or equivalent method (e.g. solve by calculator).</p> <p><b>A1:</b> Correct remaining values.</p> <p><b>(c)(ii)</b></p> <p><b>M1:</b> Correct method for one of the two required eigenvectors.</p> <p><b>A1:</b> One correct eigenvector. Allow any non-zero scalar multiple.</p> <p><b>A1:</b> Both correct eigenvectors. Allow any non-zero scalar multiple.</p> <p><b>(d)</b></p> <p><b>B1ft:</b> Correct diagonal matrix, following through on their eigenvalues.</p> <p><b>M1:</b> Normalises the eigenvectors.</p> <p><b>A1ft:</b> Forms correct matrix <b>P</b> with columns in appropriate order for their <b>D</b>, following through on their eigenvectors.</p> |                                                                                                                                                                                                                                                          |             |      |

| Question                                                                                                                                                                           | Scheme                                                                                                      | Marks                                                                                                     | AOs          |      |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------|--------------|------|
| 5(i)(a)                                                                                                                                                                            | $ x + (y - 3)i  = 2 x + yi  \Rightarrow x^2 + (y - 3)^2 = 4(x^2 + y^2)$                                     | M1                                                                                                        | 3.1a         |      |
|                                                                                                                                                                                    | $\Rightarrow x^2 + y^2 - 6y + 9 = 4x^2 + 4y^2 \Rightarrow x^2 + y^2 + 2y - 3 = 0$ in any order              | M1<br>A1                                                                                                  | 1.1b<br>1.1b |      |
|                                                                                                                                                                                    |                                                                                                             | (3)                                                                                                       |              |      |
| Alt (a)                                                                                                                                                                            | Points are twice as far from $3i$ as from $0$ so $i$ and $-3i$ are diametrically opposite points.           | M1                                                                                                        | 3.1a         |      |
|                                                                                                                                                                                    | So radius is $\frac{ i - (-3i) }{2} = 2$ and centre is $\frac{i - 3i}{2} = -i$                              | M1                                                                                                        | 1.1b         |      |
|                                                                                                                                                                                    | Hence equation is $x^2 + (y + 1)^2 = 4$                                                                     | A1                                                                                                        | 1.1b         |      |
|                                                                                                                                                                                    |                                                                                                             | (3)                                                                                                       |              |      |
| (i)(b)                                                                                                                                                                             |                            | Circle drawn with the inside shaded                                                                       | M1           | 2.2a |
|                                                                                                                                                                                    |                                                                                                             | Correct circle drawn for their equation. Implied by the position of the centre and radius. Inside shaded. | A1ft         | 3.1a |
|                                                                                                                                                                                    |                                                                                                             |                                                                                                           | (2)          |      |
| (ii)(a)                                                                                                                                                                            | A point $z$ is mapped to a point with 3 times the argument...<br>Rotate every point by 2 times the argument | B1                                                                                                        | 2.4          |      |
|                                                                                                                                                                                    | ... and with modulus as the modulus of the cube of $z$ .                                                    | B1                                                                                                        | 2.5          |      |
|                                                                                                                                                                                    |                                                                                                             | (2)                                                                                                       |              |      |
| (ii)(b)                                                                                                                                                                            |                          | Sector from $O$ along the real axis, indicated by some shading, in an anticlockwise direction             | M1           | 1.1b |
|                                                                                                                                                                                    |                                                                                                             | Correct sector, angle must be stated or implied by the diagram                                            | A1           | 2.2a |
|                                                                                                                                                                                    |                                                                                                             |                                                                                                           | (2)          |      |
| (9 marks)                                                                                                                                                                          |                                                                                                             |                                                                                                           |              |      |
| Notes:                                                                                                                                                                             |                                                                                                             |                                                                                                           |              |      |
| (i)(a)                                                                                                                                                                             |                                                                                                             |                                                                                                           |              |      |
| M1: Substitutes $z = x + yi$ into the equation and applies the modulus to obtain an equation with no $i$ 's. Must have dealt with the $i^2$ correctly. Condone not squaring the 2. |                                                                                                             |                                                                                                           |              |      |
| M1: Expands and gathers terms. Must have an $x^2$ and $y^2$ after simplifying so that it is an equation of a circle.                                                               |                                                                                                             |                                                                                                           |              |      |
| A1: Cancels common factor 3 to obtain the equation shown. ISW if they make errors trying to complete the square.                                                                   |                                                                                                             |                                                                                                           |              |      |
| Note: Any letters may be used not just $x$ and $y$                                                                                                                                 |                                                                                                             |                                                                                                           |              |      |
| Alt                                                                                                                                                                                |                                                                                                             |                                                                                                           |              |      |

