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EDEXCEL INTERNATIONAL A LEVEL MATHS: QUESTIONS BY TOPIC

Further Complex Numbers (FP2.3)

International A Level Mathematics · FP2: Further Pure Mathematics 2 (WFM02)

29 questions · 242 marks · Mark schemes included

Past papers October 2020 - June 2025 · 2 sub-topics

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HOW TO USE THIS BOOKLET

Questions are grouped by sub-topic and run from the lowest to the highest mark tariff, then oldest paper first.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme; < QUESTION brings you back.

A gold bar in the left margin marks exactly where the question sits on the original page.

Diagrams, graphs, grids and tables are printed exactly as they appear in the original papers.

Each question is filed under its main topic; longer questions may also test skills from other topics.

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QUESTIONS

29 questions · 242 marks · 2 sub-topics

SUB-TOPIC FP2.3a

DE MOIVRE'S THEOREM & ROOTS OF UNITY

Sub-topic FP2.3a · 14 questions · 130 marks · lowest tariff first



1. Solve the equation

$$z^5 - 32i = 0$$

giving each answer in the form $re^{i\theta}$ where $0 < \theta < 2\pi$

(4)

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Question 1 continued

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(Total 4 marks)

Q1

Answer box for Q1



1. (a) Express $-2\sqrt{2} - (2\sqrt{6})i$ in the form $re^{i\theta}$ where $-\pi < \theta \leq \pi$

(3)

- (b) Hence solve

$$z^5 = -2\sqrt{2} - (2\sqrt{6})i$$

Give your answers in the form $\sqrt{p}e^{i\theta}$ where $p \in \mathbb{Z}^+$ and $-\pi < \theta \leq \pi$

(3)

Question 1 continued

Lined area for writing the answer to Question 1.

(Total for Question 1 is 6 marks)

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1. In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(a) Express the complex number

$$-4 - 4\sqrt{3}i$$

in the form $r(\cos \theta + i \sin \theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$

(3)

(b) Solve the equation

$$z^3 + 4 + 4\sqrt{3}i = 0$$

giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$

(4)

Question 1 continued

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Lined area for writing the answer to Question 1 continued.



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Q1

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4. (a) Express the complex number $18\sqrt{3} - 18i$ in the form

$$r(\cos \theta + i \sin \theta) \quad -\pi < \theta \leq \pi \quad (3)$$

- (b) Solve the equation

$$z^4 = 18\sqrt{3} - 18i$$

giving your answers in the form $re^{i\theta}$ where $-\pi < \theta \leq \pi$

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Question 4 continued

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(Total 8 marks)

Q4



Question 7 continued

Lined area for writing the answer to Question 7 continued.



Question 7 continued



(Total for Question 7 is 6 marks)







Question 2 continued

(Total for Question 2 is 8 marks)



9.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(a) Use De Moivre's theorem to show that

$$\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 \quad (4)$$

(b) Hence determine the smallest positive root of the equation

$$48x^6 - 72x^4 + 27x^2 - 1 = 0$$

giving your answer to 3 decimal places. (4)

Question 9 continued

Lined area for writing the answer to Question 9 continued.



Question 9 continued

Lined area for writing the answer to Question 9 continued.



7. (a) Use de Moivre's theorem to show that

$$\tan 4\theta \equiv \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$$

(6)

(b) Use the identity given in part (a) to find the 2 positive roots of

$$x^4 + 2x^3 - 6x^2 - 2x + 1 = 0$$

giving your answers to 3 significant figures.

(3)

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Question 9 continued

Lined area for writing the answer to Question 9 continued.



Question 9 continued

Lined area for writing the answer to Question 9.

(Total for Question 9 is 9 marks)

TOTAL FOR PAPER IS 75 MARKS





Question 2 continued

Lined area for writing the answer to Question 2 continued.



(Total for Question 2 is 16 marks)



Question 7 continued

Lined area for writing the answer to Question 7 continued.



Question 7 continued



Question 7 continued

(Total for Question 7 is 11 marks)



8. Given that $z = \cos \theta + i \sin \theta$

(a) show that, for $n \in \mathbb{Z}$

$$z^n + \frac{1}{z^n} = 2 \cos n\theta \quad (2)$$

(b) Hence show that

$$\cos^4 \theta = \frac{1}{8} (\cos 4\theta + a \cos 2\theta + b)$$

where a and b are integers to be determined. (4)

