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EDEXCEL INTERNATIONAL A LEVEL MATHS: QUESTIONS BY TOPIC

Proof by Induction (FP1.7)

International A Level Mathematics · FP1: Further Pure Mathematics 1 (WFM01)

13 questions · 125 marks · Mark schemes included

Past papers October 2020 - June 2025 · 1 sub-topic

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HOW TO USE THIS BOOKLET

Questions are grouped by sub-topic and run from the lowest to the highest mark tariff, then oldest paper first.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme; < QUESTION brings you back.

A gold bar in the left margin marks exactly where the question sits on the original page.

Diagrams, graphs, grids and tables are printed exactly as they appear in the original papers.

Each question is filed under its main topic; longer questions may also test skills from other topics.

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QUESTIONS

13 questions · 125 marks · 1 sub-topic

SUB-TOPIC FP1.7a

PROOF BY INDUCTION

Sub-topic FP1.7a · 13 questions · 125 marks · lowest tariff first



9. Prove, by induction, that for $n \in \mathbb{Z}, n \geq 2$

$$4^n + 6n - 10$$

is divisible by 18

(5)



Question 9 continued

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Question 9 continued

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(Total for Question 9 is 5 marks)

TOTAL FOR PAPER IS 75 MARKS



8. Prove by induction that $4^{n+2} + 5^{2n+1}$ is divisible by 21 for all positive integers n .

(6)

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Question 8 continued

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Q8

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(Total 6 marks)

TOTAL FOR PAPER: 75 MARKS

END



9. Prove by induction that for all positive integers n

$$\sum_{r=1}^n \log(2r-1) = \log\left(\frac{(2n)!}{2^n n!}\right) \quad (6)$$



Question 9 continued

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TOTAL FOR PAPER IS 75 MARKS



8. Prove by induction that for $n \in \mathbb{Z}^+$

$$f(n) = 7^{n-1} + 8^{2n+1}$$

is divisible by 57

(6)



Question 8 continued

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(Total for Question 8 is 6 marks)



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Q9

TOTAL FOR PAPER: 75 MARKS

Question 9 continued

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Question 9 continued

Lined area for writing the answer to Question 9 continued.



Question 9 continued

Lined area for writing the answer to Question 9 continued.



Question 9 continued

Lined area for writing the answer to Question 9.

(Total for Question 9 is 10 marks)

TOTAL FOR PAPER: 75 MARKS

END



10. (i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^n = 3^{n-1} \begin{pmatrix} 2n+3 & -n \\ 4n & 3-2n \end{pmatrix} \quad (5)$$

(ii) Prove by induction that for $n \in \mathbb{Z}^+$

$$f(n) = 8^{2n+1} + 6^{2n-1}$$

is divisible by 7

(5)

Question 10 continued

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Question 10 continued

Lined area for writing the answer to Question 10.



Question 10 continued

Lined area for writing the answer to Question 10.



Question 10 continued

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(Total for Question 10 is 10 marks)

TOTAL FOR PAPER IS 75 MARKS



Question 8 continued

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Question 8 continued

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Question 8 continued

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(Total for Question 8 is 10 marks)



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Question 7 continued

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(Total 11 marks)

Q7
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8. (i) Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^n \frac{2r^2 - 1}{r^2(r+1)^2} = \frac{n^2}{(n+1)^2} \quad (6)$$

- (ii) Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 12^n + 2 \times 5^{n-1}$$

$$\text{is divisible by } 7 \tag{6}$$

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Q8

TOTAL FOR PAPER: 75 MARKS

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Question 9 continued

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(Total 12 marks)

Q9

TOTAL FOR PAPER: 75 MARKS

END





Question 8 continued

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Question 8 continued

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(Total for Question 8 is 13 marks)



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Question 9 continued

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Question 9 continued

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Q9

(Total 14 marks)

END

TOTAL FOR PAPER: 75 MARKS



MARK SCHEMES

official mark schemes, in the same order as the questions



Question Number	Scheme	Notes	Marks
9	$f(n) = 4^n + 6n - 10 \quad n \in \mathbb{Z} \quad n \geq 2$		
General guidance: Apply the way that best fits the overall approach. Condone work in e.g., n instead of k. Attempts with no induction e.g., not using $f(k)$ in an equation with $f(k+1)$ score a max of 11000. Using e.g., $f(k+2) - f(k+1)$ requires a clear indication of assuming $f(k+1)$ is true to access the last three marks. Alternative explanations are unlikely to access the last three marks unless there is a fully convincing justification of divisibility, e.g., $f(k+1) - f(k) = 3 \times 4^k + 6$ followed by “Since 3×4^k is a multiple of both 3 and 4 and hence 12, $3 \times 4^k + 6$ is divisible by 18” is not a sound argument. Attempts that involve further induction on different expressions must be complete methods to access the last 3 marks. <u>Allow use of -18 but if any different multiples of 18 are involved e.g., 36, the first A1 requires “36 is a multiple of/divisible by (but not “factor of”) 18” oe for each case</u> B1: Any correct numerical expression that is not just “18” is sufficient for this mark e.g., $16 + 12 - 10$, $28 - 10$, $4^2 + 2$. Starting with e.g., $f(3)$ scores a max of 01110. Ignore an extra evaluation of $f(1)$ but a comment on $f(1)$’s divisibility is final A0 since $n \geq 2$ Final A1: There must be evidence that true for $n = k \Rightarrow$ true for $n = k + 1$ but it could be minimal and be scored in a conclusion or a narrative or via both. So if e.g., “Assume true for $n = k \dots$” is seen in the work followed by “true for $n = k + 1$” in a conclusion this is sufficient. Condone “for all $n \in \mathbb{Z}$”, “all $n \in \mathbb{Z} \quad n > 2$”, “all $\mathbb{Z} > (or \geq) 2$” but not $n \in \mathbb{R}$			
Way 1 $f(k+1) - f(k)$	$f(2) = 4^2 + 6 \times 2 - 10 = 18$	Obtains $f(2) = 18$ with substitution	B1
	$f(k+1) = 4^{k+1} + 6(k+1) - 10$	Attempts $f(k+1)$	M1
	$f(k+1) - f(k) = 4^{k+1} + 6(k+1) - 10 - (4^k + 6k - 10)$ $= 4^{k+1} - 4^k + 6 = 3 \times 4^k + 6$ $= 3(4^k + 6k - 10) - 18k + 36$	Attempts $f(k+1) - f(k)$, uses $4^{k+1} = 4 \times 4^k$ & obtains $pf(k) + g(k)$ with $g(k)$ linear (allow constant $\neq 0$)	M1
	$f(k+1) = 4f(k) + 18(2-k)$ $f(k)$ may be written in full	Correct factorised expression Allow $4f(k) + 18 \times 2 - 18 \times k$ If $f(k+1)$ is not made the subject then e.g., “true for $f(k+1) - f(k)$ ” is also required	A1
	True for $n = 2$, if true for $n = k$ then true for $n = k + 1$ so true for all $n \in \mathbb{Z} \quad (n \geq 2)$ Minimum in bold.	Full conclusion/narrative and no errors. All marks needed but allow if B0 provided this mark was only withheld for insufficient working.	A1
			(5)
Way 2 $f(k+1) = \dots$	$f(2) = 4^2 + 6 \times 2 - 10 = 18$	Obtains $f(2) = 18$ with substitution	B1
	$f(k+1) = 4^{k+1} + 6(k+1) - 10$	Attempts $f(k+1)$	M1
	$= 4 \times 4^k + 6k - 4$ $= 4(4^k + 6k - 10) - 18k + 36$	Uses $4^{k+1} = 4 \times 4^k$ & obtains $pf(k) + g(k)$ with $g(k)$ linear (allow constant $\neq 0$)	M1
	$= 4f(k) + 18(2-k)$ $f(k)$ may be written in full	Correct factorised expression Allow $4f(k) + 18 \times 2 - 18 \times k$	A1
	True for $n = 2$, if true for $n = k$ then true for $n = k + 1$ so true for all $n \in \mathbb{Z} \quad (n \geq 2)$ Minimum in bold.	Full conclusion/narrative and no errors. All marks needed but allow if B0 provided this mark was only withheld for insufficient working.	A1
			(5)



Question Number	Scheme	Notes	Marks
9 cont.	$f(n) = 4^n + 6n - 10 \quad n \in \mathbb{Z} \quad n \geq 2$		
Way 3 $f(k+1) - mf(k)$	$f(2) = 4^2 + 6 \times 2 - 10 = 18$	Obtains $f(2) = 18$ with substitution	B1
	$f(k+1) = 4^{k+1} + 6(k+1) - 10$	Attempts $f(k+1)$	M1
	$f(k+1) - mf(k) = 4^{k+1} + 6(k+1) - 10 - m(4^k + 6k - 10)$ $= (4-m)4^k + (6-6m)k - 4 + 10m$ e.g. $m = -14 \Rightarrow 18 \times 4^k + 90k - 144$ e.g. $m = 4 \Rightarrow -18k + 36$	Attempts $f(k+1) - mf(k)$ and uses a value of m to obtain $c \times 4^k + \dots$ where c is a multiple of their 18 or uses $m = 4$	M1
	e.g., $f(k+1) = -14f(k) + 18(4^k + 5k - 8)$ $f(k+1) = 4f(k) + 18(2-k)$ $f(k)$ may be written in full	A correct factorised expression Allow $-14f(k) + 18 \times 4^k + 18 \times 5k - 18 \times 8$ If $f(k+1)$ is not made the subject then e.g., “true for $f(k+1) - mf(k)$ ” is also required	A1
	True for $n = 2$, if true for $n = k$ then true for $n = k + 1$ so true for all $n \in \mathbb{Z} \quad (n \geq 2)$ Minimum in bold.	Full conclusion/narrative and no errors. All marks needed but allow if B0 provided this mark was only withheld for insufficient working.	A1
			(5)
Way 4 $f(k) = 18M$	$f(2) = 4^2 + 6 \times 2 - 10 = 18$	Obtains $f(2) = 18$ with substitution	B1
	$f(k+1) = 4^{k+1} + 6(k+1) - 10$	Attempts $f(k+1)$	M1
	$f(k) = 18M, \quad f(k+1) = 4 \times 4^k + 6k - 4$ $= 4 \times 18M - 18k + 36$	Sets $f(k) = 18M$, uses $4^{k+1} = 4 \times 4^k$ & obtains $pf(k) + g(k)$ with $g(k)$ linear (allow constant $\neq 0$)	M1
	$f(k+1) = 18(4M + 2 - k)$	A correct factorised expression Allow $18 \times 4M + 18 \times 2 - 18 \times k$	A1
	True for $n = 2$, if true for $n = k$ then true for $n = k + 1$ so true for all $n \in \mathbb{Z} \quad (n \geq 2)$ Minimum in bold.	Full conclusion/narrative and no errors. All marks needed but allow if B0 provided this mark was only withheld for insufficient working.	A1
			(5)
PAPER TOTAL: 75			

Question Number	Scheme	Notes	Marks
	$f(n) = 4^{n+2} + 5^{2n+1}$ divisible by 21		
8	$n = 1, 4^3 + 5^3 = 189 = 9 \times 21$ (Or $n = 0, 4^2 + 5^1 = 21$)	$f(1) = 21 \times 9$ Accept $f(0) = 21$ as an alternative starting point.	B1
	Assume that for $n = k$, $f(k) = (4^{k+2} + 5^{2k+1})$ is divisible by 21 for $k \in \mathbb{Z}^+$.		
	$f(k+1) - f(k) = 4^{k+3} + 5^{2k+3} - (4^{k+2} + 5^{2k+1})$	Applies $f(k+1)$ with at least 1 power correct. May be just as $f(k+1)$, or as part of an expression in $f(k+1)$ and $f(k)$.	M1
	$= 4 \cdot 4^{k+2} + 25 \cdot 5^{2k+1} - 4^{k+2} - 5^{2k+1}$	For a correct expression in $f(k+1)$, and possibly $f(k)$, with powers reduced to those of $f(k)$.	A1
	$= 3 \cdot 4^{k+2} + 24 \cdot 5^{2k+1}$		
	$= 3f(k) + 21 \cdot 5^{2k+1}$ or $= 24f(k) - 21 \cdot 4^{k+2}$	For one of these expression or equivalent with obvious factor of 21 in each.	A1
	$f(k+1) = 4f(k) + 21 \cdot 5^{2k+1}$	Makes $f(k+1)$ the subject or gives clear reasoning of each term other than $f(k+1)$ being divisible by 21. Dependent on at least one of the previous accuracy marks being awarded.	dM1
	{ $f(k+1)$ is divisible by 21 as both $f(k)$ and 21 are both divisible by 21 }		
	If the result is true for $n = k$, then it is now true for $n = k + 1$. As the result has shown to be true for $n = 1$ (or 0) , then the result is true for all n ($\in \mathbb{Z}^+$) .	Correct conclusion seen at the end. Condone true for $n = 1$ stated earlier. Depends on both M's and A's, but may be scored if the B is lost as long as at least $f(1) = 189$ was reached (so e.g. if the 21×9 was not shown)	A1 cso
			(6)
ALT for first 4 marks	$n = 1, 4^3 + 5^3 = 189 = 9 \times 21$ (Or $n = 0, 4^2 + 5^1 = 21$)	As main scheme.	B1
	$f(k+1) - \alpha f(k) = 4^{k+3} + 5^{2k+3} - \alpha(4^{k+2} + 5^{2k+1})$	Attempts $f(k+1)$ in any equation (as main scheme).	M1
	$f(k+1) - \alpha f(k) = (4 - \alpha)4^{k+2} + (25 - \alpha)5^{2k+1}$	For a correct expression with any α , with powers reduced to match $f(k)$.	A1
	$f(k+1) - \alpha f(k) = (4 - \alpha)(4^{k+2} + 5^{2k+1}) + 21 \cdot 5^{2k+1}$ $f(k+1) - \alpha f(k) = (25 - \alpha)(4^{k+2} + 5^{2k+1}) - 21 \cdot 4^{k+2}$	Any suitable equation with powers sorted appropriately to match $f(k)$	A1
	NB: $\alpha = 0, \alpha = 4, \alpha = 25$ will make relevant terms disappear, but marks should be awarded accordingly.		
			Total 6

Question	Scheme	Marks
9	For $n = 1$, $\sum_{r=1}^1 \log(2r-1) = \log(2-1) = \log 1$ and $\log\left(\frac{(2 \times 1)!}{2^1 1!}\right) = \log 1$ So true for $n = 1$	B1
	(Assume the result is true for $n = k$, so $\sum_{r=1}^k \log(2r-1) = \log\left(\frac{(2k)!}{2^k k!}\right)$ Then) $\sum_{r=1}^{k+1} \log(2r-1) = \log\left(\frac{(2k)!}{2^k k!}\right) + \log(2(k+1)-1)$	M1
	$= \log\left(\frac{(2k)!}{2^k k!} \times (2k+1)\right)$	M1
	$= \log\left(\frac{(2k+1)!}{2^k k!} \times \frac{2k+2}{2k+2}\right) = \log\left(\frac{(2k+2)!}{2^k \times 2(k+1)!}\right)$	M1
	$= \log\left(\frac{(2k+2)!}{2^{k+1}(k+1)!}\right)$	A1
	Hence result is true for $n = k+1$. As true for $n = 1$ and have shown if true for $n = k$ then it is true for $n = k+1$, so it is true for all $n \in \mathbb{N}$ by induction.	A1
		(6)
(6 marks)		

Notes:**(a)**

B1: Checks the result for $n = 1$. Must see **both** sides (possibly in one line) identified as $\log 1$ or 0 but may not see much more than this.

