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EDEXCEL A LEVEL FURTHER MATHEMATICS: QUESTIONS BY TOPIC

Further Sequences and Series (6)

Further Pure Mathematics 2 · Topic 6

9 questions · 67 marks · Mark schemes included

Pearson Edexcel A Level Further Mathematics (9FM0), June 2019 - June 2025

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HOW TO USE THIS BOOKLET

Questions are grouped by specification sub-topic and run from the lowest mark tariff to the highest within each one.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme.

Diagrams, graphs, tables and data are printed exactly as they appeared in the original papers.

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QUESTIONS

9 questions · 67 marks · 3 sub-topics



SUB-TOPIC 6.1

RECURRENCE RELATIONS IN CONTEXT

Sub-topic 6.1 · 3 questions · 25 marks · lowest tariff first



Question 3 continued

(Total for Question 3 is 8 marks)







3. In a model for the number of subscribers to a new social media channel it is assumed that

- each week 20% of the subscribers at the start of the week cancel their subscriptions
- between the start and end of week n the channel gains $20n$ new subscribers

Given that at the end of week 1 there were 25 subscribers,

- (a) explain why the number of subscribers at the end of week n , U_n , is modelled by the recurrence relation

$$U_1 = 25 \quad U_{n+1} = 0.8U_n + 20(n+1) \quad n = 1, 2, 3, \dots \quad (2)$$

- (b) Prove by induction that for $n \geq 1$

$$U_n = 325 \left(\frac{4}{5} \right)^{n-1} + 100n - 400 \quad (5)$$

Given that 6 months after starting the channel there were approximately 1800 subscribers,

- (c) evaluate the model in the light of this information. (2)

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SUB-TOPIC 6.2

SOLVING RECURRENCE RELATIONS

Sub-topic 6.2 · 4 questions · 27 marks · lowest tariff first



2. Determine a closed form for the recurrence system

$$u_1 = 4 \quad u_2 = 6$$

$$u_{n+2} = 6u_{n+1} - 9u_n \quad (n = 1, 2, 3, \dots)$$

(5)



Question 2 continued



(Total for Question 2 is 5 marks)

6. (a) Determine the general solution of the recurrence relation

$$u_n = 2u_{n-1} - u_{n-2} + 2^n \quad n \geq 2 \quad (4)$$

(b) Hence solve this recurrence relation given that $u_0 = 2u_1$ and $u_4 = 3u_2$

6. Determine a closed form for the recurrence relation

$$\begin{aligned} u_0 &= 1 & u_1 &= 4 \\ u_{n+2} &= 2u_{n+1} - \frac{4}{3}u_n + n & n &\geq 0 \end{aligned} \quad (7)$$



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(Total for Question 6 is 7 marks)



2. Solve the recurrence system

$$\begin{array}{l} u_1 = 1 \quad u_2 = 4 \\ 9u_{n+2} - 12u_{n+1} + 4u_n = 3n \end{array}$$

(9)





Question 2 continued

(Total for Question 2 is 9 marks)



SUB-TOPIC 6.3

PROOF BY INDUCTION OF CLOSED FORMS

Sub-topic 6.3 · 2 questions · 15 marks · lowest tariff first



6. A recurrence system is defined by

$$u_{n+2} = 9(n+1)^2 u_n - 3u_{n+1} \quad n \geq 1$$

$$u_1 = -3, u_2 = 18$$

Prove by induction that, for $n \in \mathbb{N}$,

$$u_n = (-3)^n n!$$

(6)



Question 6 continued





Question 6 continued

(Total for Question 6 is 6 marks)