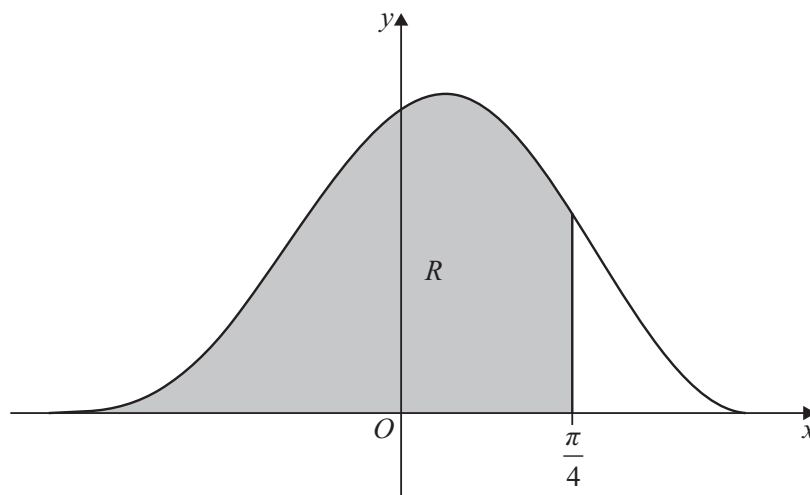


Figure 4

Figure 4 shows a sketch of the curve with equation

$$y = \cos^2 x \sqrt{1 + \sin x} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

The region R , shown shaded in Figure 4, is bounded by the curve, the x -axis and the line with equation $x = \frac{\pi}{4}$

The region R is rotated through 2π radians about the x -axis to form a solid of revolution.

(c) Use the answer to part (b) and algebraic integration to determine the exact volume of this solid.

Give the answer in the form $\frac{\pi}{160} (p + q\pi + r\sqrt{2})$ where p , q and r are integers. (6)

Question 8 continued

Lined area for writing the answer to Question 8 continued.



Question 8 continued

Lined area for writing the answer to Question 8 continued.



Question 8 continued

Lined area for writing the answer to Question 8 continued.



Question 8 continued

(Total for Question 8 is 12 marks)

TOTAL FOR PAPER IS 75 MARKS





Question 8 continued

Lined area for writing the answer to Question 8 continued.



Question 8 continued

Lined area for writing the answer to Question 8 continued.



Question 8 continued

Lined area for writing the answer to Question 8.

(Total for Question 8 is 14 marks)

TOTAL FOR PAPER IS 75 MARKS



Question 8 continued

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Question 8 continued

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Question 8 continued

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Lined area for writing the answer to Question 8.

Q8

(Total 16 marks)

TOTAL FOR PAPER: 75 MARKS

END



SUB-TOPIC FP2.3b

LOCI & TRANSFORMATIONS IN THE COMPLEX PLANE

Sub-topic FP2.3b · 15 questions · 112 marks · lowest tariff first



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1. The transformation T from the z -plane, where $z = x + iy$, to the w -plane, where $w = u + iv$, is given by

$$w = \frac{z + pi}{iz + 3} \quad z \neq 3i \quad p \in \mathbb{Z}$$

The point representing $i(1 + \sqrt{3})$ is invariant under T .

Determine the value of p .

(3)

Question 1 continued

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Q1

(Total 3 marks)



(Total for Question 1 is 4 marks)



Question 3 continued

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Question 3 continued

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(Total 6 marks)

Q3



Question 3 continued

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Question 3 continued

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- 5.** The transformation T from the z -plane to the w -plane is given by

$$w = \frac{z - 3i}{z + 2i} \quad z \neq -2i$$

The circle with equation $|z| = 1$ in the z -plane is mapped by T onto the circle C in the w -plane.

Determine

- (i) the centre of C ,
- (ii) the radius of C .

(7)

Question 5 continued

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Question 5 continued

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Question 5 continued

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Q5

(Total 7 marks)



Question 5 continued

Question 5 continued



Question 5 continued

(Total for Question 5 is 7 marks)



6. The transformation T from the z -plane to the w -plane is given by

$$w = \frac{z-i}{z+1} \quad z \neq -1$$

Given that T maps the imaginary axis in the z -plane to the circle C in the w -plane, determine

- (i) the coordinates of the centre of C
- (ii) the radius of C

(7)

Question 6 continued

Lined area for writing the answer to Question 6 continued.



Question 6 continued



Question 6 continued

(Total for Question 6 is 7 marks)



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2. The transformation T from the z -plane, where $z = x + iy$, to the w -plane, where $w = u + iv$, is given by

$$w = \frac{z + 2}{z - i} \quad z \neq i$$

The transformation T maps the circle $|z| = 2$ in the z -plane onto a circle C in the w -plane.

- Find (i) the centre of C ,
(ii) the radius of C .

(8)

Question 2 continued

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Question 2 continued

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Question 2 continued

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(Total 8 marks)

Q2

Grading box for Q2.



Question 7 continued

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Question 7 continued

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(Total 8 marks)

Q7



Question 3 continued

Question 3 continued



Question 3 continued

Lined area for writing the answer to Question 3.

(Total for Question 3 is 8 marks)



Question 6 continued



Question 6 continued

Question 6 continued

(Total for Question 6 is 8 marks)



4.

