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EDEXCEL A LEVEL FURTHER MATHEMATICS: QUESTIONS BY TOPIC

Further Complex Numbers (4)

Further Pure Mathematics 2 · Topic 4

8 questions · 68 marks · Mark schemes included

Pearson Edexcel A Level Further Mathematics (9FM0), June 2019 - June 2025

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HOW TO USE THIS BOOKLET

Questions are grouped by specification sub-topic and run from the lowest mark tariff to the highest within each one.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme.

Diagrams, graphs, tables and data are printed exactly as they appeared in the original papers.

QUESTION	SOURCE	MARKS	PAGE
3.2 DIAGONALISATION		1 Qs · 17 marks	3
1. Q8 <i>also Topic 3</i>	Option 4A: Further Pure Mathematics 2 November2021	17	4
4.1 LOCI AND REGIONS IN THE ARGAND DIAGRAM		5 Qs · 37 marks	9
2. Q1	Option 4A: Further Pure Mathematics 2 June2019	5	10
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QUESTIONS

8 questions · 68 marks · 3 sub-topics



SUB-TOPIC 3.2

DIAGONALISATION

Sub-topic 3.2 · 1 question · 17 marks · lowest tariff first



8.

$$\mathbf{A} = \begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & p \\ -6 & 6 & -4 \end{pmatrix} \quad \text{where } p \text{ is a constant}$$

Given that $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ is an eigenvector for $\mathbf{A}$

(a) (i) determine the eigenvalue corresponding to this eigenvector

(1)

(ii) hence show that $p = 2$

(2)

(iii) determine the remaining eigenvalues and corresponding eigenvectors of $\mathbf{A}$

(7)

(b) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$

(1)

(c) (i) Solve the differential equation $\dot{u} = ku$, where k is a constant.

(2)

With respect to a fixed origin O , the velocity of a particle moving through space is modelled by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

By considering $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so that $\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$

(ii) determine a general solution for the displacement of the particle.

(4)





Question 8 continued





SUB-TOPIC 4.1

LOCI AND REGIONS IN THE ARGAND DIAGRAM

Sub-topic 4.1 · 5 questions · 37 marks · lowest tariff first



(Total for Question 1 is 5 marks)



Question 2 continued

[illegible]

Question 2 continued



Question 2 continued

(Total for Question 2 is 6 marks)



8. The locus of points $z = x + iy$ that satisfy

$$\arg\left(\frac{z-8-5i}{z-2-5i}\right) = \frac{\pi}{3}$$

is an arc of a circle C .

- On an Argand diagram sketch the locus of z .
(2)
- Explain why the centre of C has x coordinate 5
(1)
- Determine the radius of C .
(2)
- Determine the y coordinate of the centre of C .
(2)

(Total for Question 8 is 7 marks)



5. The point P in the complex plane represents a complex number z such that

$$|z + 9| = 4|z - 12i|$$

Given that, as z varies, the locus of P is a circle,

(a) determine the centre and radius of this circle.

(6)

(b) Shade on an Argand diagram the region defined by the set

$$\{z \in \mathbb{C} : |z+9| < 4|z-12i|\} \cap \left\{z \in \mathbb{C} : -\frac{\pi}{4} < \arg\left(z - \frac{3+44i}{5}\right) < \frac{\pi}{4}\right\}$$

(4)



Question 5 continued



Question 5 continued

(Total for Question 5 is 10 marks)





Question 3 continued



Question 3 continued

SUB-TOPIC 4.2

TRANSFORMATIONS OF THE COMPLEX PLANE

Sub-topic 4.2 · 2 questions · 14 marks · lowest tariff first



Question 7 continued



Question 7 continued



Question 7 continued

(Total for Question 7 is 6 marks)



9. The transformation T from the z -plane to the w -plane, where $z = x + iy$ and $w = u + iv$ is given by

$$w = \frac{z}{z + 2i} \quad z \neq -2i$$

The circle with equation $|z| = 4$ is mapped by T onto the circle C .

Determine

- (i) the coordinates of the centre of C
- (ii) the radius of C

(8)



Question 9 continued



Question 9 continued

(Total for Question 9 is 8 marks)