**M1:** Identifies two diametrically opposite points on the circle by understanding the geometry of the situation.

**M1:** Finds the radius and centre of the circle from their points, dotted or solid line

**A1:** Correct equation need not expand, but should be in simplest form.

**(i)(b)**

**M1:** Circle drawn anywhere and the inside shaded

**A1ft:** Correct circle drawn for their equation. Implied by the position of the centre and radius. Inside shaded.

**(ii)(a)**

**B1:** Identifies that the argument triples in size.

**B1:** Identifies that the modulus scales according to the modulus of  $z$  cubed.

**(ii)(b)**

**M1:** A sector centre  $O$  and starting along the real axis and in an anticlockwise direction. Must be some shading to represent the region, dotted or solid line

**A1:** Correct sector shaded and the angle stated or implied



| Question        | Scheme                                                                                                                                                                                                                      | Marks       | AOs  |
|-----------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------|------|
| <b>8(a)(i)</b>  | $\begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & p \\ -6 & 6 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3-2p \\ 2 \end{pmatrix} = -1 \times \begin{pmatrix} 2 \\ 2p-3 \\ -2 \end{pmatrix}$ |             |      |
|                 | Corresponding eigenvalue is $-1$                                                                                                                                                                                            | <b>B1</b>   | 1.1b |
|                 |                                                                                                                                                                                                                             | <b>(1)</b>  |      |
| <b>(a)(ii)</b>  | $2p-3=1 \Rightarrow p=...$                                                                                                                                                                                                  | <b>M1</b>   | 1.1b |
|                 | $p=2^*$                                                                                                                                                                                                                     | <b>A1*</b>  | 1.1b |
|                 |                                                                                                                                                                                                                             | <b>(2)</b>  |      |
| <b>(a)(iii)</b> | $\det \begin{pmatrix} 5-\lambda & -2 & 5 \\ 0 & 3-\lambda & p \\ -6 & 6 & -4-\lambda \end{pmatrix} = 0$<br>$\Rightarrow (5-\lambda)((3-\lambda)(-4-\lambda)-12) - (-2)(12) + 5(6(3-\lambda)) = 0$                           | <b>M1</b>   | 1.1b |
|                 | $\Rightarrow \lambda^3 - 4\lambda^2 + \lambda + 6 = 0$                                                                                                                                                                      | <b>A1</b>   | 1.1b |
|                 | $(\Rightarrow (\lambda+1)(\lambda^2 - 5\lambda + 6) = 0 \Rightarrow (\lambda+1)(\lambda-2)(\lambda-3) = 0)$<br>Eigenvalues are $(-1), 2$ and $3$                                                                            | <b>A1</b>   | 1.1b |
|                 | Either $\begin{cases} 3x-2y+5z=0 \\ y+2z=0 \\ -6x+6y-6z=0 \end{cases}$ or $\begin{cases} 2x-2y+5z=0 \\ 2z=0 \\ -6x+6y-7z=0 \end{cases} \Rightarrow x/y/z=...$                                                               | <b>M1</b>   | 2.1  |
|                 | Either $k \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ (for $\lambda=2$ ) or $m \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (for $\lambda=3$ )                                                                                | <b>A1</b>   | 1.1b |
|                 | Both $\begin{cases} 3x-2y+5z=0 \\ y+2z=0 \\ -6x+6y-6z=0 \end{cases}$ and $\begin{cases} 2x-2y+5z=0 \\ 2z=0 \\ -6x+6y-7z=0 \end{cases} \Rightarrow x/y/z=...$                                                                | <b>M1</b>   | 2.1  |
|                 | Both $k \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ (for $\lambda=2$ ) and $m \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (for $\lambda=3$ )                                                                                 | <b>A1</b>   | 1.1b |
|                 |                                                                                                                                                                                                                             | <b>(7)</b>  |      |
| <b>(b)</b>      | E.g. $\mathbf{P} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ -2 & -1 & 0 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$                                               | <b>B1ft</b> | 2.2a |
|                 |                                                                                                                                                                                                                             | <b>(1)</b>  |      |
| <b>(c)(i)</b>   | $\dot{u} = ku \Rightarrow \int \frac{1}{u} du = k \int dt \Rightarrow \ln u = kt(+c)$                                                                                                                                       | <b>M1</b>   | 1.1b |