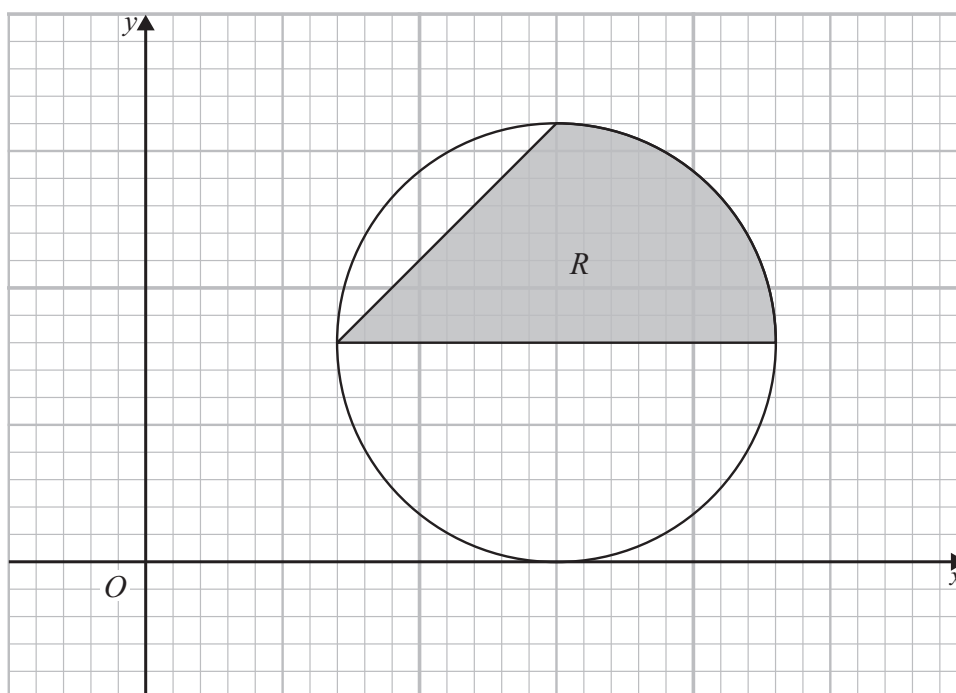


Figure 2

Figure 2 shows an Argand diagram for complex numbers of the form $z = x + iy$.

The diagram is drawn accurately, although the scale is not shown on the axes.

Complex numbers that lie in the region R , shown shaded in Figure 2, satisfy all three of the inequalities

$$|z - 15 - 8i| \leq a$$

$$0 \leq \arg(z + 1) \leq b\pi$$

$$|z + 2i| \geq |z + ci|$$

where a , b and c are real numbers.

(a) Determine the value of a , the value of b and the value of c

(3)

Given that the complex number w lies in the region R ,

(b) determine the exact range of possible values of $|w|$

(3)

(c) determine the minimum value of $\arg w$, giving the answer in radians to 3 significant figures.

(2)

Question 4 continued

Question 4 continued

Lined area for writing the answer to Question 4 continued.

(Total for Question 4 is 8 marks)



Question 6 continued

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Question 6 continued

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Question 6 continued

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(Total 9 marks)

Q6



Question 7 continued



Question 7 continued

(Total for Question 7 is 10 marks)







Question 8 continued

TOTAL FOR PAPER IS 75 MARKS

MARK SCHEMES

official mark schemes, in the same order as the questions



General Principles for Further Pure Mathematics Marking*(But note that specific mark schemes may sometimes override these general principles)***Method mark for solving 3 term quadratic:****1. Factorisation** $(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$ $(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$ **2. Formula**Attempt to use the correct formula (with values for a , b and c).**3. Completing the square**Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$ **Method marks for differentiation and integration:****1. Differentiation**Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)**2. Integration**Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)

June 2025
WFM02 Further Pure Mathematics F2
Mark Scheme

Question Number	Scheme	Notes	Marks
1(a)	$r = 4\sqrt{2}$	Correct r Accept $\pm 4\sqrt{2}$ or $\pm\sqrt{32}$ or other processed equivalents.	B1
	$\theta = -\pi + \arctan\left(\frac{2\sqrt{6}}{2\sqrt{2}}\right) = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$	Complete method to find an argument in the correct quadrant for their r	M1
	$\Rightarrow -2\sqrt{2} - (2\sqrt{6})i = 4\sqrt{2}e^{-\frac{2\pi}{3}i}$	Correct answer in the form specified. Accept with $\sqrt{32}$	A1
(3)			
(b)	$z^5 = -2\sqrt{2} - (2\sqrt{6})i$		
	$z = (4\sqrt{2})^{\frac{1}{5}} e^{\frac{\frac{2\pi}{3} + 2k\pi}{5}i}$	Attempts De Moivre correctly with either all roots indicated or simplified r with one argument. See additional notes.	M1
	$\sqrt{2}e^{\frac{2\pi}{15}i}, \sqrt{2}e^{\frac{4\pi}{15}i}, \sqrt{2}e^{\frac{2\pi}{3}i},$ $\sqrt{2}e^{\frac{8\pi}{15}i}, \sqrt{2}e^{\frac{14\pi}{15}i}$	Three correct roots provided coefficients of π for θ are single fractions, allow for $0 < \theta < 2\pi$ Allow $(4\sqrt{2})^{\frac{1}{5}}$ oe for $\sqrt{2}$ and correct roots in trig form	A1
		All five correct roots and no extra. Must have $-\pi < \theta \leq \pi$. Allow e.g., $\sqrt{2}e^{\frac{10\pi}{15}i}$ for $\sqrt{2}e^{\frac{2\pi}{3}i}$	A1
(3)			
Total 6			