M1: Makes the inductive assumption (may be implied) and applies it to the question by adding the $(k+1)^{\text{th}}$ term to the expression for the sum to k terms. Allow if there are minor slips (e.g. a missing factorial) if the intent is clear.

M1: Attempts to combine or split log terms appropriately. Not dependent, so may be scored if the wrong term is added in the previous M as long it is a log term.

M1: Introduces the relevant cancelling factors to achieve the $(2k+2)!$ term. The introduction of the factors must shown or implied in an intermediate step. Alternatively, may decompose from the $k+1$ statement to achieve the same intermediate expression.

A1: Achieves correct expression from correct work (or correctly shows equivalence).

A1: Completes the induction by demonstrating the result clearly, with suitable conclusion conveying “true for $n = 1$ ”, “assumed true for $n = k$ ” and “shown true for $n = k+1$ ”, and “hence true for all n ”. Depends on all except the B mark, though a check for $n = 1$ must have been attempted.

NB Allow the M’s and first A if n is used throughout but the steps are correct, but must have used a different variable for the final A.



Alt steps if splitting logs:

$$\sum_{r=1}^{k+1} \log(2r-1) = \log\left(\frac{(2k)!}{2^k k!}\right) + \log(2k+1) \quad \mathbf{M1}$$

$$= \log(2k)! - \log(2^k k!) + \log(2k+1) \quad \mathbf{M1}$$

$$= \log(2k+1)! - \log\left(\frac{2^{k+1}(k+1)!}{2 \times (k+1)}\right) = \log\left(\frac{(2k+1)!(2k+2)}{2^k (k+1)!}\right) \quad \mathbf{M1}$$



Question Number	Scheme	Notes	Marks
8	$f(k) = 7^{k-1} + 8^{2k+1}$		
	General guidance: Apply the Way that best fits the overall approach. Condone work in e.g., n instead of k. Allow use of -57 but if any different multiples of 57 are involved, e.g., 114, the last A1 additionally requires “114 is a multiple of/divisible by (but not “factor of”) 57” oe for each case. Ignore work re the divisibility of $f(2)$, $f(3)$ etc but starting with e.g., $f(2)$ scores a max of 01110. Final A1: There must be evidence that true for $n = k \Rightarrow$ true for $n = k + 1$ but it could be minimal and be scored in a conclusion or a narrative or via both. So if e.g., “Assume true for $n = k \dots$” is seen in the work followed by “true for $n = k + 1$” in a conclusion this is sufficient. May say “is divisible by 57” for “true”. Condone “for all $n \in \mathbb{Z}$” but not $n \in \mathbb{R}$		
Way 1 $f(k+1)$ $-f(k)$	$n = 1: f(1) = [7^0 + 8^3] = 513,$ $513 \div 57 = 9$ oe	Obtains 513 for $f(1)$ and shows 513 is divisible by 57. Allow $\frac{1+512}{57} = 9$	B1
	$[f(k+1)] = 7^{(k+1)-1} + 8^{2(k+1)+1} \{ = 7^k + 8^{2k+3} \}$	Attempts $f(k+1)$	M1
	$[f(k+1) - f(k)] =$ $7(7^{k-1}) - 7^{k-1} + 8^2(8^{2k+1}) - 8^{2k+1}$	Obtains expression for $f(k+1) - f(k)$ in 7^{k-1} and 8^{2k+1} only	M1
	$= 6(7^{k-1} + 8^{2k+1}) + 57(8^{2k+1})$ $\Rightarrow f(k+1) = 7f(k) + 57(8^{2k+1})$ or $= 63(7^{k-1} + 8^{2k+1}) - 57(7^{k-1})$ $\Rightarrow f(k+1) = 64f(k) - 57(7^{k-1})$	M1: Obtains expression for $f(k+1)$ in terms of $f(k)$. May not see $f(k+1) = \dots$ A1: Correct expression. Must see $f(k+1) = \dots$ Allow if e.g., $7f(k)$ written as $7(7^{k-1} + 8^{2k+1})$ or $7(7^{k-1}) + 7(8^{2k+1})$	M1 A1
	Shown true for $n = 1$ and if true for $n = k$ then true for $n = k + 1$ so true for all $n \in \mathbb{Z}^+$	Makes correct conclusion or narrative with no errors throughout. Minimum in bold . Requires all previous marks but can follow B0 if that mark was withheld for omitting to show that 513 is divisible by 57.	A1
(6)			
Way 2 $f(k+1) =$	$n = 1: f(1) = [7^0 + 8^3] = 513,$ $513 \div 57 = 9$ oe	Obtains 513 for $f(1)$ and shows 513 is divisible by 57. Allow $\frac{1+512}{57} = 9$	B1
	$[f(k+1)] = 7^{(k+1)-1} + 8^{2(k+1)+1} \{ = 7^k + 8^{2k+3} \}$	Attempts $f(k+1)$	M1
	$[f(k+1)] = 7(7^{k-1}) + 8^2(8^{2k+1})$	Obtains expression for $f(k+1)$ in 7^{k-1} and 8^{2k+1} only	M1
	$= 7(7^{k-1} + 8^{2k+1}) + 57(8^{2k+1})$ $\Rightarrow f(k+1) = 7f(k) + 57(8^{2k+1})$ or $= 64(7^{k-1} + 8^{2k+1}) - 57(7^{k-1})$ $\Rightarrow f(k+1) = 64f(k) - 57(7^{k-1})$	M1: Obtains expression for $f(k+1)$ in terms of $f(k)$. May not see $f(k+1) = \dots$ A1: Correct expression. Must see $f(k+1) = \dots$ Allow if e.g., $7f(k)$ written as $7(7^{k-1} + 8^{2k+1})$ or $7(7^{k-1}) + 7(8^{2k+1})$	M1 A1
	Shown true for $n = 1$ and if true for $n = k$ then true for $n = k + 1$ so true for all $n \in \mathbb{Z}^+$	Makes correct conclusion or narrative with no errors throughout. Minimum in bold . Requires all previous marks but can follow B0 if that mark was withheld for omitting to show that 513 is divisible by 57.	A1
(6)			

Question Number	Scheme	Notes	Marks
9(i)	$n = 1 \Rightarrow u_1 = 3 \times 2 - 2 \times 3 = 0$ $n = 2 \Rightarrow u_2 = 3 \times 2^2 - 2 \times 3^2 = -6$	Shows the result is true for $n = 1$ and $n = 2$. Ignore references to $n = 3$.	B1
	Substitutes $u_k = 3 \times 2^k - 2 \times 3^k$ and $u_{k+1} = 3 \times 2^{k+1} - 2 \times 3^{k+1}$ into $(u_{k+2}) = 5u_{k+1} - 6u_k = 5(3 \times 2^{k+1} - 2 \times 3^{k+1}) - 6(3 \times 2^k - 2 \times 3^k)$ (The inductive assumption may be tacit for this mark.)		M1
	$(u_{k+2}) = 15 \times 2^{k+1} - 10 \times 3^{k+1} - 18 \times 2^k + 12 \times 3^k$ $= 15 \times 2^{k+1} - 9 \times 2^{k+1} - 10 \times 3^{k+1} + 4 \times 3^{k+1}$ $= 6 \times 2^{k+1} - 6 \times 3^{k+1}$	Gathers to a correct two term expression. Accept alternative forms such as $12 \times 2^k - 18 \times 3^k$	A1
	$u_{k+2} = 3 \times 2^{k+2} - 2 \times 3^{k+2}$	Achieves this result with no errors – must be clear it is u_{k+2} but this may have been seen at the start.	A1
	If the result is true for $n = k$ and $n = k + 1$ then it is true for $n = k + 2$. As the result has been shown to be true for $n = 1$ and $n = 2$ then the result is true for all n .		A1cso
	Correct conclusion including all the bold points in some form. Depends on all previous marks.		
			(5)
(ii)	$f(n) = 3^{3n-2} + 2^{4n-1}$		
	$f(1) = 3^1 + 2^3 = 11$	Shows the result is true for $n = 1$	B1
	$f(k+1) - f(k) = 3^{3(k+1)-2} + 2^{4(k+1)-1} - 3^{3k-2} - 2^{4k-1}$ $= 27 \times 3^{3k-2} + 16 \times 2^{4k-1} - 3^{3k-2} - 2^{4k-1} = 26 \times 3^{3k-2} + 15 \times 2^{4k-1} \left(= \frac{26}{9} 3^{3k} + \frac{15}{2} 2^{4k} \right)$ Attempts $f(k+1) - f(k)$ and reaches $\alpha \times 3^{3k-2} + \beta \times 2^{4k-1}$ or $\alpha \times 3^{3k} + \beta \times 2^{4k}$ (Mark variations on the theme as appropriate here and in the following marks.)		M1
	$= 15 \times (3^{3k-2} + 2^{4k-1}) + 11 \times 3^{3k-2}$ or $= 26 \times (3^{3k-2} + 2^{4k-1}) - 11 \times 2^{4k-1}$	Correct expression with $f(k)$ evident.	A1
	$f(k+1) = 16f(k) + 11 \times 3^{3k-2}$ or $f(k+1) = 27f(k) - 11 \times 2^{4k-1}$	Makes $f(k+1)$ the subject and states divisible by 11 (oe – may be implied by conclusion), or gives full reason why $f(k+1)$ is divisible by 11. Dependent on first M	dM1
	If the result is true for $n = k$ then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n .		A1cso
	Correct conclusion including all the bold points in some form. Depends on all previous marks.		
			(5)
			Total 10

ALT 1	$f(1) = 3^1 + 2^3 = 11$	Shows the result is true for $n = 1$	B1
	$f(k+1) = 3^{3k+1} + 2^{4k+3}$ $f(k+1) = 27 \times 3^{3k-2} + 16 \times 2^{4k-1}$ or $3 \times 3^{3k} + 8 \times 2^{4k}$ Attempts $f(k+1)$ and reduces power to $\alpha \times 3^{3k-2} + \beta \times 2^{4k-1}$ or $\alpha \times 3^{3k} + \beta \times 2^{4k}$		M1
	$f(k+1) = 16 \times (3^{3k-2} + 2^{4k-1}) + 11 \times 3^{3k-2}$ or $f(k+1) = 27 \times (3^{3k-2} + 2^{4k-1}) - 11 \times 2^{4k-1}$	Correct expression	A1
	$f(k+1) = 16f(k) + 11 \times 3^{3k-2}$ or $f(k+1) = 27f(k) - 11 \times 2^{4k-1}$	States divisible by 11 (oe– may be implied by conclusion) Depends on first M	dM1
	If the result is true for $n = k$ then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n.		A1cso
	Correct conclusion including all the bold points in some form. Depends on all previous marks.		
ALT 2	$f(1) = 3^1 + 2^3 = 11$	Shows the result is true for $n = 1$	B1
	Let $3^{3k-2} + 2^{4k-1} = 11M$		
	$f(k+1) = 3^{3k+1} + 2^{4k+3}$ $f(k+1) = 27(11M - 2^{4k-1}) + 2^{4k+3}$ or $3^{3k+1} + 16(11M - 3^{3k-2})$ Attempt $f(k+1)$ and expresses in terms of M		M1
	$f(k+1) = 297M - 11 \times 2^{4k-1}$ or $176M + 11 \times 3^{3k-2}$	Correct expression	A1
	$f(k+1) = 11(27M - 2^{4k-1})$ or $11(16M + 3^{3k-2})$	Takes out a factor of 11, or gives full reason why $f(k+1)$ is divisible by 11. Depends on first M	dM1
	If the result is true for $n = k$ then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n.		A1cso
	Correct conclusion including all the bold points in some form. Depends on all previous marks.		
ALT 3	$f(1) = 3^1 + 2^3 = 11$	Shows the result is true for $n = 1$	B1
	$f(k+1) - \alpha f(k) = 3^{3k+1} + 2^{4k+3} - \alpha(3^{3k-2} + 2^{4k-1})$ Attempts $f(k+1) - \alpha f(k)$ where $\alpha = 16$ or 27 or other appropriate value.		M1
	$= (27 - \alpha)3^{3k-2} + (16 - \alpha)2^{4k-1}$ e.g. $= 11 \times 3^{3k-2} + (16 - 16) \times 2^{4k-1}$ or $= ((27 - 27) \times 3^{3k-2}) - 11 \times 2^{4k-1}$	Correct expression for their α where a common factor of 11 is clear. (E.g. $\alpha = 5$)	A1
	E.g. $f(k+1) = 16f(k) + 11 \times 3^{3k-2}$ or $f(k+1) = 27f(k) - 11 \times 2^{4k-1}$	Makes $f(k+1)$ the subject in an expression where 11 is a clear common factor and states divisible by 11 (oe – may be implied by conclusion). Dependent on first M	dM1
	If the result is true for $n = k$ then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n.		A1cso
	Correct conclusion including all the bold points in some form. Depends on all previous marks.		