MARK SCHEMES

official mark schemes



Question	Scheme	Marks	AOs
8(a)(i)	$\begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & p \\ -6 & 6 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3-2p \\ 2 \end{pmatrix} = -1 \times \begin{pmatrix} 2 \\ 2p-3 \\ -2 \end{pmatrix}$		
	Corresponding eigenvalue is -1	B1	1.1b
		(1)	
(a)(ii)	$2p-3=1 \Rightarrow p=...$	M1	1.1b
	$p=2^*$	A1*	1.1b
		(2)	
(a)(iii)	$\det \begin{pmatrix} 5-\lambda & -2 & 5 \\ 0 & 3-\lambda & p \\ -6 & 6 & -4-\lambda \end{pmatrix} = 0$ $\Rightarrow (5-\lambda)((3-\lambda)(-4-\lambda)-12) - (-2)(12) + 5(6(3-\lambda)) = 0$	M1	1.1b
	$\Rightarrow \lambda^3 - 4\lambda^2 + \lambda + 6 = 0$	A1	1.1b
	$(\Rightarrow (\lambda+1)(\lambda^2 - 5\lambda + 6) = 0 \Rightarrow (\lambda+1)(\lambda-2)(\lambda-3) = 0)$ Eigenvalues are $(-1), 2$ and 3	A1	1.1b
	Either $\begin{cases} 3x-2y+5z=0 \\ y+2z=0 \\ -6x+6y-6z=0 \end{cases}$ or $\begin{cases} 2x-2y+5z=0 \\ 2z=0 \\ -6x+6y-7z=0 \end{cases} \Rightarrow x/y/z=...$	M1	2.1
	Either $k \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ (for $\lambda=2$) or $m \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (for $\lambda=3$)	A1	1.1b
	Both $\begin{cases} 3x-2y+5z=0 \\ y+2z=0 \\ -6x+6y-6z=0 \end{cases}$ and $\begin{cases} 2x-2y+5z=0 \\ 2z=0 \\ -6x+6y-7z=0 \end{cases} \Rightarrow x/y/z=...$	M1	2.1
	Both $k \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ (for $\lambda=2$) and $m \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (for $\lambda=3$)	A1	1.1b
		(7)	
(b)	E.g. $\mathbf{P} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ -2 & -1 & 0 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$	B1ft	2.2a
		(1)	
(c)(i)	$\dot{u} = ku \Rightarrow \int \frac{1}{u} du = k \int dt \Rightarrow \ln u = kt(+c)$	M1	1.1b

	So $u = Ae^{kt}$ or $u = e^{kt+c}$	A1	1.1b
		(2)	
(ii)	$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{P} \mathbf{D} \mathbf{P}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{D} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -u \\ 2v \\ 3w \end{pmatrix}$	M1	3.1b
	$\Rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} Ae^{-t} \\ Be^{2t} \\ Ce^{3t} \end{pmatrix}$	M1	2.2a
	$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{P} \begin{pmatrix} Ae^{-t} \\ Be^{2t} \\ Ce^{3t} \end{pmatrix} = \dots$	M1	3.4
	$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2Ae^{-t} + 3Be^{2t} + Ce^{3t} \\ Ae^{-t} + 2Be^{2t} + Ce^{3t} \\ -2Ae^{-t} - Be^{2t} \end{pmatrix}$	A1	1.1b
		(4)	
(17 marks)			
Notes:			
(a)(i) B1: For the correct eigenvalue of -1 (ii) M1: Correct equation with their eigenvalue set up – need only see middle equation for this. A1*: Correct proof (full matrix calculation not necessary). (iii) M1: Applies $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ to achieve a cubic in λ (or other variable, simplification not required). Allow with p used instead of 2, and look for two correct “terms” in the expansion leading to a cubic as evidence of the expansion. A1: Correct simplified cubic. Note this may be implied by correct answers from a calculator following a correct expansion seen for the M. A1: Correct eigenvalues M1: Forms and solves eigenvector equations for at least one (other than -1) eigenvalue. A1: One correct (other) eigenvector M1: Both eigenvectors attempted. A1: Both (other) eigenvectors correct.			
(b) B1ft: A correct corresponding P and D, follow through on their answer to (a). Columns may be in different order, but should be consistent for their P and D.			
(c)(i) M1: Separates variables and attempts the integration (constant not required). A1: Correct answer for $u = \dots$, either form, including constant of integration			



(ii)

NB different orderings of the columns of $\mathbf{P}$ and $\mathbf{D}$ will give the terms in different orders here.

M1: Uses their $\mathbf{P}$ and $\mathbf{D}$ to transform system into equation in u , v and w (may be implied).

M1: Forms the solution for u , v and w using their eigenvalues.

M1: Reverses the substitution (multiplies by their $\mathbf{P}$) to get solution for x , y and z .

A1: Correct answer, in matrix form or as separate equations – award when first seen and isw.



Question	Scheme	Marks	AOs
1(a)	$(x-3)^2 + y^2 = 16((x+1)^2 + y^2)$	M1	1.1b
	$x^2 - 6x + 9 + y^2 = 16x^2 + 32x + 16 + 16y^2$ $15x^2 + 15y^2 + 38x + 7 = 0^*$	A1*	2.1
		(2)	
(b)	$15x^2 + 15y^2 + 38x + 7 = 15\left(x \pm \frac{19}{15}\right)^2 - \dots + 15y^2 + 7 = 0$	M1	2.1
	Centre is $\left(-\frac{19}{5}, 0\right)$ and radius is $\sqrt{\left(\frac{19}{15}\right)^2 - \frac{7}{15}} \left(= \frac{16}{15}\right)$	M1	2.2a
	$\max z = \frac{16}{15} + \frac{19}{15} = \frac{7}{3}$	A1	3.1a
		(3)	
(5 marks)			
Notes			
(a) M1: Obtains an equation in terms of x and y using the given information. Allow if the 4 is not squared, but i^2 must have been dealt with correctly (ie positive y^2 terms). Condone invisible brackets for the M mark. A1*: Expands and simplifies and obtains a circle equation correctly. Accept terms in different order but must include $=0$. No errors seen, so bracketing errors in solution are A0. (b) M1: Completes the square on the x term achieving $A\left(x \pm \frac{19}{15}\right)^2 - B$, or uses other appropriate method in order to attempt the radius and/or centre of the circle. Award if correct x coordinate of centre or radius is found. M1: Deduces both centre and radius for their completed square form, either seen used in work clearly as centre and radius, stated or labelled on a diagram, not just embedded within the equation. This is implied by the correct calculation being carried out for their centre and radius. A1cso: Realises the need to add distance of centre from origin to radius to achieve the correct answer. Must come from correct work. Note that completing the square as $\left(x - \frac{19}{15}\right)^2 - \dots$ can score a maximum M1M1A0			