Question 4 continued



Question 4 continued

MARK SCHEMES

official mark schemes



Question	Scheme	Marks	AOs
3(a)	$V_{n+2} = V_{n+1} + kV_n$	B1	3.3
		(1)	
(b)	$\lambda^2 - \lambda - 0.24 = 0 \Rightarrow \lambda = \dots(1.2, -0.2)$	M1	1.1b
	$V_n = a(1.2)^n + b(-0.2)^n$	A1	2.2a
	$65 = a(1.2)^1 + b(-0.2)^1$ and $71 = a(1.2)^2 + b(-0.2)^2$	B1ft	3.4
	E.g. $\begin{cases} 78 = 1.44a - 0.24b \\ 71 = 1.44a + 0.04b \end{cases} \Rightarrow 7 = -0.28b \Rightarrow b = \dots$	M1	2.1
	$a = 50, b = -25 \Rightarrow V_n = 50(1.2)^n - 25(-0.2)^n *$	A1*	1.1b
		(5)	
(c)	$50(1.2)^N > 10^6 \Rightarrow N = \dots$	M1	3.1b
	$\Rightarrow N = 55$ i.e. month 55	A1	3.2a
		(2)	
(8 marks)			
Notes			
(a) B1: A correct expression for the model using the information given (b) M1: Forms and solves the auxiliary equation for their answer to (a) with $k = 0.24$ A1: The correct closed form deduced from their solutions. This must be consistent with their equation. Note the answer is given so check carefully. This is not a follow through mark. B1ft: Applies initial conditions to their general equation – correct two equations for their general form with $V_1 = 65$ and $V_2 = 71$ M1: Attempts to solve their equations showing a correct method, reaching a value for at least one variable. It is a show that question and answers are on the paper, so method is needed. Look for one equation multiplied through to give same coefficients before attempting eliminating or substitution. If a matrix system is used the inverse must be found, not just solutions stated. A1*: Correct expression formed following suitable working with no errors seen. With fractions instead of decimals is fine. (c) M1: Selects a suitable method to solve the problem. For example, realises that in the model, $(-0.2)^n$ is negligible for large n and so attempts to solve e.g. $50(1.2)^N = 10^6$, or tries at least one value either side of $N = 55$ as a process of trial and improvement, or uses a calculator/graphical approach – implied by a value of $N = 55$ or $N = 54$ stated. A1: $N = 55$. The correct answer will imply both marks for this part. Ignore erroneous working if correct answer is stated as a restart.			
3(b)	$V_n = 50(1.2)^n - 25(-0.2)^n, V_1 = 65, V_2 = 71$		

Alt	$V_1 = 50 \times 1.2 - 25 \times -0.2 = 60 + 5 = 65$ $V_2 = 50 \times (1.2)^2 - 25 \times (-0.2)^2 = 72 - 1 = 71$	M1	1.1b
	Hence true for $n = 1$ and $n = 2$ Assume true for $n = k$ and $n = k + 1$ (for some $k > 0$)	A1	2.2a
	$V_{k+2} = V_{k+1} + 0.24V_k$ $= 50(1.2)^{k+1} - 25(-0.2)^{k+1} + 0.24(50(1.2)^k - 25(-0.2)^k)$	B1ft	3.4
	$= \frac{50}{1.2}(1.2)^{k+2} - \frac{25}{-0.2}(-0.2)^{k+2} + \frac{12}{1.2^2}(1.2)^{k+2} - \frac{6}{(-0.2)^2}(-0.2)^{k+2}$ $= \frac{125}{3}(1.2)^{k+2} + 125(-0.2)^{k+2} + \frac{25}{3}(1.2)^{k+2} - 150(-0.2)^{k+2} = \dots$	M1	2.1
	So $V_{k+2} = 50(1.2)^{k+2} - 25(-0.2)^{k+2}$ Hence true for $n = k + 2$. So the result is true for $n = 1$ and $n = 2$, and if true for $n = k$ and $n = k + 1$ then it is true for $n = k + 2$. Hence by mathematical induction, for all $n \in \mathbb{N}$ $V_n = 50(1.2)^n - 25(-0.2)^n *$	A1*	1.1b
		(5)	
Notes			
M1: Substitutes into equation for $n = 1$ and $n = 2$ to verify true for these cases. A1: Deduces true for base cases and makes a correct assumption statement. This must include two successive cases assumed true, so e.g. as in scheme, or with $k - 2$ and $k - 1$ etc, or may assume true for all (integers) $k \leq n$. But do not allow if assumed true for just k . B1ft: Substitutes the formula for k and $k + 1$ (or their successive values) into the recurrence formula, follow through their equation from part (a). M1: Rearranges to the form $a(1.2)^{(k+2)} + b(-0.2)^{k+2}$ A1*: Correct work leading to the correct equation for V_{k+2} and makes suitable inductive conclusion, including the ideas of “true for $n = 1$ and $n = 2$ ”, “if true for $n = k$ and $n = k + 1$ then true for $n = k + 2$ ” and “hence true for all integers”.			