Additional Notes:

(a)

B1: Accept positive or negative root. Allow if seen in part (b).

M1: Requires a method to achieve an argument in the correct quadrant for their r . Accept attempts at

$\arctan\left(\frac{2\sqrt{6}}{2\sqrt{2}}\right) \pm \pi$ or $\arctan\left(\frac{2\sqrt{2}}{2\sqrt{6}}\right) \pm \pi$ for attempts where $r > 0$, or equivalent methods via arccos or arcsin

(must use r as the denominator). May be implied by sight of any of $-\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{6}$ or $-\frac{5\pi}{6}$ if no method is shown.

In cases where a negative r is given, accept attempts at just $\arctan\left(\frac{2\sqrt{6}}{2\sqrt{2}}\right)$ or $\arctan\left(\frac{2\sqrt{2}}{2\sqrt{6}}\right)$ provided they

do not subsequently adjust to a different quadrant.

A1: Correct answer in the form specified, with $-\pi < \theta < \pi$. Must be given in correct form, but accept

answer with negative r , i.e. $-4\sqrt{2}e^{\frac{\pi}{3}i}$. Accept with exact processed equivalents for r .

Note: Correct answer with no working shown will score B1M1(implied)A1.



General Principles for Further Pure Mathematics Marking*(But note that specific mark schemes may sometimes override these general principles)***Method mark for solving 3 term quadratic:****1. Factorisation** $(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$ $(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$ **2. Formula**Attempt to use the correct formula (with values for a , b and c).**3. Completing the square**Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$ **Method marks for differentiation and integration:****1. Differentiation**Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)**2. Integration**Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)

Question Number	Scheme	Notes	Marks
7(a)	Condone use of e.g. $C + iS$ for $\cos x + i \sin x$ if the intention is clear.		
	$(\cos 5x \equiv) \operatorname{Re}(\cos x + i \sin x)^5 \equiv \cos^5 x + \binom{5}{2} \cos^3 x (i \sin x)^2 + \binom{5}{4} \cos x (i \sin x)^4$ Identifies the correct terms of the binomial expansion of $(\cos x + i \sin x)^5$ They may expand $(\cos x + i \sin x)^5$ completely but there must be an attempt to extract the real terms which must have the correct binomial coefficients combined with the correct powers of $\sin x$ and $\cos x$. Condone use of a different variable e.g. θ .		M1
	$(\cos 5x \equiv) \cos^5 x - 10 \cos^3 x \sin^2 x + 5 \cos x \sin^4 x$ Correct simplified expression. Condone use of a different variable e.g. θ .		A1
	$\equiv \cos x (\cos^4 x - 10 \cos^2 x \sin^2 x + 5 \sin^4 x)$ $\equiv \cos x ((1 - \sin^2 x)^2 - 10(1 - \sin^2 x) \sin^2 x + 5 \sin^4 x)$ Applies $\cos^2 x = 1 - \sin^2 x$ to obtain an expression in terms of $\sin x$ inside the bracket. Condone use of a different variable e.g. θ .		M1
	$\equiv \cos x (16 \sin^4 x - 12 \sin^2 x + 1)$	Correct expression. Must be in terms of x now. The “ $\cos 5x =$ ” is not required.	A1
			(4)
(b)	Allow use of a different variable in (b) e.g. x for <u>all</u> marks.		
	$\cos 5\theta = \sin 2\theta \sin \theta - \cos \theta$ $\Rightarrow \cos \theta (16 \sin^4 \theta - 12 \sin^2 \theta + 1) = 2 \sin^2 \theta \cos \theta - \cos \theta$ $\Rightarrow \cos \theta (16 \sin^4 \theta - 14 \sin^2 \theta + 2) = 0$ Uses the result from part (a) with $\sin 2\theta = 2 \sin \theta \cos \theta$ and collects terms		M1
	$16 \sin^4 \theta - 14 \sin^2 \theta + 2 = 0$ $\Rightarrow \sin^2 \theta = \frac{7 \pm \sqrt{17}}{16} \Rightarrow \sin \theta = \dots$ Solves for $\sin^2 \theta$ by any method including calculator and takes square root to obtain at least one value for $\sin \theta$. Depends on the first mark. May be implied by their values of $\sin \theta$ or θ . NB $\frac{7 \pm \sqrt{17}}{16} = 0.69519\dots, 0.17980\dots$		dM1
	$\sin \theta = \sqrt{\frac{7 \pm \sqrt{17}}{16}} \Rightarrow \theta = \dots$ NB $\sqrt{\frac{7 \pm \sqrt{17}}{16}} = 0.833783\dots, 0.424035\dots$ A full method to reach at least one value for θ . Depends on the previous mark. May be implied by their values of θ		ddM1
	$(\theta =) 0.986, 0.438$	Correct values and no others in range. Allow awrt these values.	A1
			(4)
			Total 8