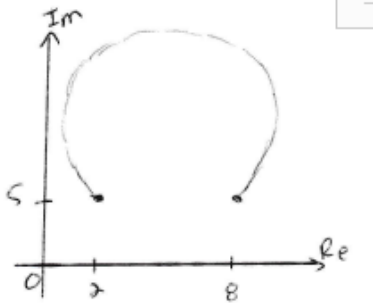
Question	Scheme	Marks	AOs	
2 (a)	$ -4-6 =10$ and $2 -4+3i =\left\{2\sqrt{4^2+3^2}=2\times 5\right\}=10$ {so -4 is on the locus} Or $ -8i-6 =\left\{\sqrt{64+36}\right\}=10$ and $2 -8i+3i =\left\{2 -5i =2\times 5\right\}=10$ {so $-8i$ is on the locus}	M1	1.1b	
	$ -4-6 =10$ and $2 -4+3i =\left\{2\sqrt{4^2+3^2}=2\times 5\right\}=10$ {so -4 is on the locus} and $ -8i-6 =\left\{\sqrt{64+36}\right\}=10$ and $2 -8i+3i =\left\{2 -5i =2\times 5\right\}=10$ {so $-8i$ is on the locus} and $ -6 =6$ and $ 3i =6$ {so the origin is also on the locus}	A1	1.1b	
		(2)		
	Alternative: Finds the Cartesian equation of the circle, any form $(x-6)^2+y^2=4x^2+4(y+3)^2$ $3x^2+3y^2+24y+12x=0$ $x^2+y^2+8y+4x=0$ $(x+2)^2+(y+4)^2=20$ Substitutes in either $(-4,0)$ or $(0,-8)$ into the equation to show it holds	M1	1.1b	
	Substitutes in $(-4,0)$ and $(0,-8)$ and $(0,0)$ into the equation to show it holds	A1	1.1b	
		(2)		
(b)+(c)		(b) Circle passing through the origin	M1	2.2a
		Correct circle, with axis intercepts	A1	1.1b
			(2)	
		(c) Circle with centre $(0,0)$ and radius 4	B1	1.1b
		Shades area inside both circles.	B1ft	2.5
			(2)	
(6 marks)				

Notes:**(a)****M1:** Verifies the equation is satisfied by at least one of the non-zero points, with evidence of correct method of the modulus seen (e.g. implied by correct value).**A1:** All three points correctly checked to be on the locus.**Alternative****M1:** Finds the Cartesian equation of the circle and verifies that the equation is satisfied by at least one of the non-zero points.**A1:** All three points correctly checked to be on the locus.**Special case: M1A0** Candidates shows that the locus of P passes through $z = -4 - 8i$

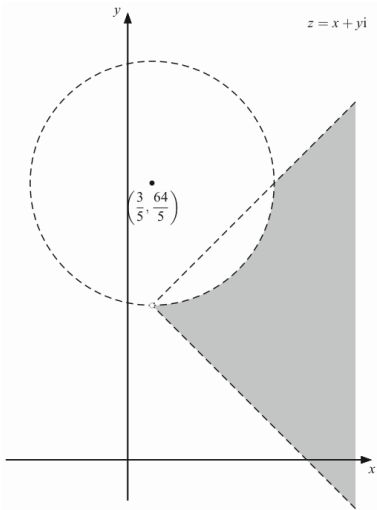
$$|-10 - 8i| = \{\sqrt{100 + 64} =\} 2\sqrt{41}$$

$$2|-4 - 5i| = \{\sqrt{16 + 25} =\} 2\sqrt{41}$$

Cartesian approach substitutes $(-4, -8)$ to show it satisfies the equation**(b)****M1:** Any complete circle passing through the origin.**A1:** Correct circle drawn and labelled. Look for centre in the third quadrant and passing through -4 , 0 and $-8i$.**(c)****B1:** Circle centre at origin and some indication that the radius is 4. Can sketch part of the circle as long as sufficient to show the overlap.**B1ft:** Area inside both 'circles' shaded. Allow as long as they have overlapping circles.