Question	Scheme	Marks	AOs
3 (a)	Implies moves from A - Has to hop to different lily pad to A - Will not hop on the same pad - They can't stay on lily pad A - Goes to B or C but not A - Can't be where it started - Can't stay on A	B1	2.4
		(1)	
(b)	$p_1 = \frac{2}{3}\left(-\frac{1}{2}\right)^1 + \left(\frac{1}{3}\right) = 0$ Minimum needed $p_1 = \frac{2}{3}\left(-\frac{1}{2}\right) + \left(\frac{1}{3}\right) = 0$	B1	2.1
	Assume result is true for $n = k$ Or $p_k = \frac{2}{3}\left(-\frac{1}{2}\right)^k + \left(\frac{1}{3}\right)$	M1	2.4
	Finds an expression for $p_{k+1} = \frac{1}{2}\left(1 - \left[\frac{2}{3}\left(-\frac{1}{2}\right)^k + \left(\frac{1}{3}\right)\right]\right)$	M1	1.1b
	$p_{k+1} = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^k$ or $\frac{1}{2} - \frac{1}{6} + \frac{2}{3}\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^k$ $p_{k+1} = \frac{1}{3} - \frac{1}{3} \times -2\left(-\frac{1}{2}\right)^{k+1}$ or $\frac{1}{2} - \frac{1}{6} - \frac{1}{3} \times -2\left(-\frac{1}{2}\right)^{k+1}$	M1	2.1
	$p_{k+1} = \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1} + \left(\frac{1}{3}\right)$	A1	1.1b
	If true for $n = k$ then it is true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all n . (Allow 'for all values')	A1	2.2a
		(6)	
(c)	As $n \rightarrow \infty, \left(-\frac{1}{2}\right)^n \rightarrow 0 \therefore p_n \rightarrow \frac{1}{3}$	B1	3.4
		(1)	
(8 marks)			
Notes:			
(a)			
B1: Explains has to move from A			
(b)			
B1: Shows the statement is true for $n = 1$. Needs to show that $p_1 = 0$ and conclusion true for $n = 1$, this statement can be recovered in their conclusion if says e.g. true for $n = 1$			
M1: Makes a statement that assumes the result is true for $n = k$. Assume (true for) $n = k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n = k$ then ...etc			
M1: Finds an expression for p_{k+1} using the recurrence relation and substitutes in for p_k			

M1: Shows a correct intermediate stage $p_{k+1} = \frac{1}{3} + \frac{2}{3} \left(-\frac{1}{2}\right) \left(-\frac{1}{2}\right)^k$

Condone $\frac{1}{3} - \frac{2}{3} \left(\frac{1}{2}\right) \left(-\frac{1}{2}\right)^k$

A1: $p_{k+1} = \frac{2}{3} \left(-\frac{1}{2}\right)^{k+1} + \left(\frac{1}{3}\right)$ cso

A1: Correct complete conclusion. This mark is dependent on previous four marks, the B mark is not required. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

Note Using $p_{k+1} = \frac{1}{2} - \frac{1}{2} p_k = \frac{1}{2} - \frac{1}{2} \left[\frac{2}{3} \left(-\frac{1}{2}\right)^k + \left(\frac{1}{3}\right) \right] = \frac{1}{2} + \frac{2}{3} \left(-\frac{1}{2}\right)^{k+1} - \left(\frac{1}{6}\right)$ scores 2nd M1 3rd M1

(c)