Question Number	Scheme	Notes	Marks
8	$\sum_{r=0}^n (r+1)(r+2)$		
(a)	$\sum_{r=0}^n r^2 + 3r + 2 = 2 + \frac{1}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) + 2n$ M1: Attempt to use at least one of the standard formulae correctly A1: For $\frac{1}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) + (2n \text{ or } 2n+2)$ A1: Fully correct expression		M1A1A1
	$\frac{1}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) + 2n + 2 = (n+1) \left[\frac{1}{6}n(2n+1) + \frac{3}{2}n + 2 \right]$ Attempt to factorise $(n+1)$ It is a “show” question so this must be seen (in any equivalent form). If their expression does not allow for factorising $(n+1)$ score M0		M1
	$\frac{1}{3}(n+1)[n^2 + 5n + 6]$	May obtain a cubic and extract a different factor ie $n+2$ or $n+3$	
	$\frac{1}{3}(n+1)(n+2)(n+3)^*$	Cso At least one intermediate step in the working must be seen.	A1*
			(5)
(a) Way 2	$\sum_{r=0}^n (r+1)(r+2) = \sum_{r=1}^{n+1} r(r+1)$ $= \sum_{r=1}^{n+1} r^2 + r = \frac{1}{6}(n+1)(n+2)(2(n+1)+1) + \frac{1}{2}(n+1)(n+2)$ M1: Attempt to use at least one of the standard formulae correctly with $n = n+1$ A1: For $\frac{1}{6}(n+1)(n+2)(2(n+1)+1)$ or $\frac{1}{2}(n+1)(n+2)$ A1: Fully correct expression		M1A1A1
	$\frac{1}{6}(n+1)(n+2)(2(n+1)+1) + \frac{1}{2}(n+1)(n+2) = (n+1) \left[\frac{1}{6}(n+1)(2n+3) + \frac{1}{2}(n+2) \right]$ Attempt to factorise $(n+1)$ (see additional comments above)		M1
	$\frac{1}{3}(n+1)[n^2 + 5n + 6]$	May obtain a cubic and extract a different factor ie $n+2$ or $n+3$	
	$\frac{1}{3}(n+1)(n+2)(n+3)^*$	Cso At least one intermediate step in the working must be seen.	A1*
(b)	Upper limit = 99	Correct upper limit	B1
	$10 \times 11 + 11 \times 12 + 12 \times 13 + \dots + 100 \times 101 = \sum_{r=0}^{99} (r+1)(r+2) - \sum_{r=0}^8 (r+1)(r+2)$ Fully correct strategy for the sum using their upper limit for the first sum and upper limit 8 for the second in the result from (a). Lower limits 0 or 1		M1
	$= \frac{1}{3}(100)(101)(102) - \frac{1}{3}(9)(10)(11)$ $= 343\,070$	Correct value	A1



			(3)
			Total 8

Question Number	Scheme	Notes	Marks
9(i)	$u_n = 5 \times 2^{n-1} - n \times 2^n$		
	$n = 1 \Rightarrow u_1 = 5 \times 2^0 - 1 \times 2 = 3$ (Shows the result is true for $n = 1$)		B1
	Assume true for $n = k$ so that $u_k = 5 \times 2^{k-1} - k \times 2^k$		
	$u_{k+1} = 2(5 \times 2^{k-1} - k \times 2^k) - 2^{k+1}$	Attempts u_{k+1} using the recurrence relationship	M1
	$= 5 \times 2^k - k \times 2^{k+1} - 2^{k+1}$	Correct expanded expression	A1
	$= 5 \times 2^k - (k+1)2^{k+1}$	Achieves this result with no errors	A1
	If the result is true for $n = k$ then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n .		A1cso
	The final mark depends on all except the B mark, though a check for $n = 1$ must have been attempted.		
			(5)
(ii)	$f(n) = 5^{n+2} - 4n - 9$		
	$f(1) = 125 - 4 - 9 = 112 = 16 \times 7$	Shows $f(1)$ is divisible by 16 Either of 112 or 16×7 must be seen	B1
	Assume true for $n = k$ so that $5^{k+2} - 4k - 9$ is divisible by 16		
	$f(k+1) = 5^{k+3} - 4(k+1) - 9$	Attempts $f(k+1)$	M1
	$= 5 \times (5^{k+2} - 4k - 9) + \dots$	Attempts to express in terms of $f(k)$	dM1
	$= 5 \times (5^{k+2} - 4k - 9) + 16k + 32$	Correct expression for $f(k+1)$	A1
	If the result is true for $n = k$ then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n .		A1cso
	The final mark depends on all except the B mark, though a check for $n = 1$ must have been attempted		(5)
			Total 10

ii ALT 1	$f(1) = 125 - 4 - 9 = 112 = 16 \times 7$	Shows $f(1)$ is divisible by 16 Either of 112 or 16×7 must be seen	B1
	Assume $5^{k+2} - 4k - 9$ is divisible by 16		
	$f(k+1) - mf(k) = 5^{k+3} - 4(k+1) - 9 - m(5^{k+2} - 4k - 9)$ Attempt $f(k+1) - mf(k)$		M1
	$= (5-m)(5^{k+2} - 4k - 9) + \dots$	Attempts to express in terms of $f(k)$	dM1
	$f(k+1) = 5 \times (5^{k+2} - 4k - 9) + 16k + 32$	Correct expression for $f(k+1)$	A1
	If the result is true for $n = k$ then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n .		A1cso
	The final mark depends on all except the B mark, though a check for $n = 1$ must have been attempted		
ii ALT 2	$f(1) = 125 - 4 - 9 = 112 = 16 \times 7$	Shows $f(1)$ is divisible by 16 Either of 112 or 16×7 must be seen	B1
	Assume $5^{k+2} - 4k - 9$ is divisible by 16		
	$f(k+1) - f(k) = 5^{k+3} - 4(k+1) - 9 - (5^{k+2} - 4k - 9)$ Attempt $f(k+1) - f(k)$		M1
	$f(k+1) - f(k) = 5 \times 5^{k+2} - 5^{k+2} - 4k - 4 - 9 + 4k + 9$ $= 4 \times 5^{k+2} - 4 = 4(5^{k+2} - 1)$ Obtains a simplified expression for the difference and attempts to prove $(5^{k+2} - 1)$ is divisible by 4 using induction		dM1
	Correct proof for $(5^{k+2} - 1)$ being divisible by 4 and states that thus as the difference is divisible by 16, $f(k+1)$ is divisible by 16		A1
	If the result is true for $n = k$ then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n .		A1 cso
	The final mark depends on all except the B mark, though a check for $n = 1$ must have been attempted		
ii ALT 3	$f(1) = 125 - 4 - 9 = 112 = 16 \times 7$	Shows $f(1)$ is divisible by 16 Either of 112 or 16×7 must be seen	B1
	$f(k)$ is divisible by 16 so set $f(k) = 16\lambda$		
	$5^{k+2} = 16\lambda + 4k + 9$		M1
	$f(k+1) = 5^{k+3} - 4(k+1) - 9$ $= 5 \times 5^{k+2} - 4k - 13 = 5(16\lambda + 4k + 9) - 4k - 13$	Expresses $f(k+1)$ in terms of λ and k and collects terms	dM1
	$= 80\lambda + 16k + 32$	Correct expression May have factor of 16 taken out	A1
	If the result is true for $n = k$ then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n .		A1cso
	The final mark depends on all except the B mark, though a check for $n = 1$ must have been attempted		



Question	Scheme	Marks
10.(i)	$n = 1 \Rightarrow 3^{n-1} \begin{pmatrix} 2n+3 & -n \\ 4n & 3-2n \end{pmatrix} = 3^0 \begin{pmatrix} 2(1)+3 & -1 \\ 4(1) & 3-2(1) \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^1$	B1
	Assume true for $n = k$ so that $\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^k = 3^{k-1} \begin{pmatrix} 2k+3 & -k \\ 4k & 3-2k \end{pmatrix}$	
	$\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^{k+1} = 3^{k-1} \begin{pmatrix} 2k+3 & -k \\ 4k & 3-2k \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}$ or $\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^{k+1} = 3^{k-1} \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2k+3 & -k \\ 4k & 3-2k \end{pmatrix}$	M1
	$= 3^{k-1} \begin{pmatrix} 10k+15-4k & -2k-3-k \\ 20k+12-8k & -4k+3-2k \end{pmatrix}$ or $3^{k-1} \begin{pmatrix} 10k+15-4k & -5k-3+2k \\ 8k+12+4k & -4k+3-2k \end{pmatrix}$ or $3^{k-1} \begin{pmatrix} 6k+15 & -3k-3 \\ 12k+12 & -6k+3 \end{pmatrix}$	A1
	$\left[= 3^k \begin{pmatrix} 2k+5 & -k-1 \\ 4k+4 & -2k+1 \end{pmatrix} \right] = 3^k \begin{pmatrix} 2(k+1)+3 & -(k+1) \\ 4(k+1) & 3-2(k+1) \end{pmatrix}$	A1
	If the result is true for $n = k$ then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n .	A1cso
		(5)
(ii)	$f(1) = 8^3 + 6 = 518 = 74 \times 7$ (so true for $n = 1$)	B1
	Assume true for $n = k$ so that $8^{2k+1} + 6^{2k-1}$ is divisible by 7	
	$f(k+1) = 8^{2k+3} + 6^{2k+1}$	M1
	$= 64 \times (8^{2k+1} + 6^{2k-1}) + \dots$ or $36 \times (8^{2k+1} + 6^{2k-1}) + \dots$	dM1
	$= 64 \times (8^{2k+1} + 6^{2k-1}) - 28 \times 6^{2k-1}$ or $36 \times (8^{2k+1} + 6^{2k-1}) + 28 \times 8^{2k+1}$	A1
	So if the result is true for $n = k$ then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n .	A1cso
		(5)
(ii) Alt	$f(1) = 8^3 + 6 = 518 = 74 \times 7$ (so true for $n = 1$)	B1
	Assume true for $n = k$ so that $8^{2k+1} + 6^{2k-1}$ is divisible by 7	
	$f(k+1) - Mf(k) = 8^{2k+3} + 6^{2k+1} - M(8^{2k+1} + 6^{2k-1})$	M1
	$= (64 - M)(8^{2k+1} + 6^{2k-1}) + \dots$ or $(36 - M)(8^{2k+1} + 6^{2k-1}) + \dots$	dM1
	$= (64 - M)(8^{2k+1} + 6^{2k-1}) - 28 \times 6^{2k-1}$ or $(36 - M)(8^{2k+1} + 6^{2k-1}) + 28 \times 8^{2k+1}$	A1
	$\Rightarrow f(k+1) - Mf(k)$ divisible by 7 $\Rightarrow f(k+1)$ divisible by 7. So if the result is true for $n = k$ then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n .	A1cso
(Total 10 marks)		

Notes

(i)

B1: Shows the result is true for $n = 1$. The LHS may just be stated, for the RHS accept as a minimum either one unsimplified term or the 3^0 seen. (Conclusion not needed here if both sides have been found correctly.)

M1: Attempts $\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^{k+1}$ either way round using the result for $n = k$.

A1: Correct unsimplified matrix. The coefficients inside may be simplified but the common factor 3 not taken out directly.

A1: Achieves this result with no errors, via $3^{k-1} \begin{pmatrix} 6k+15 & -3k-3 \\ 12k+12 & -6k+3 \end{pmatrix}$ or $3^k \begin{pmatrix} 2k+5 & -k-1 \\ 4k+4 & -2k+1 \end{pmatrix}$ (oe with simplified linear terms).

A1cso: Suitable conclusion following fully correct work. Must include in some form the points “true for $n=1$ ”, “true for $n=k$ implies true for $n=k+1$ ” and conclude true for all n in the conclusion. Depends on the preceding MAA marks and at least stating the correct matrix for $n=1$ in the initial base case check (So B0M1A1A1A1 is possible).

(ii)

B1: Shows the result is true for $n = 1$, Must express as a multiple of 7 or clearly show the factor.

M1: Attempts $f(k+1)$

dM1: Attempts to express $f(k+1)$ in terms of $f(k)$. Note they may let $f(k) = 7m$ where m is an integer and use this in the working.

A1: Correct expression for $f(k+1)$ in terms of $f(k)$ (or m)

A1cso: Suitable conclusion following fully correct work. Must include in some form the points “true for $n=1$ ”, “true for $n=k$ implies true for $n=k+1$ ” and conclude true for all n in the conclusion. Depends on the preceding MdMA marks and finding at least $f(1) = 518$ (So B0M1dM1A1A1 is possible if all that is missing is showing the factor 7 in $f(1)$).

(ii) Alt

B1: Shows the result is true for $n = 1$. Must express as a multiple of 7 or clearly show the factor.

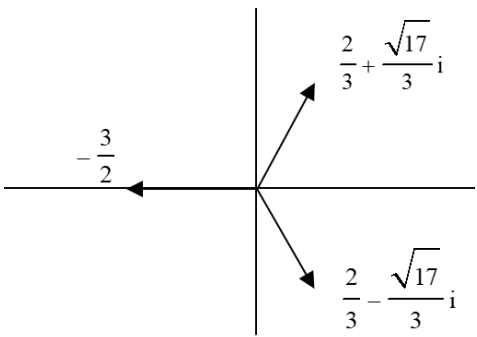
M1: Attempt $f(k+1) - Mf(k)$ for any integer M . If $M = 0$ this is the main scheme. $M = 1$ may be seen frequently, but other value are possible.

dM1: Attempts to express $f(k+1) - Mf(k)$ in terms of $f(k)$ or otherwise show a common factor of 7.

A1: Correct expression for $f(k+1) - Mf(k)$ in terms of $f(k)$ or with clear common factor of 7 shown. Note if $M = 1$ is used, the expression becomes $63 \times 8^{2k+1} + 35 \times 6^{2k-1} = 7(9 \times 8^{2k+1} + 5 \times 6^{2k-1})$ which is fine for dM1A1.

A1cso: Refers to divisibility of $f(k+1)$ and makes suitable conclusion following fully correct work. Must include in some form the points “true for $n=1$ ”, “true for $n=k$ implies true for $n=k+1$ ” and conclude true for all n in the working. Depends on the preceding MdMA marks and finding at least $f(1) = 518$ (So B0M1dM1A1A1 is possible if all that is missing is showing the factor 7 in $f(1)$).