|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |                                                                                                                                                                                                                                                                                                                                                                                                             |            |      |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|------|
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | So $u = Ae^{kt}$ or $u = e^{kt+c}$                                                                                                                                                                                                                                                                                                                                                                          | <b>A1</b>  | 1.1b |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |                                                                                                                                                                                                                                                                                                                                                                                                             | <b>(2)</b> |      |
| <b>(ii)</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      | $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{P} \mathbf{D} \mathbf{P}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{D} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -u \\ 2v \\ 3w \end{pmatrix}$ | <b>M1</b>  | 3.1b |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | $\Rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} Ae^{-t} \\ Be^{2t} \\ Ce^{3t} \end{pmatrix}$                                                                                                                                                                                                                                                                                       | <b>M1</b>  | 2.2a |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | $\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{P} \begin{pmatrix} Ae^{-t} \\ Be^{2t} \\ Ce^{3t} \end{pmatrix} = \dots$                                                                                                                                                                                                                                                                    | <b>M1</b>  | 3.4  |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | $\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2Ae^{-t} + 3Be^{2t} + Ce^{3t} \\ Ae^{-t} + 2Be^{2t} + Ce^{3t} \\ -2Ae^{-t} - Be^{2t} \end{pmatrix}$                                                                                                                                                                                                                                | <b>A1</b>  | 1.1b |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |                                                                                                                                                                                                                                                                                                                                                                                                             | <b>(4)</b> |      |
| <b>(17 marks)</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |                                                                                                                                                                                                                                                                                                                                                                                                             |            |      |
| <b>Notes:</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                                                                                                                                                                                                                                                                                                                                                                             |            |      |
| <p><b>(a)(i)</b><br/> <b>B1:</b> For the correct eigenvalue of <math>-1</math></p> <p><b>(ii)</b><br/> <b>M1:</b> Correct equation with their eigenvalue set up – need only see middle equation for this.<br/> <b>A1*:</b> Correct proof (full matrix calculation not necessary).</p> <p><b>(iii)</b><br/> <b>M1:</b> Applies <math>\det(\mathbf{A} - \lambda \mathbf{I}) = 0</math> to achieve a cubic in <math>\lambda</math> (or other variable, simplification not required). Allow with <math>p</math> used instead of 2, and look for two correct “terms” in the expansion leading to a cubic as evidence of the expansion.<br/> <b>A1:</b> Correct simplified cubic. Note this may be implied by correct answers from a calculator following a correct expansion seen for the M.<br/> <b>A1:</b> Correct eigenvalues<br/> <b>M1:</b> Forms and solves eigenvector equations for at least one (other than <math>-1</math>) eigenvalue.<br/> <b>A1:</b> One correct (other) eigenvector<br/> <b>M1:</b> Both eigenvectors attempted.<br/> <b>A1:</b> Both (other) eigenvectors correct.</p> |                                                                                                                                                                                                                                                                                                                                                                                                             |            |      |
| <p><b>(b)</b><br/> <b>B1ft:</b> A correct corresponding <b>P</b> and <b>D</b>, follow through on their answer to (a). Columns may be in different order, but should be consistent for their <b>P</b> and <b>D</b>.</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                                                                                                                                                                                                                                                                                                                                                             |            |      |
| <p><b>(c)(i)</b><br/> <b>M1:</b> Separates variables and attempts the integration (constant not required).<br/> <b>A1:</b> Correct answer for <math>u = \dots</math>, either form, including constant of integration</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         |                                                                                                                                                                                                                                                                                                                                                                                                             |            |      |

(ii)

**NB different orderings of the columns of  $\mathbf{P}$  and  $\mathbf{D}$  will give the terms in different orders here.**

**M1:** Uses their  $\mathbf{P}$  and  $\mathbf{D}$  to transform system into equation in  $u$ ,  $v$  and  $w$  (may be implied).