B1: see scheme



Question	Scheme	Marks	AOs
3(a)	Two of $U_1 = 25$ because there are 25 subscribers at the end of week 1 20% of subscribers leave so 80% so $0.8U_n$ remaining subscribers or $U_n - 0.2U_n$ - At end of week $n + 1$ there are a new $20(n + 1)$ subscribers added to those from week n	M1	3.3
	All three points above put together in conclusion Hence $U_{n+1} = 0.8U_n + 20(n+1)$, $U_1 = 25$	A1	2.4
		(2)	
(b)	$n = 1 \Rightarrow U_1 = 325 \times 1 + 100 \times 1 - 400 = 325 - 300 = 25$ {so the result is true for $n = 1$ }	B1	2.2a
	(Assume true for $n = k$ then) $U_{k+1} = 0.8U_k + 20k + 20 = 0.8 \left(325 \left(\frac{4}{5} \right)^{k-1} + 100k - 400 \right) + 20k + 20$	M1	1.1b
	$= 325 \times \frac{4}{5} \times \left(\frac{4}{5} \right)^{k-1} + 80k - 320 + 20k + 20$ $= 325 \times \left(\frac{4}{5} \right)^k + 100k - 300$	M1	1.1b
	$= 325 \times \left(\frac{4}{5} \right)^k + 100(k+1) - 400$	A1	2.1
	Hence if the result is true for $n = k$, then it is true for $n = k + 1$, and as it is true for $n = 1$, so it is true for all positive integers n or $n \geq 1$	A1	2.4
		(5)	
	Alternative $n = 1 \Rightarrow U_1 = 325 \times 1 + 100 \times 1 - 400 = 325 - 300 = 25$ so the result is true for $n = 1$	B1	2.2a
	$U_{k+1} = 325 \left(\frac{4}{5} \right)^k + 100(k+1) - 400 = \frac{4}{5} \times 325 \left(\frac{4}{5} \right)^{k-1} + 100k - 300$	M1	1.1b
	$= \frac{4}{5} \times \left(325 \left(\frac{4}{5} \right)^{k-1} + 100k - 400 \right) + 20k + 20$	M1	1.1b
	$U_{k+1} = 0.8U_k + 20(k+1)$	A1	2.1
	Hence if the result is true for $n = k$, then it is true for $n = k + 1$, and as it is true for $n = 1$, so it is true for all positive integers n or $n \geq 1$	A1	2.4
		(5)	

(c)	An attempt at either $U_{24} = 325 \times 0.8^{23} + 2400 - 400 = 2001.9\dots$ Or $U_{25} = 325 \times 0.8^{24} + 2500 - 400 = 2101.5\dots$ or $U_{26} = 325 \times 0.8^{25} + 2600 - 400 = 2201.2\dots$ Or $U_{27} = 325 \times 0.8^{26} + 2700 - 400 = 2300.98\dots$	M1	3.4
	Correct value for their number of weeks 24, 25, 26 or 27. Compares the value after 6 months with 1800 and draws a conclusion e.g. This is overestimating the actual amount by 400 people and therefore not a very good model.	A1	3.5a
		(2)	
(9 marks)			

Notes:**(a)**

M1: See scheme. Explains how the assumptions lead to at least two of the aspects indicated. Accept less formal explanations as long as the intent is clear.

A1: All three aspects explained and put together to set up the model.

(b)

B1: Checks the case for $n=1$ holds.

M1: Makes the inductive assumption (may be implied by working) and substitutes the closed form for U_k into the recurrence relation for U_{k+1} or equivalent work with different variable (e.g. n instead of k) or indexing (e.g. from $k-1$ to k).

M1: Simplifies to the point of combining the powers of $\frac{4}{5}$ to one term.

A1: For correct work leading to the form shown. The $k+1$ must be seen in the added term but allow just k for the power.

A1: For a completely correct proof (all previous marks must be gained) with a conclusion that includes all of the bold statements in the scheme or equivalents.

Alternative

B1: Checks the case for $n=1$ holds.

M1: Writes out the term U_{k+1} and starts the process to write in terms of U_k by factorising out $\frac{4}{5}$ from the first term.

M1: Factorises out $\frac{4}{5}$ to form $= \frac{4}{5} \times \left(325 \left(\frac{4}{5} \right)^{k-1} + 100k - 400 \right) + \dots$

A1: For correct work leading $U_{k+1} = 0.8U_k + 20(k+1)$

A1: For a completely correct proof (all previous marks must be gained) with a conclusion that includes all of the bold statements in the scheme or equivalents.