4(b)	Attempts that include long division: If they find a quadratic factor as in the main scheme e.g., $z^2 - \frac{4}{3}z + \frac{7}{3}$ allow M1A1 as before. Dividing it into $f(z)$ can lead to equations $B - 14 + \frac{4}{3}A + \frac{32}{3} = 0$ and $C - \frac{7}{3}(A + 8) = 0$ and these could be used with the equation $-\frac{81}{4} + \frac{9}{4}A - \frac{3}{2}B + C = 0$ from using $f(-\frac{3}{2}) = 0$ or long division by $z + \frac{3}{2}$. Score the next M for obtaining real values for all constants and the A marks as usual. Other attempts including long division that do not find the quadratic factor as per the main scheme we will score as follows. Award the first M1 for credible work to obtain enough equations with real coefficients involving A, B and C so that values could be found, followed by A1 for correct equations. The next M1 is for solving to obtain values for A, B and C and then A1 for two correct values and final A1 for all 3 correct. If divided by $z + \frac{3}{2}$ the quadratic factor is $6z^2 + (A - 9)z + B - \frac{3}{2}A + \frac{27}{2}$ It is possible to apply the quadratic formula to this and equate the answer to $\frac{2}{3} \pm \frac{\sqrt{17}}{3}i$ and generate further equations that way. (Long division by complex factors is unlikely but could lead to the other equations in Alt 1) There are potentially a lot of possible precise routes involving long division and coefficient comparison etc. but the above mark scheme principles apply.	
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Question Number	Scheme/Notes	Marks
4(c)	 May use points or lines/vectors from origin. Note that B0B1 is not possible. 1st B1: real root plotted on negative x-axis and the conjugate pair plotted in Q1 and Q4, roughly aligned vertically. Ignore both axis scales and all labelling. 2nd B1: All roots plotted correctly and labelled. Only consider scale of x/real axis so both of the conjugate pair must be clearly closer to the y/imaginary axis than the real root. Award 2nd B0 if one of the conjugate pair is less than half the distance of the other from the x/real axis. If labelled with coordinates they must be the correct way around. Accept $\left(\frac{2}{3}, \pm \frac{\sqrt{17}}{3}\right)$ or $\left(\frac{2}{3}, \pm \frac{\sqrt{17}}{3}i\right)$ if coordinates are used and accept these coordinates in vector notation or indicated by axis labels. If roots are only indicated by e.g. z_1, z_2, z_3 these must have been identified in (c) or earlier. 2nd B0 if axes labelled the wrong way round. (They do not need to be labelled or could be labelled x and y).	B1 B1
		(2)
		Total 8



Question Number	Scheme	Notes	Marks
5(a)	$r(r+1)(r+5) = r^3 + 6r^2 + 5r$	Correct expansion. May be implied	B1
	$\left\{ \sum_{r=1}^n (r^3 + 6r^2 + 5r) = \sum_{r=1}^n r^3 + 6 \sum_{r=1}^n r^2 + 5 \sum_{r=1}^n r = \frac{1}{4}n^2(n+1)^2 + 6 \times \frac{1}{6}n(n+1)(2n+1) + \frac{5}{2}n(n+1) \right\}$ M1: Having achieved two terms of the correct form from the expansion, replaces at least one of $\sum r^3$, $\sum r^2$ or $\sum r$ correctly A1: Correct expression in any form		M1A1
	$= \frac{1}{4}n(n+1)[n(n+1) + 4(2n+1) + 10] \Rightarrow \frac{1}{4}n(n+1)(n^2 + 9n + 14)$ Obtains $\frac{1}{4}n(n+1)[\dots]$ where ... is a 3TQ in n May be implied by subsequent correct work. Note they might expand first to get e.g., $\frac{1}{4}n(n^3 + 10n^2 + 23n + 14) \text{ or } \frac{1}{4}(n^4 + 10n^3 + 23n^2 + 14n) \text{ or } \frac{1}{4}n^4 + \frac{5}{2}n^3 + \frac{23}{2}n^2 + \frac{7}{2}n$ Allow factor reconstruction after solving. Condone poor algebra but if no 3TQ is seen or the expanded form is wrong and they go straight to an answer it must follow. Expect a full method if e.g., expands or partially expands $\frac{1}{4}n(n+a)(n+b)(n+c)$ and equates coefficients. Requires previous M mark.		dM1
	$\frac{1}{4}n(n+1)(n+2)(n+7)$	Correct expression. Not just values. Brackets in any order. Accept $\frac{n}{4}(n+1)(n+2)(n+7) \text{ or } \frac{n(n+1)(n+2)(n+7)}{4}$	A1
			(5)
(b)	$20 \times 21 \times 25 + 21 \times 22 \times 26 + \dots + 40 \times 41 \times 45$ $= \frac{1}{4} \times 40(40+1)(40+2)(40+7) - \frac{1}{4} \times 19(19+1)(19+2)(19+7) = \dots$ or $\frac{1}{4} \times 40(41)(42)(47) - \frac{1}{4} \times 19(20)(21)(26) = \dots$ Attempts to find a value for $\sum_{r=1}^{40} r(r+1)(r+5) - \sum_{r=1}^{19} r(r+1)(r+5)$ using $f(40) - f(19)$. We will allow one slip only (one of the eight components of the products) provided a full substitution as above is seen. There must be explicit evidence that they are using their result from part (a). So do not accept just $809340 - 51870 = \dots$ but allow e.g., $= \frac{1}{4} \times 3237360 - \frac{1}{4} \times 207480 = \dots$ Their part (a) must be of the correct form. Allow if they are using made up values for a, b and c.		M1
	$= 757\,470$	757 470 only. Allow 7.5747×10^5 Isw if subsequently rounded	A1
			(2)
			Total 7

Question Number	Scheme	Notes	Marks
6(a)	$xy = 100 \Rightarrow y = 100x^{-1} \Rightarrow \frac{dy}{dx} = -100x^{-2}$ or $\Rightarrow x \frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ or $x = 10t, y = \frac{10}{t} \Rightarrow \frac{dy}{dx} = \frac{-10t^{-2}}{10}$ Any correct expression for $\frac{dy}{dx}$. Does not need to be in terms of t . Allow for a correct $\frac{dx}{dy} = -\frac{x^2}{100}$ or $-\frac{x}{y}$ or $-\frac{100}{y^2}$ May use $-\frac{dx}{dy}$. Condone just $\frac{dy}{dx}$ or $m_{\{T\}} = -\frac{1}{t^2}$.		B1
	$m_T = -\frac{100}{100t^2} \Rightarrow m_N = t^2$ or $m_T = -\frac{\frac{10}{t}}{10t} \Rightarrow m_N = t^2$ or $m_T = -t^{-2} \Rightarrow m_N = t^2$ Correct use of the perpendicular gradient rule to obtain a normal gradient at $\left(10t, \frac{10}{t}\right)$ May come directly from $-\frac{dx}{dy}$. Starting with just $m_N = t^2$ scores 0110 maximum.		M1
	$y - \frac{10}{t} = t^2(x - 10t)$ or $y = t^2x + c, \frac{10}{t} = t^2 \times 10t + c \Rightarrow c = \dots \left\{ \frac{10}{t} - 10t^3 \right\}$	Correctly forms the normal equation with a changed gradient in terms of t . Condone late substitution of $x = 10t$ and/or $y = \frac{10}{t}$ into gradient if initial gradient not in terms of t (the previous mark is then accessible as is the A1*)	M1
	e.g., $ty - 10 = t^3x - 10t^4$ or $ty = t^3x + 10 - 10t^4 \Rightarrow t^3x - ty = 10(t^4 - 1)^*$ Obtains the given answer with intermediate line and no errors. Final answer could be reversed and terms/products could be in a different order but must have factorised $10t^4 - 10$. Allow e.g., $t^3x - ty$ written as $t(xt^2 - y)$ All previous marks are required.		A1*
			(4)



Question Number	Scheme	Notes	Marks
6(b)	$x = 0 \Rightarrow -ty = 10(t^4 - 1) \Rightarrow y = \dots \left\{ -\frac{10(t^4 - 1)}{t} \right\}$	Substitutes $x = 0$ to find y for Q May just restate their c from (a). Apply BOD throughout if the minus sign just disappears	M1
	$\frac{1}{2} \left(\frac{10(t^4 - 1)}{t} \right) \times 10t = 750$ May see x -axis intercept also used e.g., $\frac{1}{2} \times \frac{10(t^4 - 1)}{t^3} \times \left(\frac{10}{t} + \frac{10(t^4 - 1)}{t} \right) = 750$ Allow with modulus signs used	Correct method for the area of the triangle and sets = 750 or equivalent work e.g., $\left(10t^3 - \frac{10}{t} \right) \times 10t = 1500$ Allow with their y (and possibly their x -axis intercept) and allow for the sign of their y (and/or x) coordinate uncorrected and note that $-10t$ may be used with uncorrected y . There are no marks if they have Q on the x -axis instead of the y -axis	M1
	"Shoelace" methods only get credit when the determinants are processed. Look for expressions as above.		
	We will score the first 2 M marks in this order for this possible variation: $\frac{1}{2} y_Q \times 10t = 750 \Rightarrow y_Q = -\frac{150}{t}$ 1st M1: Uses a correct triangle method to find y coord. $(0, -\frac{150}{t})$ in $t^3x - ty = 10(t^4 - 1) \Rightarrow 150 = 10(t^4 - 1)$ 2nd M1: Subs. into normal equation		
	$\frac{1}{2} \left(\frac{10(t^4 - 1)}{t} \right) \times 10t = 750$ or $\frac{1}{2} \left(10t^3 - \frac{10}{t} \right) \times 10t = 750$ or $\frac{1}{2} \left(-\frac{10(t^4 - 1)}{t} \right) \times -10t = 750$ or $\frac{1}{2} \left(\frac{10}{t} - 10t^3 \right) \times -10t = 750$ $\Rightarrow \{50t^4 - 50 = 750 \Rightarrow t^4 = \dots\{16\}$ Reaches $t^4 = \dots$ (or $t^2 = \dots$ or $t = \dots$) from a correct equation. Allow if $\dots$ is negative. If they additionally work with an incorrect equation then ignore this work. Modulus signs must have been removed although this could happen later in the working, but do be vigilant with attempts where e.g., the "16" has clearly not been obtained appropriately.		M1
	$\left\{ \Rightarrow t = \pm 2 \Rightarrow \left(10 \times \pm 2, \frac{10}{\pm 2} \right) \Rightarrow (20, 5), (-20, -5) \text{ only} \right.$ A1: One correct pair of coordinates A1: Both correct and no others including complex solutions. Allow for $x = \dots, y = \dots$ but must be clearly paired correctly Allow $\pm(20, 5)$ or $(\pm 20, \pm 5)$ Score A1 A0 for e.g., $(\pm 20, \pm \frac{10}{2})$ Do not accept correct coordinates if they have come from an incorrect equation. Note that the second pair of coordinates might be deduced.		A1 A1
			(5)
			Total 9

Question Number	Scheme	Notes	Marks
7(i)(a)	A rotation of $240^\circ/\frac{4\pi}{3}$ (anti/counter clockwise) about/around/at centre (0, 0)/origin/<i>O</i> M1: Any rotation. Condone "rotate" for all marks A1: Fully correct description. Condone missing degrees symbol. If direction is not mentioned assume anticlockwise so allow rotation of $120^\circ/\frac{2\pi}{3}$ clockwise about <i>O</i>. $-120^\circ/\frac{2\pi}{3}$ (anticlockwise) or $-240^\circ/\frac{4\pi}{3}$ clockwise are also acceptable. Must be a single transformation.		M1A1
			(2)
(b)	$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$	Correct matrix. Allow if they omit (b) and this matrix is seen in (c).	B1
			(1)
(c)	$\{C = AB = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\}$	Forms a correct product of matrices with their B Condone a clear miscopy of A	M1
	$\begin{pmatrix} -1 & \frac{\sqrt{3}}{2} \\ -\sqrt{3} & -\frac{1}{2} \end{pmatrix}$	Correct matrix. Any equivalent. No evidence of incorrect method for product or any incorrect matrices.	A1
			(2)

Question Number	Scheme	Notes	Marks
7(ii)(a)	$\begin{pmatrix} k & -2 \\ -1 & 2k \end{pmatrix} \begin{pmatrix} k \\ k \end{pmatrix} = \begin{pmatrix} k^2 - 2k \\ -k + 2k^2 \end{pmatrix} \left\{ = \begin{pmatrix} 35 \\ 91 \end{pmatrix} \right\}$	Multiplies to obtain one correct element. May just see e.g., $\begin{pmatrix} k & -2 \\ -1 & 2k \end{pmatrix} \begin{pmatrix} k \\ k \end{pmatrix} = \begin{pmatrix} k^2 - 2k \\ -k + 2k^2 \end{pmatrix}$	M1
	$k^2 - 2k - 35 = 0 \Rightarrow k = \dots \quad \text{or} \quad 2k^2 - k - 91 = 0 \Rightarrow k = \dots$ Attempts to solve one correct quadratic equation. Usual rules. One root correct if no working. This mark can also be awarded if they solve a correct 3TQ following combining correct equations e.g., $3k^2 - 3k - 126 \{ = k^2 - k - 42 = 0 \} = 0 \Rightarrow k = \dots \{7, -6\}$ If they eliminate to a correct equation that isn't a 3TQ then one solution must be correct e.g., $3k^2 - 147 = 0 \Rightarrow k^2 = 49 \Rightarrow k = \{\pm\}7$ or $-3k + 21 = 0 \Rightarrow k = 7$ If reduced to a linear equation allow all marks otherwise 1100 max if the evidence is that only one quadratic has been solved (e.g., 2 solutions from one equation offered even if one is "rejected" etc. rather than crossed out - ignore all crossed out work)		M1
	$k^2 - 2k - 35 = 0 \Rightarrow k = \dots \{-5, 7\}$ and $2k^2 - k - 91 = 0 \Rightarrow k = \dots \{-\frac{13}{2}, 7\}$ Or one of the above with one of $3k^2 - 3k - 126 = 0 \{ \Rightarrow k^2 - k - 42 = 0 \} \Rightarrow k = \dots \{7, -6\}$ or $3k^2 - 147 = 0 \{ \Rightarrow k^2 = 49 \} \Rightarrow k = \dots \{7, -7\}$ Or $\Rightarrow -3k + 21 = 0 \Rightarrow k = 7$ Factorisations shown below	Attempts to solve two correct quadratic equations (or one correct linear). Usual rules. One root correct if no working. Allow if they obtain 7 from one equation and verify correctly that it works in the second equation (or shows that the other root doesn't work). It is valid to solve one of the equations with a combined equation. Allow if equations not "extracted" from matrices e.g., $\begin{pmatrix} k^2 - 2k \\ -k + 2k^2 \end{pmatrix} = \begin{pmatrix} 35 \\ 91 \end{pmatrix}$	M1
	$k^2 - 2k - 35 = (k+5)(k-7), 2k^2 - k - 91 \Rightarrow (2k+13)(k-7), k^2 - k - 42 \Rightarrow (k+6)(k-7), k^2 - 49 \Rightarrow (k+7)(k-7)$		
	$k = 7$ following use of 2 correct equations. No incorrect values of k or incorrect factorisations etc seen. No other values offered. Two correct equations (which could be "unextracted") followed by 7 only is sufficient. Condone poor matrix work such as using (k, k) or matrix multiplications written the wrong way around provided correct elements/equations are obtained.		A1
	(4)		
	Alternative using inverse: $\mathbf{M} \begin{pmatrix} k \\ k \end{pmatrix} = \begin{pmatrix} 35 \\ 91 \end{pmatrix} \Rightarrow \begin{pmatrix} k \\ k \end{pmatrix} = \mathbf{M}^{-1} \begin{pmatrix} 35 \\ 91 \end{pmatrix} \text{ and } \mathbf{M}^{-1} = \frac{1}{2k^2 - 2} \begin{pmatrix} 2k & 2 \\ 1 & k \end{pmatrix} \Rightarrow \begin{pmatrix} k \\ k \end{pmatrix} = \frac{1}{2k^2 - 2} \begin{pmatrix} 70k + 182 \\ 35 + 91k \end{pmatrix}$ M1: For $70k + 182$ or $35 + 91k$ $k(2k^2 - 2) = 70k + 182 \Rightarrow k^3 - 36k - 91 = (k-7)(k^2 + 7k + 13) = 0 \Rightarrow k = 7 \left\{ \text{or } \frac{-7 \pm \sqrt{31}}{2} \right\}$ $k(2k^2 - 2) = 35 + 91k \Rightarrow 2k^3 - 93k - 35 = (k-7)(2k^2 + 14k + 5) = 0$ $\Rightarrow k = 7 \left\{ \text{or } \frac{-7 \pm \sqrt{39}}{2} \text{ or awrt } -0.38 \text{ or } -6.6 \right\}$ M1: Obtains a correct root from a correct equation M1: Obtains a correct root from two correct equations or verifies as mentioned above OR $70k + 182 = 35 + 91k \Rightarrow k = 7$ M1 M1: Obtains $k = 7$ from this linear equation A1: $k = 7$ and no incorrect values/factorisations etc. Condone poor matrix work such as using (k, k) or matrix multiplications written the wrong way around provided correct elements/equations are obtained.		