Question	Scheme	Marks	AOs
8 (a)	 The major arc of a circle drawn anywhere The end points of their arc (2, 5) and (8, 5) and the arc drawn above the coordinates	B1	1.1b
		B1	1.1b
		(2)	
(b)	The centre lies on the perpendicular bisector/midpoint/equidistant of 2 and 8	B1	2.4
		(1)	
(c)	A complete method to find the radius of the circle $\sin\left(\frac{\pi}{3}\right) = \frac{3}{r} \Rightarrow r = \dots$ Or $6^2 = r^2 + r^2 - 2 \times r \times r \times \cos\left(\frac{2\pi}{3}\right) \Rightarrow r = \dots$ Or $h = \frac{3}{\tan\left(\frac{\pi}{3}\right)} = \sqrt{3} \Rightarrow r = \sqrt{(\sqrt{3})^2 + 3^2} = \dots$ or $\tan[\arg(x + yi - (8 + 5i)) - \arg(x + yi - (2 + 5i))] = \tan\left(\frac{\pi}{3}\right)$ $\frac{\tan[\arg(x + yi - (8 + 5i))] - \tan[\arg(x + yi - (2 + 5i))]}{1 + \tan[\arg(x + yi - (8 + 5i))] \tan[\arg(x + yi - (2 + 5i))]} = \sqrt{3}$ $\frac{\frac{y-5}{x-8} - \frac{y-8}{x-2}}{1 + \frac{y-5}{x-8} \times \frac{y-8}{x-2}} = \sqrt{3}$ Leading to an equation of a circle by completing the square $(x - a)^2 + (y - b)^2 = r^2$ leading to $r = \dots$	M1	3.1a
	$r = \frac{6}{\sqrt{3}}$ or $2\sqrt{3}$ o.e.	A1	1.1b
		(2)	
(d)	$y = 5 + h$ where $h = \frac{3}{\tan(\frac{\pi}{3})}$ or $h = '2\sqrt{3}' \cos\left(\frac{\pi}{3}\right)$ or $h = \sqrt{('2\sqrt{3}')^2 - 3^2}$	M1	3.1a

	$y = 5 + \sqrt{3}$	A1	2.2a
		(2)	
(7 Marks)			
Notes:			
(a) B1: Major arc drawn anywhere B1: Correct end points for their arc drawn above the end points, condone written as complex numbers			
(b) B1: States perpendicular bisector or midpoint of 2 and 8. Condone “in between 2 and 8” if they write $\frac{2+8}{2} = 5$ Note: $\frac{2+8}{2} = 5$ on its own is B0 In between 2 and 8 on its own is B0			
(c) M1: A complete method to find the radius of the circle A1: Correct radius			
(d) M1: Any correct complete strategy. If they attempt to find the height (even if incorrect method) in part (c) then $y = 5 +$ their height A1: Correct answer			

Question	Scheme	Marks	AOs	
5(a)	$z = x + iy \Rightarrow x + 9 + iy = 4 x + (y - 12)i $	M1	1.1b	
	$\Rightarrow (x + 9)^2 + y^2 = 16(x^2 + (y - 12)^2)$	M1 A1	1.1b 1.1b	
	$\Rightarrow 15x^2 + 15y^2 - 18x - 384y = 81 - 16 \times 12^2$ $\Rightarrow x^2 + y^2 - \frac{6}{5}x - \frac{128}{5}y = -\frac{741}{5}$			
	$\Rightarrow \left(x - \frac{3}{5}\right)^2 - \left(\frac{3}{5}\right)^2 + \left(y - \frac{64}{5}\right)^2 - \left(\frac{64}{5}\right)^2 = -\frac{741}{5}$ $\left(\Rightarrow \left(x - \frac{3}{5}\right)^2 + \left(y - \frac{64}{5}\right)^2 = 16\right)$	M1	2.1	
	centre $\frac{3}{5} + \frac{64}{5}i$ or radius 4	A1	2.2a	
	centre $\frac{3}{5} + \frac{64}{5}i$ and radius 4	A1	2.2a	
		(6)		
(b)		Circle, with centre in correct quadrant for their answer to (a)	M1	1.1b
		Pair of rays at roughly 45° to horizontal, with source in first quadrant OR on the circle.	M1	1.1b
		Correct circle and rays, circle with centre in first quadrant and spanning only quadrant 1 and 2 and pair of rays at roughly 45° to horizontal, meeting at the bottom point of the circle	A1	3.1a
		Region between rays and outside circle shaded	B1ft	3.1a
		(4)		
(10 marks)				
Notes:				
(a)				
M1: Applies $z = x + iy$ to the given equation. Use of other letters, eg $z = u + iv$ is fine.				
M1: Squares and uses modulus to achieve $(x + a)^2 + y^2 = K(x^2 + (y + b)^2)$				
A1: Correct equation, need not be expanded. Award when first seen.				
M1: Expands, gathers terms and completes the square.				

A1: Either centre or radius correct. Accept coordinates for centre.

A1: Correct centre and radius. Accept coordinates for centre.

(b)

M1: Sketches their circle on an Argand diagram. Look for the centre being in the correct quadrant for their answer to (a).