(c)

M1: Evaluates U_{24} U_{25} U_{26} or U_{27} . Allow attempts that deduce $\left(\frac{4}{5} \right)^n \rightarrow 0$ and just evaluate

$100 \times 26 - 400 = 2200$

A1: Correct value and appropriate conclusion.



Question	Scheme	Marks	AOs
2	Aux equation is $r^2 - 6r + 9 = 0 \Rightarrow r = \dots$	M1	1.1b
	$(r - 3)^2 = 0 \Rightarrow r = 3$	A1	1.1b
	General form is $u_n = (A + Bn)3^n$	M1	1.1a
	$\begin{cases} 4 = 3(A + B) \\ 6 = 9(A + 2B) \end{cases} \Rightarrow A = \dots, B = \dots$	M1	2.1
	$u_n = \left(2 - \frac{2}{3}n\right)3^n$ o.e $u_n = (6 - 2n)3^{n-1}$	A1	1.1b
		(5)	
(5 marks)			
Notes:			
M1: Forms and solves the auxiliary equation. A1: Correct single root of 3. M1: Selects the correct form for u_n for their roots of the equation. (If distinct real roots were found allow for $u_n = A\alpha^n + B\beta^n$) M1: Uses the values of u_1 and u_2 with the corresponding values of n used to form and uses a correct method to solve simultaneous equations to find the constants. If no method is shown, use of calculator, for solving the simultaneous equations the values must be correct for their equations. A1: Correct closed form, isw			

Question	Scheme	Marks	AOs
6 (a)	$l^2 - 2l + 1 = 0 \Rightarrow l = \dots \{\lambda = 1\}$	M1	1.1b
	$u_n = A + Bn$	A1	2.2a
	$u_n = \mu(2)^n \Rightarrow \mu(2)^n = 2\mu(2)^{n-1} - \mu(2)^{n-2} + (2)^n$ $\Rightarrow 4\mu(2)^{n-2} = 4\mu(2)^{n-2} - \mu(2)^{n-2} + 4(2)^{n-2}$ $\Rightarrow \mu = \dots \{4\}$	M1	1.1b
	$u_n = A + Bn + 4(2)^n$ or $u_n = A + Bn + (2)^{n+2}$	A1	1.1b
		(4)	
(b)	Uses the information to find the values of the constants For example $u_0 = 2u_1 \Rightarrow A + 4 = 2(A + B + 4(2)) \Rightarrow \dots \{A + 2B = -12\}$ $u_4 = 3u_2 \Rightarrow A + 4B + 4(2)^4 = 3(A + 2B + 4(2)^2) \Rightarrow \dots \{2A + 2B = 16\}$ Solves simultaneous equations to find values for A and B. Alternatively $u_2 = 2u_1 - u_0 + 2^2 = u_0 - u_0 + 4 = 4$ leading to $u_4 = 3u_2 = 3 \times 4 = 12$ $u_3 = 2u_2 - u_1 + 2^3 = 2 \times 4 - \left(\frac{A + 4}{2}\right) + 8 = 16 - \left(\frac{A + 4}{2}\right)$ $u_4 = 2u_3 - u_2 + 2^4 = 2\left(16 - \left(\frac{A + 4}{2}\right)\right) - 4 + 16 = 12$ Leading $A = \dots u_2 = '28' + 2B + 4(2^2) = 4 \Rightarrow B = \dots$	M1	3.1a
	$u_n = 28 - 20n + 4(2)^n$ or $u_n = 28 - 20n + (2)^{n+2}$	A1	1.1b
		(2)	
	(6 marks)		
Notes:			
(a) M1: Forms and solves the auxiliary equation. A1: Correct complementary function. M1: Correct form for the particular solution, substitutes into the recurrence relation to find the PS. A1: Correct general solution			
(b) Note: They must have two constants to score any marks in this part M1: A complete method to find the constants using the information given. Form two equations and solves simultaneously to find values for A and B. A1: Correct solution.			