Question Number	Scheme	Notes	Marks
7(ii)(b)	$\left\{ \begin{array}{cc} "7" & -2 \\ -1 & 2 \times "7" \end{array} \right\} \Rightarrow "7" \times 2 \times "7" - (-1)(-2) \{=96\}$	Correct numerical expression for det M correct for any of their values of k . May be implied by sight of "96" $\times$ 336 (or e.g. 32256). Allow an invented value of k	M1
	$\Rightarrow \frac{7}{2} \left\{ \text{or } 3.5, 3\frac{1}{2}, \frac{336}{96} \right\}$	Correct value and <u>no others</u> . Any equivalent	A1
	If points and transformed points are used a correct numerical expression for the area scale factor or its reciprocal must be achieved for the M mark		(2)
			Total 11



Question Number	Scheme	Notes	Marks
7	$\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$		
7(a)	$n = 1, \text{LHS}=1^2 = 1, \text{RHS}=\frac{1}{6}.2.3 = 1$	Shows both LHS=1 and RHS=1 Accept LHS = 1 but must see at least $\frac{1}{6}.2.3 = 1$ for RHS.	B1
	Assume true for $n = k$		
	When $n = k + 1$ $\sum_{r=1}^{k+1} r^2 = \frac{k}{6}(k+1)(2k+1) + (k+1)^2$	Adds $(k+1)^2$ to result for $n = k$	M1
	$= \frac{(k+1)}{6}(k(2k+1) + 6(k+1))$	Attempt to factorise by $\frac{(k+1)}{6}$	dM1
	$= \frac{(k+1)}{6}(k+2)(2k+3)$ $= \frac{(k+1)}{6}((k+1)+1)((2(k+1)+1))$	Either factorised form. SC allow dM1A0 for fully factorising to a cubic expression and going direct to the fully factorised expression with no intermediate quadratic seen.	A1
	True for $n = 1$. If true for $n = k$ then true for $n = k + 1$ therefore true for all n .	Complete proof with no errors and these 4 statements seen anywhere. Depends on both M's and the A, but may be scored if the B is lost as long as some indication of true for $n = 1$ is given.	A1cso
			(5)
7(b)	$\sum_{r=1}^n (r^2 + 2) = \sum_{r=1}^n r^2 + \sum_{r=1}^n 2$ $= \frac{n}{6}(n+1)(2n+1) + \dots$	Split into the addition of 2 sums and applies the result of (a).	M1
	$= \frac{n}{6}(n+1)(2n+1) + 2n$	Correct expression.	A1
	$= \frac{n}{6}(2n^2 + 3n + 13)$	Factorises out the $\frac{n}{6}$ - must have a common factor n to achieve this mark; Simplifies to correct answer.	M1; A1
	$(a = 2, b = 3, c = 13)$		
			(4)
7(c)	$\sum_{r=10}^{25} (r^2 + 2) = S_{25} - S_9$ $= \frac{25}{6}.(2 \times 25^2 + 3 \times 25 + 13) - \frac{9}{6}(2 \times 9^2 + 3 \times 9 + 13)$	Attempts $S_{25} - S_9$ or $S_{25} - S_{10}$ with some substitution.	M1
	$= \frac{25}{6} \times 1338 - \frac{9}{6} \times 202 = 5575 - 303 = 5272$	For 5272	A1
	Note: Answer only (from calculator) is M0A0 as question requires use of part (b).		
			(2)
			Total 11

(a) Way 2	$y^2 = 4ax \Rightarrow 2y \frac{dy}{dx} = 4a$ $3x - 4y + 48 = 0 \Rightarrow \frac{dy}{dx} = \frac{3}{4}$ $\Rightarrow 2y \times \frac{3}{4} = 4a$	Uses differentiation to obtain the gradient of C and substitutes the gradient of l to obtain an equation connecting y and a , or connecting x and a , or an equation in t	M1
	$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = a^{\frac{1}{2}}x^{-\frac{1}{2}} \Rightarrow a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{3}{4}$		
	$x = at^2, y = 2at \Rightarrow \frac{dy}{dx} = \frac{1}{t}$ $\frac{1}{t} = \frac{3}{4}$		
	$y = \frac{8a}{3} \text{ or } x = \frac{16a}{9}$	Correct y value, or correct x value (possibly implied in subsequent work, particularly if using the parametric equations)	A1
	$y^2 = 4ax \Rightarrow \frac{64a^2}{9} = 4ax \Rightarrow x = \frac{16a}{9}$ $3 \times \frac{16a}{9} - 4 \times \frac{8a}{3} + 48 = 0 \Rightarrow a = \dots$	Uses $y^2 = 4ax$ or l to find a value for x (or y) and substitutes their x and y into the other equation to find a value for a If using parameter t , substitutes their value for t into $3(at^2) - 4(2at) + 48 = 0$ and solves to find a value for a	M1
	$x = \frac{4y - 48}{3} = \frac{32a}{9} - 16$ $\frac{64a^2}{9} = 4a \left(\frac{32a}{9} - 16 \right) \Rightarrow a = \dots$		
	$y^2 = 4ax \Rightarrow y^2 = \frac{64a^2}{9} \Rightarrow y = \frac{8a}{3}$ $3 \times \frac{16a}{9} - 4 \times \frac{8a}{3} + 48 = 0 \Rightarrow a = \dots$		
	$y = \frac{3x + 48}{4} = \frac{4a}{3} + 12$ $\left(\frac{4a}{3} + 12 \right)^2 = 4a \left(\frac{16a}{9} \right) \Rightarrow a = \dots$		
	$3(at^2) - 4(2at) + 48 = 0$ $3 \left(\frac{16a}{9} \right) - 4 \left(\frac{8a}{3} \right) + 48 = 0 \Rightarrow a = \dots$		
	$a = 9 *$	CSO	A1*

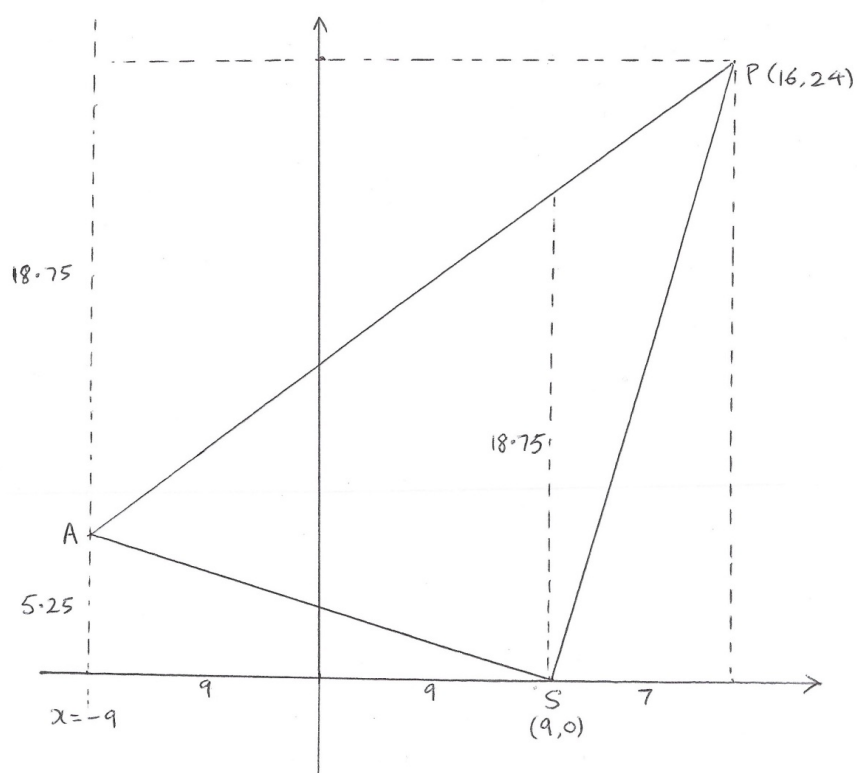
(b)	$a = 9 \Rightarrow 3y^2 - 144y + 1728 = 0 \Rightarrow y = 24$ $9x^2 - 288x + 2304 = 0 \Rightarrow x = 16$	Uses $a = 9$ to solve their 3TQ to obtain the repeated root for x or y .	M1
	$x = 16$ and $y = 24$	Correct values or coordinates.	A1
			(2)

(b) Way 2 follows (a)Way2	$a = 9 \Rightarrow x = \dots$ or $y = \dots$	Substitutes $a = 9$ into their expression for x or y , OR substitutes $a = 9$ into at^2 to find x , or into $2at$ to find y .	M1
	$x = 16$ and $y = 24$	Correct values or coordinates.	A1

(c) Way 1	Focus is at (9, 0)	Correct focus (could be seen on a sketch or implied in working)	B1
	$x = -9 \Rightarrow 3(-9) - 4y + 48 = 0 \Rightarrow y = 5.25$	Correct method with the correct directrix to find the y coordinate of A	M1
	E.g. Trapezium – 2 triangles $= \frac{1}{2} \left(\frac{21}{4} + 24 \right) \times 25 - \frac{1}{2} \times 18 \times \frac{21}{4} - \frac{1}{2} \times 7 \times 24 = \frac{1875}{8}$ Fully correct triangle area method (condone one slip if the intention seems clear)		dM1
	$= \frac{1875}{8} (234.375)$	Correct area (exact)	A1
			(4)
			Total 10

(c) Way 2	Focus is at (9, 0)	Correct focus (could be seen on a sketch or implied in working)	B1
	$x = 9 \Rightarrow 3(9) - 4y + 48 = 0 \Rightarrow y = 18.75$	Correct method to find the y coordinate when $x = 9$, but also requires correct directrix at some stage of the solution	M1
	E.g. $A = \frac{1}{2}(18.75 \times 18) + \frac{1}{2}(18.75 \times (16 - 9))$	Fully correct triangle area method (condone one slip if the intention seems clear)	dM1
	$= \frac{1875}{8} (234.375)$	Correct area (exact)	A1
(c) Way 3	Focus is at (9, 0)	Correct focus (could be seen on a sketch or implied in working)	B1
	$x = -9 \Rightarrow 3(-9) - 4y + 48 = 0 \Rightarrow y = 5.25$	Correct method with the correct directrix to find the y coordinate of A	M1
	E.g. $\begin{array}{r rrrr} \frac{1}{2} & 9 & -9 & 16 & 9 \\ & 0 & 5.25 & 24 & 0 \end{array}$ $= \frac{1}{2} 47.25 - 216 - 84 - 216 $	Fully correct area method (condone one slip if the intention seems clear)	dM1
	$= \frac{1875}{8} (234.375)$	Correct area (exact)	A1

(c) Way 4	Focus is at (9, 0)	Correct focus (could be seen on a sketch or implied in working)	B1
	$x = -9 \Rightarrow 3(-9) - 4y + 48 = 0 \Rightarrow y = 5.25$	Correct method with the correct directrix to find the y coordinate of A	M1
	E.g. Rectangle – 3 triangles $(25 \times 24) - \frac{1}{2}(18 \times 5.25) - \frac{1}{2}(7 \times 24) - \frac{1}{2}(25 \times 18.75)$ Fully correct triangle area method (condone one slip if the intention seems clear)		dM1
	$= \frac{1875}{8} (234.375)$	Correct area (exact)	A1



Question Number	Scheme	Notes	Marks
8(i)	$\sum_{r=1}^n \frac{2r^2-1}{r^2(r+1)^2} = \frac{n^2}{(n+1)^2}$		
	$\frac{2(1)^2-1}{1^2(1+1)^2} = \frac{1}{4}, \frac{1^2}{(1+1)^2} = \frac{1}{4}$	Getting $\frac{1}{2^2}$ or $\frac{1}{4}$ from each is the minimum	B1
	Assume $\sum_{r=1}^k \frac{2r^2-1}{r^2(r+1)^2} = \frac{k^2}{(k+1)^2}$		
	$\sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{k^2}{(k+1)^2} + \frac{2(k+1)^2-1}{(k+1)^2(k+2)^2}$ Assumes the result is true for say $n = k$ and adds the next term		M1
	$\frac{k^2(k+2)^2 + 2(k+1)^2 - 1}{(k+1)^2(k+2)^2}$	Attempts common denominator	dM1
		Correct expression	A1
	$\frac{k^4 + 4k^3 + 4k^2 + 2k^2 + 4k + 1}{(k+1)^2(k+2)^2} = \frac{k^4 + 4k^3 + 6k^2 + 4k + 1}{(k+1)^2(k+2)^2} = \frac{(k+1)^4}{(k+1)^2(k+2)^2}$		
	$\frac{(k+1)^2}{(k+2)^2}$	Achieves this result with intermediate working and no errors	A1
	If the result is <u>true for</u> $n = k$, then it is <u>true for</u> $n = k + 1$. As the result has been shown to be <u>true for</u> $n = 1$, then the result is <u>true for all</u> n .	The underlined features should be seen. Some may appear earlier in the solution.	A1cso
			(6)
(ii)	$f(n) = 12^n + 2 \times 5^{n-1}$		
	$f(1) = 12 + 2 \times 1 = 14$	This is sufficient	B1
	$f(k+1) = 12^{k+1} + 2 \times 5^k$	Attempt $f(k+1)$	M1
	$f(k+1) - f(k) = 12^{k+1} + 2 \times 5^k - 12^k - 2 \times 5^{k-1}$ $f(k+1) - f(k) = 11 \times 12^k + 22 \times 5^{k-1} + 10 \times 5^{k-1} - 24 \times 5^{k-1}$	Working with $f(k+1) - f(k)$	
	$= 11 \times (12^k + 2 \times 5^{k-1}) - 14 \times 5^{k-1}$	$11 \times (12^k + 2 \times 5^{k-1})$ or $11f(k)$	A1
		$-14 \times 5^{k-1}$	A1
	$f(k+1) = 12f(k) - 14 \times 5^{k-1}$	Makes $f(k+1)$ the subject Dependent on at least one of the A marks	dM1
	If the result is <u>true for</u> $n = k$, then it is <u>true for</u> $n = k + 1$. As the result has been shown to be <u>true for</u> $n = 1$, then the result is <u>true for all</u> n .	The underlined features should be seen. Some may appear earlier in the solution.	A1cso
			(6)
			Total 12