Note that it is possible to do 7(b) by changing to $\cos \theta$ e.g.

$$\cos \theta (16 \sin^4 \theta - 12 \sin^2 \theta + 1) = \cos \theta (16(1 - \cos^2 \theta)^2 - 12(1 - \cos^2 \theta) + 1)$$

$$\cos \theta (16(1 - \cos^2 \theta)^2 - 12(1 - \cos^2 \theta) + 1) = 2 \sin^2 \theta \cos \theta - \cos \theta$$

$$16 \cos^4 \theta - 18 \cos^2 \theta + 4 = 0$$

$$\cos^2 \theta = \frac{9 \pm \sqrt{17}}{16} \Rightarrow \cos \theta = \sqrt{\frac{9 \pm \sqrt{17}}{16}}$$

$$(\theta =) 0.986, 0.438$$

This is acceptable as they used part (a) and can be scored as:

M1: Uses part (a) with $\sin^2 \theta = 1 - \cos^2 \theta$ and $\sin 2\theta = 2 \sin \theta \cos \theta$ and collects terms.

dM1: Solves for $\cos^2 \theta$ by any method including calculator and takes square root to obtain at least one value for $\cos \theta$. Depends on the first mark. May be implied by their values of $\cos \theta$ or θ .

$$\text{NB } \frac{9 \pm \sqrt{17}}{16} = 0.82019..., 0.30480...$$

dM1: A full method to reach at least one value for θ .
Depends on the previous mark. May be implied by their values of θ

$$\text{NB } \sqrt{\frac{9 \pm \sqrt{17}}{16}} = 0.905645..., 0.552092...$$

$$\text{A1: } (\theta =) 0.986, 0.438$$



4. All A marks are 'correct answer only' (cao), unless shown, for example, as A1 ft to indicate that previous wrong working is to be followed through. After a misread however, the subsequent A marks affected are treated as A ft, but manifestly absurd answers should never be awarded A marks.
5. For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.
6. If a candidate makes more than one attempt at any question:
 - If all but one attempt is crossed out, mark the attempt which is NOT crossed out provided it is not cursory.
 - If either all attempts are crossed out or none are crossed out, score for their best attempt.
7. Ignore wrong working or incorrect statements following a correct answer unless the mark scheme indicates otherwise.
8. Mark question parts separately unless the scheme indicates otherwise.

Usual rules for the method mark for solving a 3 term quadratic:

(Note: There may be schemes where the below does not apply)

If no method is shown then one root must be obtained that is consistent with their equation.

1. Factorisation

$(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$

$(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$

2. Formula

Complete attempt to use the correct formula with values for a, b and c leading to $x = \dots$ (may be unsimplified).

3. Completing the square (where $a = 1$, otherwise must divide by a first – allow equivalent work if a is a square number)

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$

Question Number	Scheme	Marks
9(a)	$(\cos 6\theta =) \operatorname{Re}(\cos \theta + i \sin \theta)^6 = \cos^6 \theta - 15 \cos^4 \theta \sin^2 \theta + 15 \cos^2 \theta \sin^4 \theta - \sin^6 \theta$	M1A1
	$\cos^6 \theta - 15 \cos^4 \theta \sin^2 \theta + 15 \cos^2 \theta \sin^4 \theta - \sin^6 \theta$ $= \cos^6 \theta - 15 \cos^4 \theta (1 - \cos^2 \theta) + 15 \cos^2 \theta (1 - \cos^2 \theta)^2 - (1 - \cos^2 \theta)^3$	M1
	$= \cos^6 \theta - 15 \cos^4 \theta + 15 \cos^6 \theta + 15 \cos^2 \theta (1 - 2 \cos^2 \theta + \cos^4 \theta) - (1 - 3 \cos^2 \theta + 3 \cos^4 \theta - \cos^6 \theta)$ $\cos 6\theta \equiv 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1^*$	A1*
		(4)
(b)	$48x^6 - 72x^4 + 27x^2 - 1 = 0$ $x = \cos \theta \Rightarrow \frac{3}{2} \cos 6\theta + \frac{1}{2} = 0 \Rightarrow \cos 6\theta = \dots \left(-\frac{1}{3}\right)$	M1
	$\cos 6\theta = -\frac{1}{3} \Rightarrow 6\theta = \dots \Rightarrow \theta = \dots$ (pv $6\theta = 109.5^\circ, 1.91 \text{ rad}, \theta = 18.25^\circ, 0.318 \text{ rad}$)	dM1
	$x = \cos \theta = \dots$ (commonly 0.950)	dM1
	$x = 0.204$	A1
		(4)
		Total 8

Notes:

(a)

M1: Attempts the binomial expansion of $(\cos \theta + i \sin \theta)^6$ and extracts the real terms. Binomial coefficients should be correct but allow if there are sign errors. Ignore errors on imaginary terms. Allow use of $(c+is)^6$ notation.