Question	Scheme	Marks	AOs
6	$\lambda^2 = 2\lambda - \frac{4}{3} \Rightarrow 3\lambda^2 - 6\lambda + 4 = 0 \Rightarrow \lambda = \dots$	M1	1.1b
	$\lambda = \frac{6 \pm \sqrt{36 - 48}}{6} = \frac{3 \pm i\sqrt{3}}{3} = \frac{2}{\sqrt{3}} \left(\frac{\sqrt{3}}{2} \pm \frac{1}{2}i \right) = \frac{2\sqrt{3}}{3} e^{\pm i\frac{\pi}{6}}$	A1	1.1b
	CF is $u_n = A \left(\frac{3+i\sqrt{3}}{3} \right)^n + B \left(\frac{3-i\sqrt{3}}{3} \right)^n$ o.e. or $\left(\frac{2\sqrt{3}}{3} \right)^n \left(P \cos \left(\frac{\pi n}{6} \right) + Q \sin \left(\frac{\pi n}{6} \right) \right)$	A1ft	2.2a
	$u_n = kn + l \Rightarrow k(n+2) + l = 2k(n+1) + 2l - \frac{4}{3}(kn+l) + n$ $\Rightarrow \left(1 - \frac{1}{3}k \right)n - \frac{1}{3}l = 0 \Rightarrow k = \dots, l = \dots$ ($k=3, l=0$)	M1	1.1b
	$u_n = A \left(\frac{3+i\sqrt{3}}{3} \right)^n + B \left(\frac{3-i\sqrt{3}}{3} \right)^n + 3n$ o.e. or $u_n = \left(\frac{2\sqrt{3}}{3} \right)^n \left(P \cos \left(\frac{\pi n}{6} \right) + Q \sin \left(\frac{\pi n}{6} \right) \right) + 3n$	A1	1.1b
	$u_0 = 1 \Rightarrow A + B = 1$ $u_1 = 4 \Rightarrow A + B + (A-B) \frac{i\sqrt{3}}{3} = 1 \Rightarrow (A-B) \frac{i\sqrt{3}}{3} = 0 \Rightarrow A = \dots, B = \dots$ $u_0 = 1 \Rightarrow P + 0Q = 1$ $u_1 = 4 \Rightarrow \frac{2}{\sqrt{3}} \left(P \frac{\sqrt{3}}{2} + \frac{Q}{2} \right) = 1 \Rightarrow P = \dots \Rightarrow Q = \dots$ Solves simultaneous equations to find values for A and B or P and Q	M1	3.1a
	$\left(A = B = \frac{1}{2} \text{ or } P = 1, Q = 0 \right)$ $\Rightarrow u_n = \frac{1}{2} \left(\frac{3+i\sqrt{3}}{3} \right)^n + \frac{1}{2} \left(\frac{3-i\sqrt{3}}{3} \right)^n + 3n$ o.e. Or $u_n = \left(\frac{2\sqrt{3}}{3} \right)^n \cos \left(\frac{\pi n}{6} \right) + 3n$	A1	1.1b
		(7)	
(7 marks)			

Notes:

M1: Forms and solves the auxiliary equation.

A1: Correct roots – either Cartesian or polar form, award when first seen and isw.

A1ft: Correct complementary function, follow through on their first complex roots. (so A1A0 if roots initially correct but error simplifying leads to wrong CF). Note: use of power $n + 1$ or $n - 1$ in Cartesian form is also fine

M1: Correct form for the particular solution and a complete method to find the PS.

A1: Correct general solution, either form

M1. Substitutes $n = 1$ and sets equal to 4 **and** substitutes $n = 0$ and sets equal to 1. To find the values of the constants

A1: Correct solution, any form.