ALT 1	$f(1) = 12 + 2 \times 1 = 14$	This is sufficient	B1
	$f(k+1) = 12^{k+1} + 2 \times 5^k$	Attempt $f(k+1)$	M1
	$f(k+1) = 12(12^k + 2 \times 5^{k-1}) + 2 \times 5 \times 5^{k-1} - 12 \times 2 \times 5^{k-1}$		
	$f(k+1) = 12(12^k + 2 \times 5^{k-1}) - 14 \times 5^{k-1}$	$12(12^k + 2 \times 5^{k-1})$ or $12f(k)$	A1
		$-14 \times 5^{k-1}$	A1
	$f(k+1) = 12f(k) - 14 \times 5^{k-1}$	Dependent on at least one of the A marks	dM1
	If the result is <u>true for $n = k$</u> , then it is <u>true for $n = k+1$</u> . As the result has been shown to be <u>true for $n = 1$</u> , then the result is <u>true for all n</u> .	The underlined features should be seen. Some may appear earlier in the solution.	A1cso
ALT 2	$f(1) = 12 + 2 \times 1 = 14$	This is sufficient	B1
	Let $12^k + 2 \times 5^{k-1} = 7M$		
	$f(k+1) = 12^{k+1} + 2 \times 5^k$	Attempt $f(k+1)$	M1
	$f(k+1) = 12(7M - 2 \times 5^{k-1}) + 2 \times 5^k$	OR: $f(k+1) = 5(7M) + 7 \times 12^k$	
	$f(k+1) = 84M - 14 \times 5^{k-1}$	$84M$	OR: $35M$ A1
	OR: $f(k+1) = 35M + 7 \times 12^k$	$-14 \times 5^{k-1}$	$+7 \times 12^k$ A1
	$f(k+1) = 12f(k) - 14 \times 5^{k-1}$ OR: $f(k+1) = 5f(k) + 7 \times 12^k$	Dependent on at least one of the A marks	dM1
	If the result is <u>true for $n = k$</u> , then it is <u>true for $n = k+1$</u> . As the result has been shown to be <u>true for $n = 1$</u> , then the result is <u>true for all n</u> .	The underlined features should be seen. Some may appear earlier in the solution.	A1cso
ALT 3	$f(1) = 12 + 2 \times 1 = 14$	This is sufficient	B1
	$f(k+1) = 12^{k+1} + 2 \times 5^k$	Attempts $f(k+1)$	M1
	Working with $f(k+1) - mf(k)$ $f(k+1) - mf(k) = 12^{k+1} + 2 \times 5^k - m(12^k + 2 \times 5^{k-1})$ $f(k+1) - f(k) = (12-m) \times 12^k + 2 \times (12-m) \times 5^{k-1} + 10 \times 5^{k-1} - 24 \times 5^{k-1}$		
	$= (12-m) \times (12^k + 2 \times 5^{k-1}) - 14 \times 5^{k-1}$	$(12-m) \times (12^k + 2 \times 5^{k-1})$ or $(12-m)f(k)$	A1
		$-14 \times 5^{k-1}$	A1
	$f(k+1) = 12f(k) - 14 \times 5^{k-1}$	Makes $f(k+1)$ the subject Dependent on at least one of the A marks	dM1
	If the result is <u>true for $n = k$</u> , then it is <u>true for $n = k+1$</u> . As the result has been shown to be <u>true for $n = 1$</u> , then the result is <u>true for all n</u> .	The underlined features should be seen. Some may appear earlier in the solution.	A1cso

ALT 4	$f(1) = 12 + 2 \times 1 = 14$	This is sufficient	B1
	$f(k+1) = 12^{k+1} + 2 \times 5^k$	Attempts $f(k+1)$	M1
	$f(k+1) - 5f(k) = 12^{k+1} + 2 \times 5^k - 5(12^k + 2 \times 5^{k-1})$	Working with $f(k+1) - 5f(k)$	
	$= 7 \times 12^k + 2 \times 5^k - 2 \times 5^k$	7×12^k	A1
		$2 \times 5^k - 2 \times 5^k$ (or zero)	A1
	$f(k+1) = 5f(k) + 7 \times 12^k$	Makes $f(k+1)$ the subject Dependent on at least one of the A marks	dM1
	If the result is <u>true for $n = k$</u> , then it is <u>true for $n = k+1$</u> . As the result has been shown to be <u>true for $n = 1$</u> , then the result is <u>true for all n</u> .	The underlined features should be seen. Some may appear earlier in the solution.	A1cso

NOTES:**Part (i)**

This approach may be seen:

Assume result is true for $n = k$ and $n = k+1$

Subtract: (sum to $(k+1)$ terms) minus (sum to k terms)

Show that this is equal to the $(k+1)$ th term

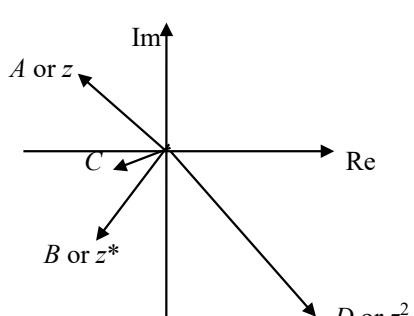
Please send any such response to Review.

Part (ii)

Apart from the given alternatives, other versions will work and can be marked equivalently.

If in any doubt, send to Review.



Question Number	Scheme	Marks
6(a)	$\lambda = 4$	B1 (1)
(b)	$\arctan \frac{3}{4}$ or $\arctan \frac{-3}{4}$ (Second quadrant so $\arg z = 2.498\dots = 2.5$ (rad))	M1 A1 (2)
(c)(i)	$\frac{z+3i}{2-4i} = \frac{-4+6i}{2-4i} \times \frac{2+4i}{2+4i}$ or $\frac{z+3i}{2-4i} = a+ib \Rightarrow -4+6i = (a+ib)(2-4i)$ $= \frac{-8+12i-16i+24i^2}{4+16} = -\frac{8}{5} - \frac{1}{5}i$ Accept e.g. $\frac{-32-4i}{20}$ Or $2a+4b = -4, 2b-4a = 6 \Rightarrow a = \dots, b = \dots$	M1 dM1A1
(ii)	$z^2 = (-4+3i)^2 = 16-24i+9i^2 = 16-24i-9$ $= 7-24i$	M1 A1ft (5)
(d)		B1 B1ft B1ft (3)
[11]		
(a) B1 (b) M1 A1 (c)(i) M1 dM1 A1 (ii) M1 A1ft (d) B1 B1ft	Correct answer. No working needed. For $\arctan\left(\pm \frac{3}{4}\right)$ with their "4". Can be awarded from $\tan \theta = \pm \frac{3}{4} \Rightarrow \theta = \dots$ or by implication if correct value for either $\arctan$ or correct final answer (rounded or not rounded, may be degrees) is seen. Cao 2.5 Multiplies numerator and denominator by complex conjugate of denominator. Award if denominator of $4+16$ or 20 is seen instead of product. May still have z at this stage, or even allow with $\lambda+3i$ as numerator. Alternatively, sets equal to $a+ib$ and cross multiplies. Using their λ or 4 substitutes correctly for z, fully expands the numerator and uses $i^2 = -1$ Alt, uses $i^2 = -1$, equates real and imaginary terms and solves their equations for a and b Correct answer only, as shown or single fraction accepting equivalent fractions or with exact decimals $(-1.6 - 0.2i)$. Squaring an expression of form $k+3i$ (with a real value for k) to get 3 terms (may be implied) and uses $i^2 = -1$ Correct answer, follow through their $\lambda > 0$ (ie for "$\lambda^2 - 9$" - "6λ" i must be negative i term) NB: Penalise once only (in the first mark due) for mislabelling or failing to label points as long as they look to be placed correctly. Award if lines/arrows not included. Points may be labelled by letter, name or their Cartesian coordinates (which may be given on the axes). Plots z in second quadrant and z^* as mirror image in the Real axis. Both must be labelled. Plot and label C for their solution to (c)(ii) It must be the correct side of B for their	

B1ft

answers) and a correct relative scale (so noticeably closer to O than their B if correct values).

Plot and label their D (- 24 need not be to scale, but should be further from O than their B).



Question Number	Scheme	Marks
7(a)	$\begin{vmatrix} 4 & -5 \\ -3 & 2 \end{vmatrix} = 8 - 15 = -7 \Rightarrow \text{Area } T' = \pm 7 \times 23 = \dots$ $\text{Area } T' = 161$	M1 A1 (2)
(b)	$\begin{pmatrix} 4 & -5 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3p+2 \\ 2p-1 \end{pmatrix} = \begin{pmatrix} 17 \\ -18 \end{pmatrix} \text{ or } \begin{pmatrix} 3p+2 \\ 2p-1 \end{pmatrix} = \frac{1}{8-15} \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 17 \\ -18 \end{pmatrix}$ $4(3p+2) - 5(2p-1) = 17 \text{ or } -3(3p+2) + 2(2p-1) = -18 \text{ or}$ $3p+2 = -\frac{1}{7}(34-90) \text{ or } 2p-1 = -\frac{1}{7}(51-72)$ (e.g. $2p+13=17 \Rightarrow \dots$) $p=2$	M1 A1 (2)
(c)	Rotation; through 90° clockwise (or 270° anticlockwise) about origin	B1;B1 (2)
(d)	CA = B $\mathbf{A}^{-1} = -\frac{1}{7} \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} \text{ or } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 4 & -5 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 4a-3b & -5a+2b \\ 4c-3d & -5c+2d \end{pmatrix}$ $\mathbf{C} = -\frac{1}{7} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} = \dots \text{ or } \begin{cases} 4a-3b=0 & -5a+2b=1 \\ 4c-3d=-1 & -5c+2d=0 \end{cases} \Rightarrow \dots$ $\mathbf{C} = -\frac{1}{7} \begin{pmatrix} 3 & 4 \\ -2 & -5 \end{pmatrix} \text{ or } \begin{pmatrix} -\frac{3}{7} & -\frac{4}{7} \\ \frac{2}{7} & \frac{5}{7} \end{pmatrix} \text{ oe}$	B1 M1 A1 (3)
[9]		
(a)	Attempts to find the determinant of M and use as a scale factor. Accept if a slip in calculation is made and accept if negative is used for this mark. Dividing by the determinant is M0.	
M1		
A1	Correct answer only. No working needed (correct answer implies the method).	
(b)		
M1	Form a matrix equation using either A or an attempt at $\mathbf{A}^{-1}$, obtain a linear equation and solve for p .	
A1	Correct value for p , obtained from a correct equation. (No need to check in other equation.)	
(c)		
B1	Rotation, rotates, rotate or rotating (oe) Accept “turn”	
B1	Correct angle (degrees or radians) with direction specified and about origin or (0, 0)	
(d)		
B1	Correct matrix for $\mathbf{A}^{-1}$ May have been found in (b) but must be used in (d). Alternatively, correct CA with unknowns for entries of C .	
M1	Multiply B by $\mathbf{A}^{-1}$ on the right . Alternatively, sets CA equal to B and solves equations.	
A1	Correct matrix C (isw after a correct answer).	

Question Number	Scheme	Marks
8(a)	$200t^3 = 25$ or $\left(\frac{25}{x}\right)^2 = 40x$ or $y^2 = 40\left(\frac{25}{y}\right)$ $t = \frac{1}{2}$ or $x = \frac{5}{2}$ or $y = 10$ $\left(\frac{5}{2}, 10\right)$	M1 A1 A1 (3)
(b)	$y^2 = 40x \Rightarrow x = \frac{y^2}{40} \Rightarrow \frac{dx}{dy} = \frac{2y}{40}$ or $2y \frac{dy}{dx} = 40 \Rightarrow \frac{dy}{dx} = \frac{40}{2y}$ or $\frac{dy}{dx} = \frac{\sqrt{10}}{\sqrt{x}}$ or by parametric differentiation: $\frac{dy}{dx} = \frac{1}{t}$ at $(10, 20)$: $\frac{dy}{dx} = 1$ or $\frac{dx}{dy} = 1$ Grad normal = -1 $y - 20 = -(x - 10)$ $x + y - 30 = 0$	B1 M1 A1 M1 A1 (5)
(c)	$xy = 25$ $x + y - 30 = 0$ $x + \frac{25}{x} - 30 = 0$ or $\frac{25}{y} + y - 30 = 0$ or $\frac{5}{t} + 5t - 30 = 0$ $x^2 - 30x + 25 = 0$ or $y^2 - 30y + 25 = 0$ or $5t^2 - 30t + 5 = 0$ $x = 15 \pm \sqrt{200}$ or $y = 15 \pm \sqrt{200}$ or $t = 3 \pm 2\sqrt{2}$ (or exact equivalents) eg $x = 15 + \sqrt{200} \Rightarrow y = 15 - \sqrt{200}$ $(15 + 10\sqrt{2}, 15 - 10\sqrt{2})$ $(15 - 10\sqrt{2}, 15 + 10\sqrt{2})$	M1 dM1 A1 ddM1 A1A1 (6)
(a)	Attempt an equation in a single variable.	
M1	Correct value for x , y or t	
A1	Correct values for x and y . Need not be in coordinate brackets. No other points seen.	
(b)		
B1	Any correct expression involving the derivative, $\frac{dy}{dx}$ or $\frac{dx}{dy}$, for P	
M1	Attempt to obtain value of their derivative at $(10, 20)$. May be from an incorrect curve.	
A1	Correct gradient of normal.	
M1	Equation of normal by any complete method. Must involve a sign change of their derivative and have numerical gradient. Can be scored from an incorrect starting equation/point.	
A1	Correct equation in form demanded (though terms may be in different order).	
(c)		
M1	Use the equation of H and their equation of the normal from (b) to obtain an equation in a single variable	
dM1	Obtains a 3TQ and attempts to solve by any valid means.	
A1	Correct values for x or y or t . Must be exact but need not be fully simplified (but discriminant must be evaluated).	
ddM1	Use at least one of their values for x or y to obtain a value for the other coordinate or t to find at least one set of coordinates. (Can be scored with inexact values.)	
A1	Either pair of coordinates correct. Allow if unsimplified.	
A1	Second pair of coordinates correct and no extra solutions and both pairs in simplest form (as shown in scheme). Need not be coordinates as long as correctly paired.	
	Award A1A0 if $x = 15 \pm 10\sqrt{2}$, $y = 15 \pm 10\sqrt{2}$ is given	