A1: Correct extracted expression. Need not be set equal to $\cos 6\theta$ at this stage.

M1: Applies $\sin^2 \theta = 1 - \cos^2 \theta$ to obtain an expression in $\cos \theta$ only.

A1*: Sets equal to $\cos 6\theta$ and fully expands the square and cube brackets (must be shown) then completes with no errors to obtain the given answer.

(b)

Note: The question says hence and thus methods via solving the cubic in x^2 and square rooting the smallest root score no marks.

M1: Attempts to use the result from part (a) and obtains $\cos 6\theta = k, |k| < 1, k \neq 0$

dM1: Solves to obtain at least one value for θ . If the $6\theta = \arccos\left(-\frac{1}{3}\right)$ is seen, accept any value

that follows for the attempt, but if this method is not shown answers must be correct for their value of $\cos(6\theta)$. The principal values are shown in the scheme. Other values that may be seen are

In radians: for 6θ 4.33, 8.19, 10.66, 14.48, 16.94 ; for θ 0.73, 1.37, 1.78, 2.41, 2.82

In degrees: for 6θ 251°, 469°, 611°, 829°, 1331°; for θ 41.8°, 78.2°, 102°, 138°, 222°;

dM1: Depends on first M. Reverses the substitution to find at least one value for x .

Full list of values for reference: 0.950, 0.746, 0.204, -0.204, -0.746, -0.950

A1: Awt 0.204 and must be seen identified as the answer, not just given in a list.

Note those who find $\cos 6\theta = -\frac{1}{3}$ but go on to find $6\theta = \arccos \frac{1}{3} = 1.23$ (70.5°), and so

$\theta = 0.205$ (11.8°) etc (0.979 as primary answer) may score the M1M0M1A0 but watch out for

those who use $\pi - \arccos \frac{1}{3}$ as these can score full marks.



9(a) Alt	$\left(z + \frac{1}{z}\right)^6 = z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$ $= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$ $= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$	M1A1
	$= 2\cos 6\theta + 12(2\cos^2 2\theta - 1) + 30(2\cos^2 \theta - 1) + 20$ $= 2\cos 6\theta + 12(2(2\cos^2 \theta - 1)^2 - 1) + 30(2\cos^2 \theta - 1) + 20$	M1
	$64\cos^6 \theta = 2\cos 6\theta + 12(8\cos^4 \theta - 8\cos^2 \theta + 1) + 60\cos^2 \theta - 30 + 20$ $\Rightarrow \cos 6\theta \equiv 32\cos^6 \theta - 48\cos^4 \theta + 18\cos^2 \theta - 1^*$	A1*
		(4)

Notes:

(a)

M1: Attempts the binomial expansion of $(z + z^{-1})^6$, groups terms (may be implied) and attempts to replace the $z^n + z^{-n}$ to achieve an expression in cosines. Allow if there are errors in the “ $2\cos(n\theta)$ ”.

A1: Correct expression in cosines. Need not be set equal to $64\cos^6 \theta$ at this stage.

M1: Applies $\cos 2A = 2\cos^2 A - 1$ repeatedly to write the $\cos 4\theta$ and $\cos 2\theta$ in terms of $\cos \theta$ only.

A1*: Sets equal to $64\cos^6 \theta$, expands all brackets and rearranges to the given expression with no errors to obtain the given answer.

Question Number	Scheme	Marks
	Alternative for first 4 marks of 7(a): $\sin 4\theta = \frac{1}{2i}(z^4 - z^{-4}) = \frac{1}{2i}((\cos \theta - i\sin \theta)^4 - (\cos \theta + i\sin \theta)^4)$ $= \frac{1}{2i}(\cos^4 \theta + 4i\cos^3 \theta \sin \theta - 6\cos^2 \theta \sin^2 \theta - 4i\cos \theta \sin^3 \theta + \sin^4 \theta)$ $- \frac{1}{2i}(-\cos^4 \theta + 4i\cos^3 \theta \sin \theta + 6\cos^2 \theta \sin^2 \theta - 4i\cos \theta \sin^3 \theta - \sin^4 \theta)$ $= 4\cos^3 \theta \sin \theta - 4\cos \theta \sin^3 \theta$ Similar work leads to $\cos 4\theta = \cos^4 \theta - 6\cos^2 \theta \sin^2 \theta + \sin^4 \theta$ Remaining 2 marks as main scheme	M1 M1 A1 A1
M1 A1 M1 A1	For the expression derived from de Moivre for either $\sin 4\theta$ or $\cos 4\theta$ Both shown and correct Attempt the binomial expansion for either, reaching a simplified expression Both simplified expressions correct	