**M1:** Forms the solution for  $u$ ,  $v$  and  $w$  using their eigenvalues.

**M1:** Reverses the substitution (multiplies by their  $\mathbf{P}$ ) to get solution for  $x$ ,  $y$  and  $z$ .

**A1:** Correct answer, in matrix form or as separate equations – award when first seen and isw.



| Question                                                                                                                                                                                                                                                                                                                                                                                                  | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                  | Marks                  | AOs          |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------|--------------|
| <b>2 (a)</b>                                                                                                                                                                                                                                                                                                                                                                                              | $\begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = 0$ leading to $(1-\lambda)(4-\lambda) - 4 = 0$                                                                                                                                                                                                                                                                                                                        | <b>M1</b>              | 1.1b         |
|                                                                                                                                                                                                                                                                                                                                                                                                           | Solves $\lambda^2 - 5\lambda = 0 \Rightarrow \lambda = \dots$                                                                                                                                                                                                                                                                                                                                                                           | <b>dM1</b>             | 1.1b         |
|                                                                                                                                                                                                                                                                                                                                                                                                           | $\lambda = 5, 0$                                                                                                                                                                                                                                                                                                                                                                                                                        | <b>A1</b>              | 1.1b         |
|                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                                                                                                                                                                                                                                                                                                                                                                                         | <b>(3)</b>             |              |
| <b>(b)</b>                                                                                                                                                                                                                                                                                                                                                                                                | $\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \text{their '0'} \begin{pmatrix} x \\ y \end{pmatrix}$ leading to $x - 2y = 0 \Rightarrow x = 2y$<br>or<br>$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \text{their '5'} \begin{pmatrix} x \\ y \end{pmatrix}$<br>leading to $x - 2y = 5x \Rightarrow 2y = -4x$ or $-2x + 4y = 5y \Rightarrow y = -2x$ | <b>M1</b>              | 3.1a         |
|                                                                                                                                                                                                                                                                                                                                                                                                           | Eigenvectors $\lambda = 0: \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda = 5: \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ oe                                                                                                                                                                                                                                                                                                            | <b>A1</b><br><b>A1</b> | 2.2a<br>2.2a |
|                                                                                                                                                                                                                                                                                                                                                                                                           | <b>eg</b> $\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{pmatrix}$ <b>or</b> $\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$                                                                                                                                                                                                                        | <b>B1ft</b>            | 2.2a         |
|                                                                                                                                                                                                                                                                                                                                                                                                           | <b>eg</b> $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$                                                                                                                                                                                                                              | <b>A1ft</b>            | 2.5          |
|                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                                                                                                                                                                                                                                                                                                                                                                                         | <b>(5)</b>             |              |
| <b>(8 marks)</b>                                                                                                                                                                                                                                                                                                                                                                                          |                                                                                                                                                                                                                                                                                                                                                                                                                                         |                        |              |
| <b>Notes:</b>                                                                                                                                                                                                                                                                                                                                                                                             |                                                                                                                                                                                                                                                                                                                                                                                                                                         |                        |              |
| <b>(a)</b><br><b>M1:</b> Attempts $\begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = 0$<br><b>dM1:</b> Dependent on the previous method mark. Solves their 3TQ to find a value for $\lambda$<br><b>A1:</b> Correct values for $\lambda$                                                                                                                                                    |                                                                                                                                                                                                                                                                                                                                                                                                                                         |                        |              |
| <b>(b)</b><br><b>Note this appears on open as M1A1A1A1A1 but is being marked as M1A1A1B1A1</b><br><b>M1:</b> Uses $\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \text{'their } \lambda \text{' } \begin{pmatrix} x \\ y \end{pmatrix}$ to form an equation of the form $ax = by$ for at least one of their eigenvalues. May be implied by a correct eigenvector. |                                                                                                                                                                                                                                                                                                                                                                                                                                         |                        |              |

**A1:** Deduces one correct eigenvector for one of the (correct) eigenvalues. Accept any non-zero multiple.

**A1:** Deduces both correct eigenvectors for the correct eigenvalues. Accept any non-zero multiples.

**B1ft:** Note: not dependent on the method mark. Deduces a correct matrix **P** or **D**, following through on their eigenvalues or **non-zero** eigenvectors. This may be scored for a correct **D** even if no attempt at the eigenvectors has been made, or a correct f.t. **P** even if the method for eigenvectors was incorrect.

**A1ft: Depends on the M having been scored.** Correct matrices **D** and **P** which are consistent. Follow through on their eigenvalues and **non-zero** eigenvectors.