[14]



Question Number	Scheme	Marks
9(i)	$n=1 \quad u_1 = 3 \times \frac{2}{3} - 1 = 1 \quad (\text{so true for } n=1 \text{ } (\dagger))$ Assume true for $n=k$ ie $u_k = 3\left(\frac{2}{3}\right)^k - 1 \quad (\dagger)$ $u_{k+1} = \frac{1}{3}(2u_k - 1) = \frac{1}{3}\left(2\left(3\left(\frac{2}{3}\right)^k - 1\right) - 1\right) = \frac{1}{3}\left(6\left(\frac{2}{3}\right)^k - 2 - 1\right)$ $= \frac{1}{3}\left(2 \times 3\left(\frac{2}{3}\right)^{k+1} - 2 - 1\right)$ $= \frac{1}{3} \times 2 \times 3\left(\frac{2}{3}\right)^{k+1} - \frac{1}{3}(-2-1)$ $= 3\left(\frac{2}{3}\right)^{k+1} - 1$ $\therefore$ if true for $n=k$, also true for $n=k+1 \quad (\dagger)$ (True for $n=1$) so $u_n = 3\left(\frac{2}{3}\right)^n - 1$ is true for all $n \in \mathbb{Z}^+$	B1 M1A1 dM1 A1 A1 cso (6)
(ii)	$f(1) = 2^3 + 3^3 = 8 + 27 = 35 \text{ (Multiple of 7)} \quad (\text{so true for } n=1 \text{ } (\dagger))$ Assume $f(k)$ is a multiple of 7 $f(k) = 2^{k+2} + 3^{2k+1}$ is a multiple of 7 $(\dagger)$ $f(k+1) - Mf(k) = 2^{k+3} + 3^{2k+3} - M(2^{k+2} + 3^{2k+1})$ $= 2^{k+2}(2-M) + 3^{2k+1}(3^2 - M)$ $= (2-M)(2^{k+2} + 3^{2k+1}) + 3^{2k+1} \times 7 \text{ or } (9-M)(2^{k+2} + 3^{2k+1}) - 7 \times 2^{k+2} \text{ oe}$ $\therefore f(k+1) = 2f(k) + 7 \times 3^{2k+1}$ oe e.g. $9f(k) - 7 \times 2^{k+2}$ Or e.g. $7 \times 3^{2k+1}$ is a multiple of 7, so if $f(k)$ is a multiple of 7 <u>then $f(k+1)$ is also a multiple of 7</u> If the result is true for $n=k$ it is also true for $n=k+1 \quad (\dagger)$ As the result has been shown to be true for $n=1$, it is true for all $n \in \mathbb{Z}^+$	B1 M1 A1 dM1 A1 A1 cso (6) [12]

9(i)	
B1	Check that the formula gives 1 when $n = 1$ Working must be shown. (Need not state true for $n = 1$ for this mark – but see final A)
M1	(Assume true for $n = k$ and) attempts to substitute the formula for u_k into $u_{k+1} = \frac{1}{3}(2u_k - 1)$ or equivalent with suffixes increased. Allow slips.
A1	Correct substitution.
dM1	Obtain an expression with $\left(\frac{2}{3}\right)^{k+1}$ and no other k . Alternatively, expands u_{k+1} to a matching expression (ie work from both directions).
A1	Correct expression when $n = k + 1$ At least one intermediate stage of working must be shown and no errors (though notational slips may be condoned). If working from both directions, it is for correct work to reach matching expressions.
A1cso	Correct concluding statements following correct solution which has included each of the points (†) at some stage during the working. Depends on all except the first B mark (e.g. if they think they have checked $n = 1$ but have really checked $n = 2$). Note: Allow the M's and first two A's for students who go from $k+1$ to $k+2$ but treat it as k to $k + 1$.
(ii)	
B1	Checks the case $n = 1$. Minimum statement of $f(1) = 35$
M1	Attempts an expression for $f(k + 1) - Mf(k)$ with any value of M . Need not be simplified. Most likely with $M = 1$ but may be seen with other values of M . With $M = 0$, $f(k + 1) = 2^{k+3} + 3^{2k+3}$ is all that is required.
A1	A correct expression with terms 2^{k+2} and 3^{2k+1} clearly identified.
dM1	Attempts to extract/identify $f(k)$ within a correct expression to give terms divisible by 7. With $M = 0$ look for $f(k + 1) = 2 \times (2^{k+2} + 3^{3k+1}) + 7 \times 3^{2k+1}$ or $9 \times (2^{k+2} + 3^{3k+1}) - 7 \times 2^{k+2}$ or similar for other value of M .
A1	One of the correct expressions for $f(k+1)$ shown (or with powers of 2 and 3) or full reason why $f(k+1)$ is divisible by 7, following a suitable expression.
A1cso	Correct concluding statements following correct solution which has included each of the points (†) at some stage during the working. Depends on all previous marks.

Question Number	Scheme	Notes	Marks
8(a)	$\sum_{r=1}^{2n} (2r^2 - 1) = 2 \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{2n} 1$ $= 2 \times \frac{2n}{6} (2n+1)(2(2n)+1) - \frac{2n}{1}$ $\left\{ = \frac{2n}{3} (2n+1)(4n+1) - 2n = \frac{16n^3}{3} + 4n^2 - \frac{4n}{3} \right\}$	<u>M1</u>: Uses $2n$ for k in $2 \times \frac{k}{6} (k+1)(2k+1)$ Must replace k with $2n$ at least once. Must be otherwise correct i.e., only allow n or $2n$ for k Note that failing to replace the first k with $2n$ leads to $\frac{n}{3}(\dots)$ Award M0 if it is quite clear that the wrong formula for the sum of the squares has been used.	M1
		<u>B1</u> : Correct $\sum_{r=1}^{2n} 1 = 2n$ seen/used	B1
	$= \frac{2n}{3} (8n^2 + 6n - 2) \text{ or } \frac{4n}{3} (4n^2 + 3n - 1)$ Obtains ...n(3TQ in n) from a cubic with no constant. If 3TQ allow fractional coefficients. Allow implication of this mark with $\frac{4}{3}(4n^3 + 3n^2 - n)$ or $\frac{2}{3}(8n^3 + 6n^2 - 2n)$ but not just $\frac{16n^3}{3} + 4n^2 - \frac{4n}{3}$. Must not have had a constant term. Can be scored if n instead of $2n$ used throughout the sum of squares formula $\left(\Rightarrow \frac{n}{3}(n^2 + 3n - 5) \right)$ but next ddM0. Condone poor algebra.		M1
	$= \frac{4}{3} n(n+1)(4n-1)$ Allow e.g., $\frac{4n(4n-1)(n+1)}{3}$	Obtains ...n(factorised 3TQ in n) Apply usual quadratic rules for the factorisation. If e.g., $\frac{2n}{3}(\dots) \rightarrow \frac{4n}{3}(\dots)$ work must be on a consistent 3TQ. Factors must have all real & exact terms. Previous 2 method marks required.	ddM1
		Correct result. Allow minor recovered algebraic/bracketing slips.	A1
	Note that e.g., if $2 \times \frac{2n}{6} (2n+1)(2(2n)+1) - 2n$ or $\frac{16n^3}{3} + 4n^2 - \frac{4n}{3}$ or $\frac{2n}{3}(2n+1)(4n+1) - 2n$ is immediately followed by $\frac{4}{3}n(n+1)(4n-1)$ score 11000. Allow expanding the given answer $\frac{4}{3}n(n+1)(an+b)$ and equating coefficients for the last three marks: $\frac{16n^3}{3} + 4n^2 - \frac{4n}{3} = \frac{4}{3}an^3 + \frac{4}{3}(a+b)n^2 + \frac{4}{3}bn \Rightarrow a = 4, b = -1 \Rightarrow \frac{4}{3}n(n+1)(4n-1)$ M1: Correct form for expansion of given answer and obtains a value for either a or b ddM1: Obtains values for both a and b A1: $\frac{4}{3}n(n+1)(4n-1)$ (allow just $a = 4, b = -1$ if $\frac{4}{3}n(n+1)(an+b)$ seen) Summation formulae must be used. No credit for attempts using induction or e.g., setting up simultaneous equations with 2 values of n, unless there is work that can score as above.		(5)

Question Number	Scheme	Notes	Marks
8(b)	$\sum_{r=1}^1 r(3r-2)^2$ or LHS = $1(3 \times 1 - 2)^2 = 1$ and $\frac{n^2(n+1)(9n-7)}{4}$ or RHS = $\frac{1^2 \times 2 \times 2}{4} = 1$ Achieves 1 from two numerical expressions (both not just "1"). No requirement here to say "true" etc. Allow as minimum, e.g., $1 \times 1 = \frac{4}{4} = 1$. "(When $n = 1$), both =1" is B0 but final A1 available.		B1
	{Assume true for $n = k$, then} $\sum_{r=1}^{k+1} r(3r-2)^2 = \sum_{r=1}^k r(3r-2)^2 + (k+1)(3(k+1)-2)^2$ $= \frac{k^2(k+1)(9k-7)}{4} + (k+1)(3k+1)^2$	Adds an attempt at the $(k+1)$ th term to an attempt at the sum to k terms. Allow clear copying slips (e.g., losing the squared from the k^2) but must be a recognisable attempt at forming $\frac{k^2(k+1)(9k-7)}{4} + (k+1)(3(k+1)-2)^2$ So e.g., ... + $k(3k-2)^2$ is M0	M1
	$= \frac{(k+1)}{4}(9k^3 - 7k^2 + 4(9k^2 + 6k + 1)) = \frac{(k+1)}{4}(9k^3 + 29k^2 + 24k + 4)$ $\left\{ \text{or } = (k+1) \left(\frac{9}{4}k^3 + \frac{29}{4}k^2 + 6k + 1 \right) \right\}$ OR $= \frac{1}{4}(9k^4 + 38k^3 + 53k^2 + 28k + 4) \left\{ \text{or } = \frac{9}{4}k^4 + \frac{19}{2}k^3 + \frac{53}{4}k^2 + 7k + 1 \right\}$		dM1
	Reaches $\frac{(k+1)}{4}$ (4 term cubic in k) or $(k+1)$ (4 term cubic in k) OR expands to a 5 term quartic. Must collect terms. Condone poor algebra. Previous mark required		
	$= \frac{1}{4}(k+1)((k+1)(9k^2 + 20k + 4))$ or $\frac{1}{4}(k+1)^2(k+2)(9k+2) \Rightarrow \frac{(k+1)^2((k+1)+1)(9(k+1)-7)}{4}$ Correctly reaches the result completely in terms of $k+1$ [but allow $k+2$ or $k+1+1$ for $((k+1)+1)$] with an intermediate step . Condone poor notation and allow the odd recovered algebraic slip/poor bracketing provided they reach the correct $\frac{k+1}{4}$ (4 term cubic in k) or $(k+1)$ (4 term cubic in k) or 5 term quartic and there are no subsequent errors. Factorisation may be achieved via calculator use. Allow a final answer of $\frac{1}{4}(k+1)^2(k+2)(9(k+1)-7)$ and with brackets in any order. Meet in the middle approaches must clearly join and the final expression in $k+1$ must be seen in the work		A1
	<u>True for $n = 1$</u> and <u>if true for $n = k$ then</u> it is also <u>true for $n = k + 1$</u> so by induction it is <u>true for (all) n</u> Acceptable proof (i.e., must have scored the previous A) and narrative/conclusion. Requires previous three marks and can only follow B0 if B0 was given for insufficient evidence of substitution. There must be no errors if one substitution was attempted and must have reached both sides = 1. "Assume (true) for $n = k$ " or "If true for $n = k$ " in narrative followed by "true for $n = k + 1$ " is sufficient for the "if...then". Allow suitable surrogates for "true". Allow stating the result from the question paper or " P_n is true" <u>with added reference to n</u> . Condone $n \in \square$ but not $n \in \square$. Allow work in n rather than k throughout. There is no credit for attempts using summation formulae instead of induction unless there is work that can score via the scheme above		A1

(5)



Question	Scheme	Marks
4(a)	$4 - 3i$	B1
		(1)
(b)	$(x - (4 + 3i))(x - (4 - 3i)) = \dots$	M1
	$x^2 - 8x + 25$	A1
		(2)
(c)	E.g. Product of roots is 225, so product of real roots is $\frac{225}{25} = 9$ Or $x^4 + Ax^3 + Bx^2 + Cx + 225 = (x^2 - 8x + 25)(x^2 + \dots + 9)$	M1
	Hence (as root is positive) repeated real root is 3	A1
		(2)
(d)	$(x^2 - 8x + 25)(x^2 - 6x + 9)$ $= x^4 - 6x^3 + 9x^2 - 8x^3 + 48x^2 - 72x + 25x^2 - 150x + 225$	M1
	$= x^4 - 14x^3 + 82x^2 - 222x + 225$ Two correct middle term coefficients	A1
	So $A = -14$, $B = 82$ and $C = -222$ (or accept in the quartic)	A1
		(3)

(8 marks)**Notes:****(a)****B1:** For $4 - 3i$ **(b)****M1:** Correct strategy to find a quadratic factor. May expand as shown in scheme, or may look for sum of roots and product of roots first and then write down the factor.**A1:** Correct quadratic factor. Can be written down – give M1A1 if correct, M0A0 if incorrect.

Ignore “= 0” with their quadratic factor.

Alt for (b):**M1:** Product of complex roots is 25, so product of real roots is $\frac{225}{25} = 9$, so the (positive) real root is**A1:** $x^2 - 6x + 9$ or $(x - 3)^2$ **(c)****M1:** A complete strategy to deduce the real root or its square. May consider product of roots, as in scheme, or may first attempt to factorise/long division to find the other quadratic factor – award at the point the quadratic factor with real roots is found. May have been seen in (b)**A1:** Real root is 3. (No need to see rejection of the negative possibility.)

Not a “show that” so award M1A1 if correct root is written down with no working.