Question Number	Scheme	Notes	Marks
8(a)	For intersection when $x > 0$: $\frac{15x}{x+4} = x-2$ $15x = x^2 + 2x - 8$ $x^2 - 13x - 8 = 0$ $\Rightarrow x = \frac{13 + \sqrt{201}}{2}$	M1: Solves 3TQ obtained from a correct equation (usual rules) for $x > 0$ A1: Correct exact positive value (condone negative root also included)	M1 A1
	For intersections when $x < 0$: $\frac{15x}{-x+4} = x-2$ $15x = -x^2 + 6x - 8$ $x^2 + 9x + 8 = 0$ $(x+8)(x+1) = 0$ $\Rightarrow x = -8, -1$	M1: Solves 3TQ obtained from a correct equation (usual rules) for $x < 0$ A1: Correct values	M1 A1
	$-8 < x < -1, \quad x > \frac{13 + \sqrt{201}}{2}$	M1: For $a < x < b, x > c$ oe with cvs $a < b < 0 < c$ from the correct equations. A1: Fully correct set of values in any form	M1 A1
(6)			
(b)	$\frac{15x}{x+4} = -x+2$ $15x = -x^2 - 2x + 8$ $x^2 + 17x - 8 = 0$ $\Rightarrow x = \frac{-17 + \sqrt{321}}{2}$	M1: Solves 3TQ obtained from a correct equation (usual rules) A1: Correct exact positive value (condone negative root also included)	M1 A1
	$x < "-8", \quad x > "\frac{13 + \sqrt{201}}{2}"$	Correct regions fit their $a < 0$ and $c > 0$	B1ft
	$x < "-8", \quad "-1" < x < \frac{-17 + \sqrt{321}}{2}, \quad x > "\frac{13 + \sqrt{201}}{2}"$	Correct set of values in any form. Allow fit on their cvs a, b and c provided $a < b < 0$ and $c > \frac{-17 + \sqrt{321}}{2} (= 0.458...)$	A1ft
(4)			
Total 10			

Additional notes.

“Usual Rules” : as well as the general information, we will accept here solutions by calculator having reached a quadratic equation but they must be correct solutions for the stated quadratic if no method is shown (you may need to check).

(a)

Accept with any inequalities used instead of equalities for the solving of the intersection points. For methods via putting over common denominators score the Ms for the process of putting over a common denominator and solving for the numerator = 0.

May see attempts via squaring

$\frac{15x}{|x|+4} = x-2 \Rightarrow |x| = \frac{15x}{x-2} - 4 = \frac{11x+8}{x-2} \Rightarrow x^2 = \left(\frac{11x+8}{x-2}\right)^2$ then cross multiplies (or puts over a common denominator $(x-2)^2$) to get

$\Rightarrow x^2(x^2 - 4x + 4) - (121x^2 + 176x + 64) = x^4 - 4x^3 - 117x^2 - 176x - 64 = 0$. This will probably then be solved by calculator, which as algebra has been used will be accepted. Score M1 for the process reaching quartic, A1 correct quartic, M1 for solving the quartic (look for attempt to factorise to linear terms with correct product of roots, or accept answer from calculator – may need to check), A1 for at least the three relevant roots (and condone if $x=2$ is also include for this mark).

For the final M they must be producing a solution set of the correct form from their CVs from attempts at the correct equations (but may have had incorrect method for solving), so if extra values from denominators are used it will be M0 (regardless of whether the form of the answer is correct).

FYI $\frac{-13 - \sqrt{201}}{2} \approx -13.6$ is being used often as one of the limits, but note it is less than -8 , so

$-8 < x < \frac{-13 - \sqrt{201}}{2}$ used is M0 but you may condone the " $-8 < x$ " for the B1ft in (b).

Final A1: Accept any correct equivalent notation. Accept with "and" or "or" written between inequalities, but if using set notation must use the union. E.g. in interval notation $(-8, -1) \cup \left(\frac{13 + \sqrt{201}}{2}, \infty\right)$. Use of square brackets is A0.

(b)

Attempts at squaring may be seen again here. They will need a full process leading solving a quartic. Again for the A, accept if other roots are also included at this stage.

B1ft: For the follow through same "outside" ends of the region matching their answer to (b) in their **final** answer.

A1ft: Accept answers of the correct form providing their answer to (a) was of the correct form as specified in the scheme and their c is $c > \frac{-17 + \sqrt{321}}{2} (= 0.458...)$.