Question	Scheme	Marks	AOs
2	Auxiliary equation is $9r^2 - 12r + 4 = 0$, so $r = \dots$	M1	1.1b
	$(3r - 2)^2 = 0 \Rightarrow r = \frac{2}{3}$ is repeated root.	A1	1.1b
	Complementary function is $x_n = (A + Bn)\left(\frac{2}{3}\right)^n$ or $A\left(\frac{2}{3}\right)^n + Bn\left(\frac{2}{3}\right)^n$	M1	2.2a
	Try particular solution $y_n = an + b \Rightarrow 9(a(n+2) + b) - 12(a(n+1) + b) + 4(an + b) = 3n$	M1	2.1
	$\Rightarrow an + 6a + b = 3n \Rightarrow a = \dots, b = \dots$	dM1	1.1b
	$a = 3, b = -18$	A1	1.1b
	General solution is $u_n = x_n + y_n = (A + Bn)\left(\frac{2}{3}\right)^n + 3n - 18$	B1ft	2.2a
	$\left. \begin{aligned} u_1 = 1 &\Rightarrow 1 = \left(\frac{2}{3}\right)(A + B) - 15 \\ u_2 = 4 &\Rightarrow 4 = \left(\frac{4}{9}\right)(A + 2B) - 12 \end{aligned} \right\} A = \dots, B = \dots$	M1	2.1
	$u_n = 12(n+1)\left(\frac{2}{3}\right)^n + 3n - 18$ oe	A1	1.1b
		(9)	
(9 marks)			
Notes:			
M1: Forms and solves the auxiliary equation. A1: Correct (repeated) root found. M1: Forms the correct complementary function for their (real) root(s) to the equation, $(A + Bn)r^n$ if repeated root, or allow $Ar_1^n + Br_2^n$ if distinct real roots are found. M1: Attempts to use a particular solution of the correct form (ie $an + b$ or a higher order polynomial in n containing this) in the recurrence relation. dM1: Expands and solves for a and b A1: Correct values for a and b B1ft: Forms the general solution as the sum of their complementary function and a particular solution of correct form with their a and b M1: Applies the initial values and solves for the constants A1: Correct answer.			

Question	Scheme	Marks	AOs
6	$n = 1: u_1 = (-3)^1 \times 1! = -3$ $n = 2: u_2 = (-3)^2 \times 2! = 9 \times 2 = 18$ Hence true for $n = 1$ and $n = 2$	B1	2.2a
	Assume true for some $n = k$ and $n = k + 1$, so $u_k = (-3)^k k!$ and $u_{k+1} = (-3)^{k+1} (k+1)!$	M1	2.4
	Then $u_{k+2} = 9(k+1)^2 ((-3)^k k!) - 3((-3)^{k+1} (k+1)!)$	M1	1.1b
	$= (-3)^k k! [9(k+1)^2 - 3(-3)(k+1)]$	M1	1.1b
	$= (-3)^k k! [9(k+1)(k+1+1)] = (-3)^k \times (-3)^2 \times (k+1)(k+2)k!$ $= (-3)^{k+2} (k+2)!$	A1	2.1
	Hence if true for $n = k$ and $n = k + 1$ then true for $n = k + 2$. As also true for $n = 1$ and $n = 2$, then true for all $n \in \mathbb{N}$ by mathematical induction.	A1	2.4
		(6)	
(6 marks)			
B1: Checks the closed form works for $n = 1$ and $n = 2$ M1: Makes the inductive assumption. May use e.g. $n = k - 2$ and $n = k - 1$ instead and show true for $n = k$. It must be clear it is the closed forms they are assuming, not a recurrence form. M1: Substitutes expression for $n = k$ and $n = k + 1$ (or equivalents) into the recurrence formula. M1: Takes out common factors of at least $(-3)^k k!$ in their expression, or equivalent for their assumed true values. Treatment of the (-3) must be correct, but condone invisible brackets if recovered. Note: they may well take out more at this stage, which is fine, e.g. $u_{k+2} = 9(k+1)^2 ((-3)^k k!) - 3((-3)^{k+1} (k+1)!) = (-3)^{k+2} (k+1)! [(k+1)+1]$ A1: Simplifies correctly to the required form for their assumed true values. A1: Correct conclusion made. Depends on all three M's and the A being gained. Must convey the ideas of 1) true for $n = 1$ and $n = 2$, 2) if true for two successive cases, it is also true for the next case and 3) a suitable conclusion that it is true for all positive n.			