(d)

M1: Attempts to expand the two quadratic factors – one of which must have a repeated root, so $(x^2 \pm 9)$ scores M0. (Alternative, may apply $-(\text{sum of roots})$ to find A , pair sum to find B etc – accept method for at least two constants.)

A1: Two correct values of the three. Accept as embedded in a quartic equation.

A1: All three correct. Accept as embedded in their quartic equation.

If their answers are wrong a correct method would get M1A0A0 but w/o some working score M0



Question	Scheme	Marks
5(a)	Two of: Rotation; about O ; through $60^\circ \left(\frac{\pi}{3}\right)$ (anticlockwise)	M1
	All of: Rotation about O through $60^\circ \left(\frac{\pi}{3}\right)$ (anticlockwise)	A1
		(2)
(b)	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	B1
		(1)
(c)	$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	M1
	$= \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ QP correctly found	A1
		(2)
(d)	$3\mathbf{R} = \begin{pmatrix} -\frac{3\sqrt{3}}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3\sqrt{3}}{2} \end{pmatrix}$ or correctly deals with 3 as a multiple.	B1ft
	Required matrix is	
	$(3\mathbf{R})^{-1} = \frac{1}{\begin{pmatrix} -\frac{3\sqrt{3}}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{3\sqrt{3}}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{3\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \end{pmatrix}} \begin{pmatrix} \frac{3\sqrt{3}}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{3\sqrt{3}}{2} \end{pmatrix} = \dots$	M1
	Or $(\mathbf{R})^{-1} = \frac{1}{\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \end{pmatrix}} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \dots$	
	$(3\mathbf{R})^{-1} = \frac{1}{-9} \begin{pmatrix} \frac{3\sqrt{3}}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{3\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{\sqrt{3}}{6} \end{pmatrix}$	A1
		(3)
(8 marks)		



Notes:**(a)****M1:** Two aspects of the type, centre of rotation and angle correct. Accept equivalent angles or angle in radians. (E.g. 300° clockwise is fine). Assume anticlockwise unless otherwise stated.**A1:** Fully correct description. Accept just 60° for the angle, but 60° clockwise is incorrect**(b)****B1:** Correct matrix.**(c)****M1:** Attempts to multiply **Q** and **P** in the correct order.**A1:** **QP** correct**(d)****B1ft:** Multiplies all elements of their matrix by 3, or multiplies all elements of their $\mathbf{R}^{-1}$ by $\frac{1}{3}$ **M1:** Attempts the inverse of their $3\mathbf{R}$ or $\mathbf{R}$. This must be a complete method – ie must transpose and evaluate the determinant and use it. Alternatively, they may attempt an inverse from first principles. Award this mark if a slip is made in solving their simultaneous equations.**A1:** Correct answer. Accept alternative forms

Question	Scheme	Marks
6(a)	(i) $\alpha + \beta = -\frac{5}{A}$	B1
	(ii) $\alpha\beta = -\frac{12}{A}$	B1
		(2)
(b)	$\left(\alpha - \frac{3}{\beta}\right) + \left(\beta - \frac{3}{\alpha}\right) = (\alpha + \beta) - 3\left(\frac{\alpha + \beta}{\alpha\beta}\right) = -\frac{5}{A} - 3\left(\frac{-5}{A}\right) \times \frac{-A}{12}$	M1
	$-\frac{5}{A} - \frac{15}{12} = \frac{5}{4} \Rightarrow A = \dots$	dM1
	$A = -2$	A1
		(3)
(c)	$\left(\alpha - \frac{3}{\beta}\right)\left(\beta - \frac{3}{\alpha}\right) = \alpha\beta - 6 + \frac{9}{\alpha\beta} = -\frac{12}{A} - 6 + \frac{9}{-12/A}$	M1
	$-\frac{12}{-12/A} - 6 - \frac{9 - 2}{12} = \frac{B}{4} \Rightarrow B = \dots$	dM1
	$B = 6$	A1
		(3)
(8 marks)		
Notes:		
(a) (i) B1: Correct expression for $\alpha + \beta$ (ii) B1: Correct expression for $\alpha\beta$ (b) M1: Attempts the sum of roots for second equation in terms of A using results from (a). Allow slips in signs. dM1: Equates the sum of roots to $\frac{5}{4}$ and solves for A. Depends on the previous M mark. A1: $A = -2$ (c) M1: Attempts the product of roots for second equation in terms of A using results from (a). Allow slips in signs. May be using their value of A or A itself dM1: Equates the product of roots to $\frac{B}{4}$ and solves for B using their value of A. Depends on first M mark of (c). A1: $B = 6$		

Question	Scheme	Marks
7(a)	$\frac{dy}{dx} = -\frac{36}{x^2}$ oe	B1
	$m_t = -\frac{36}{4^2} \Rightarrow m_n = \frac{16}{36} = \frac{4}{9}$	M1
	Normal is $y - 9 = \frac{4}{9}(x - 4)$	M1
	$\Rightarrow 9y - 81 = 4x - 16 \Rightarrow 4x - 9y + 65 = 0^*$	A1*
		(4)
(b)	Normal meets H again when $4x - 9 \times \frac{36}{x} + 65 = 0$ or $4 \times \frac{36}{y} - 9y + 65 = 0$	M1
	$\Rightarrow 4x^2 + 65x - 324 = 0 \Rightarrow x = \dots$ or $9y^2 - 65y - 144 = 0 \Rightarrow y = \dots$	dM1
	$\Rightarrow Q = \left(-\frac{81}{4}, -\frac{16}{9}\right)$	A1
	At $x = -\frac{81}{4}$, $\frac{dy}{dx} = -\frac{36}{\left(-\frac{81}{4}\right)^2} = \dots$ so tangent is	M1
	$y - \left(-\frac{16}{9}\right) = -\frac{64}{729}\left(x - \left(-\frac{81}{4}\right)\right)$	
	$y = -\frac{64}{729}x - \frac{32}{9}$	A1(5)
(9 marks)		

Notes:
(a)
B1: Correct expression for $\frac{dy}{dx}$, or any equivalent correct expression including it, such as

$$x \frac{dy}{dx} + y = 0 \text{ or } \frac{dy}{dx} = -\frac{1}{t^2}$$

M1: Attempts negative reciprocal gradient at the point P
M1: Uses their normal (changed from tangent) gradient and $P(4,9)$ to find the equation of the normal. Look for $y - 9 = m_n(x - 4)$ Working must be shown for their constant if $y = mx + c$ is used as this is a “show that” question.

A1*: Correct equation achieved from correct working with intermediate step.

(b)
M1: Substitutes hyperbola equation into the **given** normal to obtain an equation in one variable. Other valid means of obtaining an equation in a single variable are acceptable.

dM1: Gathers terms and solves the 3 term quadratic to find a value $\neq 4$ for x or $\neq 9$ for y . Solution by calculator allowed if correct roots (or values $\neq 4$ for x or $\neq 9$ for y) are shown

A1: Correct coordinates of intersection.

M1: Uses their x value to find the gradient at Q and then uses the intersection point with their gradient to form the equation of the line.

A1: Correct equation.


Question	Scheme						Marks													
8(a)	<table border="1"><tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr><tr><td>$f(x)$</td><td>0.5</td><td>-1.2401</td><td>-0.2885</td><td>0.1508</td><td>0.5840</td></tr></table>						x	1	2	3	4	5	$f(x)$	0.5	-1.2401	-0.2885	0.1508	0.5840	One correct Both correct	B1 B1
	x	1	2	3	4	5														
	$f(x)$	0.5	-1.2401	-0.2885	0.1508	0.5840														
(2)																				
(b)	Identify an interval where the sig changes and mention the change of sign						M1													
	f is continuous on $[3,4]$, not on $[1,2]$ hence the root is in $[3,4]$						A1													
							(2)													
(c)	$f(3.5) = -0.064... < 0$ so root in $[3.5,4]$						M1													
	$f(3.75) = 0.0435... > 0$						M1													
	Hence root is in the interval $[3.5, 3.75]$						A1													
							(3)													
(d)	E.g. $\frac{f(-0.5)-f(-1)}{-0.5-(-1)} = \frac{f(-0.5)-0}{-0.5-\beta}$ or $\frac{\beta-(-1)}{0-f(-1)} = \frac{\beta-(-0.5)}{0-f(-0.5)}$ etc						M1													
	$\Rightarrow \beta = -0.5 - \frac{0.5 \times f(-0.5)}{f(-0.5)-f(-1)} = ...$ or $\beta = \frac{-0.5f(-1)+f(-0.5)}{(f(-1)-f(-0.5))} = ...$ etc						dM1													
	$= -0.5 - \frac{0.5 \times 0.5786...}{0.5786... - (-0.875)} = -0.6990... = -0.699$ (awrt)						A1													
							(3)													
(10 marks)																				

Notes:

Accept open or closed intervals throughout the question where relevant and intervals described by inequalities.

(a)

B1: One correct value of the two missing.

B1: Both values correct.

(b)

M1: Identifies at least one of the intervals on which a sign change occurs – must mention sign changing.

A1: Correct interval with reason given. Accept reasons such as f not defined at $\frac{5}{3}$ in $[1,2]$ or $x = \frac{5}{3}$ is an asymptote as reason for dismissing this interval.

(c)

M1: Evaluates f at the midpoint of their chosen interval from (b) and selects interval of length 0.5 in which the root lies. This mark can be awarded if the interval was incorrect (even if no change of sign in that interval)

M1: Evaluates f at the midpoint of their interval of length 0.5, and considers the signs or chooses the “correct” interval of length 0.25. There must have been a change of sign in their initial interval for this mark to be awarded.

A1: Correct interval selected with all values correct to at least 1 s.f. rounded or truncated. No extra intervals included.



(d)

M1: Correct interpolation strategy. Accept any correct statement such as the one shown. Sign errors imply an incorrect formula unless they follow a correct general statement.

dM1: Rearranges to find β and evaluates.

A1: Accept awrt -0.699 following correct working.



Question	Scheme	Marks
9(a)	For $n = 1$, $\sum_{r=1}^1 r^3 = 1$ and $\frac{1}{4}(1^2)(1+1)^2 = \frac{1}{4} \times 1 \times 4 = 1$ So true for $n = 1$	B1
	(Assume the result is true for $n = k$, so $\sum_{r=1}^k r^3 = \frac{1}{4}k^2(k+1)^2$) Then $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$	M1
	$= \frac{1}{4}(k+1)^2[k^2 + 4(k+1)] = \frac{1}{4}(k+1)^2[k^2 + 4k + 4]$	M1
	$= \frac{1}{4}(k+1)^2(k+2)^2$	A1
	$\left[= \frac{1}{4}(k+1)^2((k+1)+1)^2 \right]$ Hence result is true for $n = k+1$. As true for $n = 1$ and have shown if true for $n = k$ then it is true for $n = k+1$, so it is true for all $n \in \mathbb{N}$ by induction.	A1
		(5)
(b)	$\sum_{r=1}^n r(r+1)(r-1) = \sum_{r=1}^n r^3 - r$	B1
	$= \frac{1}{4}n^2(n+1)^2 - \frac{1}{2}n(n+1)$	M1
	(Please note the mark above is incorrectly labelled as A1 on e-PEN)	
	$= \frac{1}{4}n(n+1)[n^2 + n - 2] = \frac{1}{4}n(n+1)(n+...)(n+...)$	M1
	$= \frac{1}{4}n(n+1)(n-1)(n+2)$	A1
		(4)
(c)	$\sum_{r=n}^{2n} r^2 = \frac{1}{6}(2n)(2n+1)(2(2n)+1) - \frac{1}{6}(n-1)(n)(2(n-1)+1)$	M1 A1
	$3 \sum_{r=1}^n r(r+1)(r-1) = 17 \sum_{r=n}^{2n} r^2$ $\Rightarrow \frac{3}{4}n(n+1)(n-1)(n+2) = \frac{17}{6}n(2(8n^2 + 6n + 1) - (2n^2 - 3n + 1))$ $\Rightarrow 18(n+1)(n-1)(n+2) = 68(14n^2 + 15n + 1) = 68(14n+1)(n+1)$	dM1

	$\Rightarrow 18(n-1)(n+2) = 68(14n+1)$ $\Rightarrow 18n^2 - 934n - 104 = 0 \Rightarrow n = \dots$	ddM1
	$n = 52$	A1
		(5)

(14 marks)**Notes:****(a)**

B1: Checks the result for $n = 1$. Should see a clear substitution into both sides, accept minimum of seeing $\frac{1}{4} \times 1 \times 4$, or $\frac{1}{4} \times 1 \times 2^2$, or $\frac{1}{4} \times 1 \times (1+1)^2 = 1$ for right hand side.

M1: (Makes or assumes the inductive assumption, and) adds $(k+1)^3$ to the result for $n = k$

M1: Attempts to take at least $(k+1)^2$ as a factor out of the expression. Allow if an expansion to a quartic is followed by the factorised expression.

A1: Reaches the correct expression for $n = k+1$ from correct working with sufficient working seen, so expect at least seeing the quadratic before a factorised form.. Need not see the “ $k+1$ ” explicitly for this mark.

A1: Completes the induction by demonstrating the result clearly, with suitable conclusion conveying “true for $n = 1$ ”, “assumed true for $n = k$ ” and “shown true for $n = k+1$ ”, and “hence true for all n ”. All these statements (or equivalents) must be seen in their conclusion (not simply scattered through the work). Depends on all except the B mark, though a check for $n = 1$ must have been attempted.

(b)

B1: Correct expansion.

M1: Applies the standard formula for $\sum r$ and the result from (a) to their sum.

If the expansion is given as $\sum r^3 - r^2$, allow the use of $\sum r^2$ instead of $\sum r$

M1: Takes out the common factors n and $(n+1)$ and attempts to simplify to required form OR factorises their quartic.

A1: Correct answer. (Ignore A , B and C listed explicitly.) Correct answer can be obtained from a cubic or a quartic. Award M1A1 in either of these cases.

(c)

M1: Attempts to apply $\sum_{r=n}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2$ with the standard result for $\sum r^2$ Accept with n

instead of $n-1$ in second expression.

A1: Correct expression for the RHS seen, no need to be simplified.

dm1: Applies the summations to the equation in the question and cancels/factorises out the factor n . Depends on the first M mark of (c)

ddM1: Simplifies the quadratic factor of the right hand side and cancels/factors out the $n+1$ and solves the resulting quadratic. Note M1M1 is implied by sights of the correct roots $-\frac{1}{9}, 52, 0, -1$ of the quartic. Depends on both previous M marks in (c)

A1: Correct answer. Must reject other roots. The correct answer obtained from a quartic or cubic equation solved by calculator gains all relevant marks.