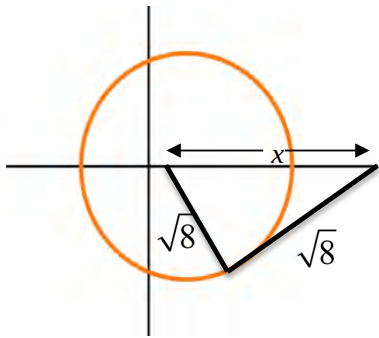
M1: Pair of rays added to the sketch, at angles $\frac{\pi}{4}$ above and below the horizontal with vertex in the first quadrant OR somewhere on the circle. Need not stem from base of circle for this mark if it stems from the first quadrant, but if not in the first quadrant it must stem from the circle.

A1: Circle (or arc) in correct position, centre in first quadrant that would span quadrants 1 and 2, with pair of rays at roughly 45° to horizontal, meeting at the bottom point of the circle.

B1ft: Area outside the circle and between the rays (minor sector) shaded provided the rays span approximately a 90° sector.

NB Only the region is asked for, so allow the marks above if only the relevant part of the circle is shown.



Question	Scheme	Marks	AOs
3(a)	$(x-5)^2 + (y+4)^2 = 4[(x-2)^2 + (y+1)^2]$ or $\sqrt{(x-5)^2 + (y+4)^2} = 2\sqrt{(x-2)^2 + (y+1)^2}$	M1	1.1b
	$(x-5)^2 + (y+4)^2 = 4[(x-2)^2 + (y+1)^2]$	A1	1.1b
	$3x^2 - 6x + 3y^2 - 21 = 0$ $x^2 - 2x + y^2 - 7 = 0 *$	A1*	2.1
		(3)	
(b)	$y = x - a$	B1	2.2a
	$x^2 - 2x + (x-a)^2 - 7 = 0$ leading to a 3TQ	M1	2.1
	$2x^2 - (2+2a)x + a^2 - 7 = 0$	A1	1.1b
	$b^2 - 4ac = (2+2a)^2 - 4(2)(a^2 - 7) = 0$	M1	3.1a
	Solve 3TQ $4a^2 - 8a - 60 = 0 \Rightarrow a = \dots$	dM1	1.1b
	$a = 5$ only [$a = -3$ must be clearly rejected]	A1	3.2a
		(6)	
	Alternative $x^2 - 2x + y^2 - 7 = 0 \Rightarrow (x-1)^2 + y^2 = 8$	B1	2.2a
	Centre (1, 0) & radius = $\sqrt{8}$	M1 A1	2.1 1.1b
			
	E.g. $x^2 = '8' + '8' \Rightarrow x = \dots\{4\}$	M1	3.1a
	$a = '1' + '4'$	dM1	1.1b
	$a = 5$	A1	3.2a
		(6)	
(9 marks)			

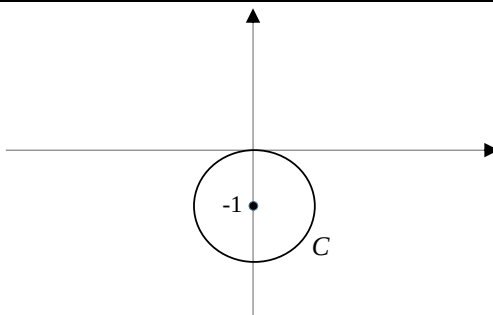
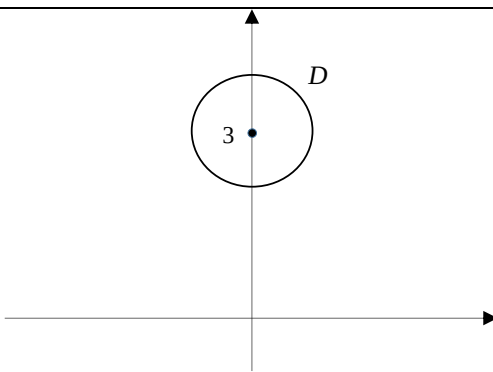
Notes**(a)****M1:** Obtains an equation in terms of x and y using the given information. Condone $(x-5)^2 + (y+4)^2 = 2[(x-2)^2 + (y+1)^2]$ for this mark and condone e.g. $(y \pm 4)^2$ as slip but **must** be + between the brackets.**A1:** Obtains any correct equation without the square root.**A1*:** Obtains printed answer, including the “=0”, with no errors via an expanded intermediate step.**(b)****B1:** Deduces the equation of the tangent. Allow for $y = x + c$ used (ie correct gradient identified).Note an incorrect tangent $y = -x + a$ will give the same answers as below but must lose the A marks as from incorrect working.**M1:** Substitutes their equation of the tangent into the circle to form a 3TQ. This may be in terms of $y = \pm x + c$ for this mark.**A1:** Correct quadratic equation in x (or y) (may have c instead of $-a$), e.g.