Note if they assume the eigenvector for  $\lambda = 0$  is **0** then do not allow this for the follow through mark(s) (though the first may be gained for correct **D**).

SC: If they mislabel **P** and **D** then allow A1ftA0ft for both correct but the wrong order, but A0A0 if only one is “correct” but wrong order.



| Question    | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | Marks | AOs  |
|-------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------|------|
| <b>1(a)</b> | $\det \begin{pmatrix} -1-\lambda & a \\ 3 & 8-\lambda \end{pmatrix} = 0 \Rightarrow (-1-\lambda)(8-\lambda) - 3a = 0 \Rightarrow \dots$                                                                                                                                                                                                                                                                                                                                                                       | M1    | 1.1b |
|             | $\lambda^2 - 7\lambda - 3a - 8 = 0$ o.e.                                                                                                                                                                                                                                                                                                                                                                                                                                                                      | A1    | 1.1b |
|             |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               | (2)   |      |
| <b>(b)</b>  | $\mathbf{A}^2 - 7\mathbf{A} - 3a\mathbf{I} - 8\mathbf{I} = 0 \Rightarrow \mathbf{A}^3 = 7\mathbf{A}^2 + (3a+8)\mathbf{A}$                                                                                                                                                                                                                                                                                                                                                                                     | M1    | 1.1b |
|             | $\Rightarrow \mathbf{A}^3 = 7(7\mathbf{A} + (3a+8)\mathbf{I}) + (3a+8)\mathbf{A}$<br>Or<br>$\mathbf{A}^3 = 7 \begin{pmatrix} -1 & a \\ 3 & 8 \end{pmatrix}^2 + (8+3a) \begin{pmatrix} -1 & a \\ 3 & 8 \end{pmatrix}$<br>$= 7 \begin{pmatrix} 1+3a & 7a \\ 21 & 3a+64 \end{pmatrix} + \begin{pmatrix} -8-3a & 8a+3a^2 \\ 24+9a & 64+24a \end{pmatrix}$                                                                                                                                                         | M1    | 2.1  |
|             | $\Rightarrow \mathbf{A}^3 = (3a+8+49)\mathbf{A} + 7(3a+8)\mathbf{I} \Rightarrow 3a+57=1 \Rightarrow a = \dots$<br>Or<br>$\begin{pmatrix} b-1 & a \\ 3 & b+8 \end{pmatrix} = \begin{pmatrix} 18a-1 & 3a^2+57a \\ 171+9a & 512+27a \end{pmatrix} \Rightarrow \text{e.g. } 3=171+9a \Rightarrow a = \dots$<br>Or<br>$\mathbf{A}^3 = \begin{pmatrix} -1 & a \\ 3 & 8 \end{pmatrix} + \begin{pmatrix} 18a & 3a^2+56a \\ 168+9a & 504+27a \end{pmatrix} \Rightarrow \text{e.g. } 18a=504+27a \Rightarrow a = \dots$ | M1    | 1.1b |
|             | $\Rightarrow a = -\frac{56}{3}, b = -336$                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | A1    | 1.1b |
|             |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               | (4)   |      |
|             | <b>(6 marks)</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |       |      |

**Notes:****(a)****M1:** Correct method to find the characteristic equation for  $\mathbf{A}$ , condone missing  $= 0$ **A1:** Correct simplified characteristic equation.**(b)****Note: this question asks the candidates to use the Cayley-Hamilton theorem so any other approach that doesn't score the first method mark scores no marks****M1:** Uses the Cayley-Hamilton theorem with their equation and multiplies through by  $\mathbf{A}$  to find an equation for  $\mathbf{A}^3$ **M1:** Substitutes for  $\mathbf{A}^2$  to obtain an equation for  $\mathbf{A}^3$  in terms of  $a$ ,  $\mathbf{I}$  and  $\mathbf{A}$ . Alternatively substitutes in the matrix  $\mathbf{A}$  and attempts to square**M1:** Equates coefficient(s) of  $\mathbf{A}$  to 1 and proceeds to find a value for  $a$ . Alternatively equates elements to find a value for  $a$ .**A1:** Correct values for  $a$  and  $b$ .**Special case:** Missing matrix  $\mathbf{I}$  from their working can score maximum of M1 (if multiply by  $\mathbf{A}$  correctly) M1M1A0 unless implied  $\mathbf{I}$  from their working this can score all marks.