Question	Scheme	Marks	AOs
4(a)	$n = 1: u_1 = 200(2)^1 - \frac{1}{10}(4)^1 = 399.6$ $n = 2: u_2 = 200(2)^2 - \frac{1}{10}(4)^2 = 798.4$ (Hence true for $n = 1$ and $n = 2$)	B1	2.2a
	Assume true for some $n = k$ and $n = k + 1$, so $u_k = 200(2)^k - \frac{1}{10}(4)^k$ and $u_{k+1} = 200(2)^{k+1} - \frac{1}{10}(4)^{k+1}$	M1	2.4
	Then $u_{k+2} = 6 \left[200(2)^{k+1} - \frac{1}{10}(4)^{k+1} \right] - 8 \left[200(2)^k - \frac{1}{10}(4)^k \right]$	M1	1.1b
	e.g. $u_{k+2} = \left[3(200(2)^{k+2}) - 2(200(2)^{k+2}) \right] + \left[-\frac{6}{10}(4)^{k+1} + \frac{2}{10}(4)^{k+1} \right]$ $u_{k+2} = 200(2)^{k+2} - \frac{4}{10}(4)^{k+1}$ or $u_{k+2} = \left[3(200(2)^{k+2}) - 2(200(2)^{k+2}) \right] + \left[-\frac{3}{20}(4)^{k+2} + \frac{1}{20}(4)^{k+2} \right]$	M1	1.1b
	$u_{k+2} = 200(2^{k+2}) - \frac{1}{10}(4^{k+2})$	A1	2.1
	Hence <u>if true for $n = k$ and $n = k + 1$ then true for $n = k + 2$</u> . As also <u>true for $n = 1$ and $n = 2$</u> , then <u>true for all $n \in \mathbb{N}$</u> by mathematical induction.	A1	2.4
		(6)	
(b)	$u_n = 200(2^n) - \frac{1}{10}(4^n) = 200(2^n) - \frac{1}{10}(2^n)^2 = 0$ Leading to $2^n = \dots\{2000\}$ Alt : reaches $4^n > 2000 \times 2^n \Rightarrow n \log 4 > \log 2000 + n \log 2$	M1	3.1a
	Solves $2^n = 2000 \Rightarrow n = \log_2 2000 = \dots\{10.96\}$ $2^n = 2000 \Rightarrow n = \frac{\log 2000}{\log 2} = \dots\{10.96\}$ Alt: Solve the linear equation to find n	dM1	1.1b
	$u_{11} = -9830.4$ or $-\frac{49152}{5}$	A1	2.2a
		(3)	
(9 marks)			



Notes**(a)**

B1: Checks the closed form works for $n = 1$ and $n = 2$. Allow if they use the recurrence to find u_3 and check for $n = 2$ and $n = 3$, but a consecutive pair must be checked.

M1: Makes the inductive assumption. If not explicitly made, accept just stating “ $n = k$ and $n = k + 1$ ” as making the assumption these are true – or implied by use of the relevant formulae, as long as the assumption is made clear in the conclusion. May use e.g. $n = k - 2$ and $n = k - 1$ instead and show true for $n = k$. It must be clear it is the closed forms they are assuming, not a recurrence form.

M1: Substitutes expression for $n = k$ and $n = k + 1$ (or equivalents) into the recurrence formula.

M1: Uses algebra in an attempt to achieve the required result

e.g. Uses the coefficients of 6 and 8 to

- write as 2^n terms as 2^{k+2} and simplify.
- write as 4^n terms as 4^{k+1} or 4^{k+2} and simplify.

Note this is a method mark so you may score for the attempt even if some of the working is incorrect as long as the intent to reach the correct form is clear and at least one bit of indec work is correct.

A1: Completes the process correctly to the required form

A1: Correct conclusion made at the end. Depends on all three M's and the A being gained and an attempt at both $n = 1$ and $n = 2$ having been shown true. Must convey the underlined ideas of

- true for $n = 1$ and $n = 2$
- if true for two successive cases, it is also true for the next case
- a suitable conclusion that it is true for all positive n

though accept equivalent wordings for these.

Note Accept work with n instead of k throughout the inductive step.

(b)

M1: Sets closed form $= 0$ or < 0 and solves to set up an inequality, or to find a non-zero value, for 2^n . Alternatively if the 4^n is not written in terms of 2^n score for a correct process for taking logs to get a linear equation in n . Be tolerant with incorrect inequalities for the M marks.

dM1: Solves $2^n = a$ where $a > 0$ by any valid means. May be by inspection. In the Alt it is for proceeding to a value for n from the linear equation.

A1: Deduces the first negative term of the sequence