$$2x^2 - (2 - 2c)x + c^2 - 7 = 0$$

M1: Uses the discriminant equal to 0 to produce a quadratic in a or c **M1:** Solves their 3TQ in a . If working with c they must return to find a value of a for this mark.**A1cso:** Deduces the correct value $a = 5$, the solution $a = -3$ must clearly be rejected – e.g. accept underling 5 as the answer. Must be from correct work.**Alternative****B1:** Correct completion of the square seen or implied.**M1:** Finds the centre and radius of the circle**A1:** Correct centre and radius**M1:** Uses a right-angled triangle with their radius and tangent to find the length x shown in the diagram. E.g. recognises the tangent length has the same length as the radius and uses Pythagoras tofind the length x , or uses trigonometry in the right-angled triangle such as $\sin \frac{\pi}{4} = \frac{\sqrt{8}}{x}$ leading to x **M1:** Finds the value of a by adding their x -coordinate of the centre to their value of x **A1cso:** $a = 5$ Must be from correct work.**(b) Alt II**

Use of Differentiation may also be seen.

B1: $2x - 2 + 2y \frac{dy}{dx} = 0$ Correct derivative statement.**M1A1:** $\frac{dy}{dx} = \tan\left(-\frac{3}{4}\pi\right) = 1 \Rightarrow$ radial equation is $2x - 2 + 2y = 0$ Any correct equation.**M1:** $y = 1 - x \Rightarrow x^2 - 2x + (1 - x)^2 - 7 = 0 \Rightarrow x = \dots, y = \dots$ substitutions into the circle equation (either variable) and solves to find the points of intersection of tangent and circle $[(3, -2) \text{ and/or } (-1, 2)]$ **dM1:** Tangent is $y - (-2) = x - 3 \Rightarrow y = x - 5 \Rightarrow a = \dots$ Correct process to find a including selecting the correct point (below the x -axis) to use.**A1cso:** Correct a Must be from correct work.

Question	Scheme	Marks	AOs
7(a)	$w = \frac{3iz - 2}{z + i} \Rightarrow z = \frac{2 + wi}{3i - w}$	M1	2.1
	$ z + i = 1 \Rightarrow \left \frac{2 + wi}{3i - w} + i \right = 1$	M1	2.1
	$\left \frac{2 + wi - 3 - iw}{3i - w} \right = 1 \Rightarrow \left \frac{1}{w - 3i} \right = 1$	A1	1.1b
	$ w - 3i = 1 \Rightarrow u^2 + (v - 3)^2 = 1$	A1	2.2a
		(4)	
(b)		B1	1.1b
		B1ft	1.1b
		(2)	
(6 marks)			
Notes			
(a) M1: Attempts to solve the problem by attempting to make z the subject of the formula M1: Uses the given locus to obtain an equation in w A1: Obtains a correct simplified equation in terms of w A1: Deduces the correct form of the equation (allow x , y or u , v etc.) (b) B1: The circle C correctly positioned, passing through the origin coordinates of centre labelled. Accept as coordinates or marked on axes. B1ft: Their D correctly positioned with the centre correctly labelled (accept as coordinates or marked on axes). Accept both drawn on the same diagram. Allow S.C. B1B0 for two circles in correct respective positions but with no labelling.			

7(a) ALT 1	$w = \frac{3iz - 2}{z + i} \Rightarrow w(z + i) = 3i(z + i) + 3 - 2$ Attempts to isolate $z + i$ terms	M1	2.1
	$(z + i)(w - 3i) = 1 \Rightarrow (z + i)(w - 3i) = 1 \Rightarrow (w - 3i) = 1$ Gathers $z + i$ terms and applies $(z + i) = 1$	M1	2.1
	As main scheme	A1	1.1b
	As main scheme	A1	2.2a
7(a) ALT 2	$w = \frac{3iz - 2}{z + i} \Rightarrow z = \frac{2 + wi}{3i - w} \text{ as main scheme}$	M1	2.1
	$x + yi = \frac{2 - v + ui}{-u - (v - 3)i} \times \frac{-u + (v - 3)i}{-u + (v - 3)i} = \dots = \frac{u - (u^2 + v^2 - 5v + 6)i}{u^2 + (v - 3)^2}$ $\Rightarrow \left(\frac{u}{u^2 + (v - 3)^2} \right)^2 + \left(\frac{-(u^2 + v^2 - 5v + 6)}{u^2 + (v - 3)^2} + 1 \right)^2 = 1$ Applies Cartesian coordinates to both sides, extracts x and y terms and attempts to apply $x^2 + (y + 1)^2 = 1$	M1	2.1
	$\left(\frac{u}{u^2 + (v - 3)^2} \right)^2 + \left(\frac{3 - v}{u^2 + (v - 3)^2} \right)^2 = 1$ Correct expression with $y + 1$ term combined and simplified.	A1	1.1b
	$\Rightarrow u^2 + (v - 3)^2 = 1$	A1	2.2a
7(a) ALT 3	$u + iv = \frac{3i(x + iy) - 2}{x + iy + i} \times \frac{x - (y + 1)i}{x - (y + 1)i} = \frac{f(x, y) + g(x, y)i}{x^2 + (y + 1)^2}$ Applies Cartesian coordinates to expression and use complex conjugate of denominator to reach Cartesian form.	M1	2.1
	$x^2 + (y + 1)^2 = 1 \Rightarrow u + iv = x + (3x^2 + 3y^2 + 5y + 2)i$ $\Rightarrow u = x \text{ and } v = 3x^2 + 3(y + 1)^2 - y - 1 = a + by$ Uses $x^2 + (y + 1)^2 = 1$ in their equation and extract u and v as linear terms in x and y	M1	2.1
	$u = x \text{ and } v = 2 - y$ Correct u and v	A1	1.1b
	$\Rightarrow u^2 + (2 - v + 1)^2 = 1 \Rightarrow u^2 + (v - 3)^2 = 1$ Uses $x^2 + (y + 1)^2 = 1$ again to find correct equation.	A1	2.2a

Note that there may be attempts via identifying images of points on a diameter. If seen, send to review.



Question	Scheme	Marks	AOs
9	$w(z + 2i) = z \Rightarrow wz + 2wi = z \Rightarrow 2wi = z - wz$ $\Rightarrow 2wi = z(1 - w) \Rightarrow z = \frac{2wi}{(1 - w)}$	M1 A1	3.1a 1.1b
	$ z = 4 \Rightarrow \left \frac{2wi}{(1 - w)} \right = 4 \Rightarrow 2wi = 4 1 - w $	dM1	1.1b
	$w = u + vi$ $ 2ui - 2v = 4 (1 - u) - vi \Rightarrow (2u)^2 + (2v)^2 = 16[(1 - u)^2 + v^2] \text{ oe}$	ddM1	2.1
	$\{4u^2 + 4v^2 = 16 - 32u + 16u^2 + 16v^2\}$ $12u^2 - 32u + 12v^2 + 16 = 0 \text{ oe}$	A1	1.1b
	$u^2 - \frac{8}{3}u + v^2 + \frac{4}{3} = 0 \Rightarrow \left(u - \frac{4}{3}\right)^2 - \frac{16}{9} + v^2 + \frac{4}{3} = 0$	M1	1.1b
	Centre $\left(\frac{4}{3}, 0\right)$ or radius $= \frac{2}{3}$	A1	1.1b
	Centre $\left(\frac{4}{3}, 0\right)$ and radius $= \frac{2}{3}$	A1	2.2a
		(8)	
(8 marks)			
Notes:			
M1: A complete method of rearranging to make z the subject. Condone slips rearranging A1: Correct expression for z NB if the only error is incorrect sign, this will correct when taking modulus – allow recovering of later marks, losing just this first A mark. dM1: Dependent on the previous method mark. Sets $z = 4$ and forms a linear equation ddM1: Dependent on previous method mark. Uses $w = u + vi$ and Pythagoras to form an equation in u and v only. Condone $(2u)^2 + (2v)^2 = 4[(1 - u)^2 + v^2]$ or slips in signs on $1 - (u + vi)$. They may factor out the 2 first, which is fine. A1: Correct equation M1: Having reached a circle equation, completes the square on u (and v if appropriate). A1: Correct centre or radius of C from correct work A1: Centre and radius both correct from correct work. Accept centre as coordinates or as $\frac{4}{3} + 0i$			
Alt I	$w(z + 2i) = z \Rightarrow wz + 2wi = z \Rightarrow 2wi = z - wz$ $\Rightarrow 2wi = z(1 - w) \Rightarrow z = \frac{2wi}{(1 - w)}$	M1 A1	3.1a 1.1b
	$z = \frac{2ui - 2v}{1 - u - vi} \times \frac{1 - u + vi}{1 - u + vi} = \dots$	dM1	1.1b

	$ z = 4 \Rightarrow \left(\frac{2v}{(u-1)^2 + v^2} \right)^2 + \left(\frac{2u^2 + 2v^2 - 2u}{(u-1)^2 + v^2} \right)^2 = 16$ $\Rightarrow 12v^4 + 24u^2v^2 - 56uv^2 + 28v^2 + 12u^4 - 56u^3 + 92u^2 - 64u + 16 = 0$ $\Rightarrow 4(v^2 + u^2 - 2u + 1)(3v^2 + 3u^2 - 8u + 4) = 0$	ddM1	2.1
	$\{4u^2 + 4v^2 = 16 - 32u + 16u^2 + 16v^2\}$ $12u^2 - 32u + 12v^2 + 16 = 0$ oe	A1	1.1b
	$u^2 - \frac{8}{3}u + v^2 + \frac{4}{3} = 0 \Rightarrow \left(u - \frac{4}{3}\right)^2 - \frac{16}{9} + v^2 + \frac{4}{3} = 0$	M1	1.1b
	Centre $\left(\frac{4}{3}, 0\right)$ or radius $= \frac{2}{3}$	A1	1.1b
	Centre $\left(\frac{4}{3}, 0\right)$ and radius $= \frac{2}{3}$	A1	1.1b
		(8)	
Notes: M1: A complete method of rearranging to make z the subject. Condone slips rearranging A1: Correct expression for z dm1: Dependent on the previous method mark. Applies $w = u + vi$ and rationalises the denominator with correct conjugate used. ddM1: Dependent on previous method mark. Extracts real and imaginary components, sets $ z = 4$ and applies $\text{Re}^2 + \text{Im}^2 = 16$ (condone “=4”) and proceeds to simplify to a quadratic expression in u and v . This mark is not likely to be successfully earned but is possible. The most likely approach is shown but identifying and cancelling a factor $(u-1)^2 + v^2$ is also possible. A1: Correct equation M1: Having reached a circle equation, completes the square on u (and v if appropriate). A1: Correct centre or radius of C from correct work A1: Centre and radius both correct from correct work. Accept centre as coordinates or as $\frac{4}{3} + 0i$			
9 Alt II	$w = \frac{x+iy}{x+iy+2i} \times \frac{x-(y+2)i}{x-(y+2)i} = \dots$ $= \frac{x^2 - x(y+2)i + ixy + y(y+2)}{x^2 + (y+2)^2}$	M1	3.1a
		A1	1.1b
	$ z = 4 \Rightarrow x^2 + y^2 = 16 \Rightarrow w = \frac{x^2 + y^2 + 2y - i(xy + 2x - xy)}{x^2 + y^2 + 4y + 4} = \frac{16 + 2y - 2xi}{4y + 20}$	dM1	1.1b
	$u = \frac{8+y}{2y+10} = \frac{1}{2} + \frac{3}{2y+10}, v = \frac{-x}{2y+10}$ $\Rightarrow u^2 + v^2 = \frac{(8+y)^2 + x^2}{(2y+10)^2} = \frac{64 + 16y + y^2 + x^2}{4(y+5)^2} = \frac{64 + 16y + 16}{4(y+5)^2}$ $= \frac{16y + 80}{4(y+5)^2} = \frac{16(y+5)}{4(y+5)^2} = \frac{4}{y+5} = 4 \times \frac{2}{3} \left(u - \frac{1}{2}\right)$	ddM1	2.1



	$\Rightarrow u^2 + v^2 = \frac{8}{3} \left(u - \frac{1}{2} \right)$	A1	1.1b
	$u^2 - \frac{8}{3}u + v^2 + \frac{4}{3} = 0 \Rightarrow \left(u - \frac{4}{3} \right)^2 - \frac{16}{9} + v^2 + \frac{4}{3} = 0$	M1	1.1b
	Centre $\left(\frac{4}{3}, 0 \right)$ or radius $= \frac{2}{3}$	A1	1.1b
	Centre $\left(\frac{4}{3}, 0 \right)$ and radius $= \frac{2}{3}$	A1	2.2a
		(8)	
(8 marks)			
Notes:			
M1: Replaces z by $x + iy$ and rationalises the denominator. A1: Correct expression. dm1: Dependent on the previous method mark. Uses $z = 4 \Rightarrow x^2 + y^2 = 16$ in the equation to simplify coefficients ddM1: Dependent on previous method mark. Extracts the real and imaginary parts of the equation and attempts to find $u^2 + v^2$ eliminates all x's and y's A1: Correct equation. M1: Having reached a circle equation, completes the square on u (and v if appropriate). A1: Correct centre or radius of C from correct work A1: Centre and radius both correct from correct work			
9 Alt III	$ z = 4 \Rightarrow \pm 4, \pm 4i$ on circle so e.g. $\frac{\pm 4}{\pm 4 + 2i}, \frac{\pm 4i}{\pm 4i + 2i}$ on C	M1	3.1a
	$\Rightarrow \frac{2}{3}, 2, \frac{4}{5} \pm \frac{2}{5}i$ on C	A1	1.1b
	Perpendicular bisector of points $\frac{2}{3}, 2$ is $x = \frac{4}{3}$	dM1	1.1b
	Perpendicular bisector of points $\frac{4}{5} \pm \frac{2}{5}i$ is $y = 0$	ddM1	2.1
	$x = \frac{4}{3}$ and $y = 0$	A1	1.1b
	Centre is intersection of these lines = ...	M1	1.1b
	Centre $\left(\frac{4}{3}, 0 \right)$ or radius $= \frac{2}{3}$	A1	1.1b
	Centre $\left(\frac{4}{3}, 0 \right)$ and radius $= \frac{2}{3}$	A1	2.2a
		(8)	
(8 marks)			

Notes:

M1: Identifies at least 2 points on the original circle and finds the images on C

A1: Correct simplified two points on C

dM1: Dependent on the previous method mark. Finds the perpendicular bisector of two of their points found on C

ddM1: Dependent on previous method mark. Finds a third point on C (or two more points on C) and attempts the perpendicular bisector for another pair of points.

A1: Correct equations for the bisectors.

M1: Solves the equations of the bisectors to find the centre of the circle.

A1: Correct centre or radius of C from correct work

A1: Centre and radius both correct from correct work.

There may be variations on approach (e.g. using diametrically opposite points to work out centre and radius). If solutions are not fully correct by such approaches, use the review system for guidance.