Question Number	Scheme	Notes	Marks
9(a)	$\cos 5\theta + i \sin 5\theta = (\cos \theta + i \sin \theta)^5$ $= \cos^5 \theta + {}^5C_1 \cos^4 \theta (i \sin \theta) + {}^5C_2 \cos^3 \theta (i \sin \theta)^2 + {}^5C_3 \cos^2 \theta (i \sin \theta)^3 + {}^5C_4 \cos \theta (i \sin \theta)^4 + (i \sin \theta)^5$		M1
	$(\sin 5\theta =) 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta$	M1: Equates imaginary parts A1: Correct expression	M1 A1
	$= 5 \sin \theta (1 - \sin^2 \theta)^2 - 10 \sin^3 \theta (1 - \sin^2 \theta) + \sin^5 \theta$ $= 5 \sin \theta - 10 \sin^3 \theta + 5 \sin^5 \theta - 10 \sin^3 \theta + 10 \sin^5 \theta + \sin^5 \theta$ $= 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$	M1: Uses $\cos^2 \theta = 1 - \sin^2 \theta$ and reaches an expression of the correct form A1: Correct expression or $a = 16, b = -20, c = 5$ $\sin 5\theta$ must have been seen at some point	M1 A1
(5)			
(b)	$\left(\text{For } \theta = \frac{k\pi}{5}, k \in \mathbb{Z}, \sin 5\theta = \sin k\pi = 0 \Rightarrow \right)$ $16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta = 0$	Sets their answer to (a) = 0 Could be implied	M1
	$\sin \theta (16 \sin^4 \theta - 20 \sin^2 \theta + 5) = 0$ $\Rightarrow \sin^2 \theta = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(16)(5)}}{2(16)} = \frac{20 \pm \sqrt{80}}{32}$	Solves their quadratic in $\sin^2 \theta$ to find both roots (usual rules)	M1
	$\Rightarrow \sin^2 \frac{k\pi}{5} = \frac{5 \pm \sqrt{5}}{8}, 0$	A1: $\frac{5 \pm \sqrt{5}}{8}$ oe B1: 0	A1, B1
(4)			
Total 9			

Additional Notes

(a)

M1: Attempts to expand $(\cos \theta + i \sin \theta)^5$ to obtain an expression of the correct form for the three imaginary terms. May use $c + is$ notation. Need only see the imaginary terms, condone missing or incorrect real terms. Binomial coefficients may be as expressions or values, if values only used they must be correct. The powers of i may be processed in the initial expansion. Allow invisible brackets if recovered.

M1: Extracts the imaginary terms from the expansion. The equating to $\sin 5\theta$ may be tacit at this stage.

A1: Correct expression for $\sin 5\theta$ (though the $\sin 5\theta$ need not be seen yet) in terms of $\sin \theta$ and $\cos \theta$.

M1: Uses $\cos^2 \theta = 1 - \sin^2 \theta$ to eliminate $\cos \theta$ from the expression to get to an expanded equation in $\sin \theta$ only. Terms need not be gathered at this stage, but the simplification may be done in one step.

A1: Correct answer including the $\sin 5\theta$ having been seen identified with the expression at some stage if not included at the end. Accept values for a, b and c stated following reaching a suitable equation in $\sin \theta$

(b)

M1: Recognition of the correct process seen by either their answer to (a) equal to zero, which by be implied, or by an attempt to solve it (the “=0” being implied, accept one correct root for their equation as evidence).

M1: Correct method to solve the quadratic in $\sin^2 \theta$ after factoring out the $\sin \theta$ - both roots must be given. Condone for this mark if they mistakenly mislabel the solution to the quadratic in $\sin^2 \theta$ as $\sin \theta$, but do not

allow the A if they go on square their values. Note answer stated directly from the quintic score M0. They must at least identify the quadratic in $\sin^2 \theta$ before reaching the answers (from calculator).

A1: Correct answers only but accept equivalent forms.

B1: For 0 identified as a value for $\sin^2 \frac{k\pi}{5}$ - this can come from anywhere (method not required).



Question Number	Scheme	Marks
1(a)	$\frac{2}{\sqrt{r} + \sqrt{r-2}} \times \frac{\sqrt{r} - \sqrt{r-2}}{\sqrt{r} - \sqrt{r-2}}$	M1
	$\frac{2(\sqrt{r} - \sqrt{r-2})}{r - (r-2)} = \sqrt{r} - \sqrt{r-2} *$	A1*
		(2)
(b)	$r = 2: \sqrt{2} - \sqrt{2-2} (= \sqrt{2} - 0) \quad \dots r = n-2: \sqrt{n-2} - \sqrt{n-4}$ $r = 3: \sqrt{3} - \sqrt{3-2} (= \sqrt{3} - 1) \quad r = n-1: \sqrt{n-1} - \sqrt{n-3}$ $r = 4: \sqrt{4} - \sqrt{4-2} (= 2 - \sqrt{2}) \quad r = n: \sqrt{n} - \sqrt{n-2}$	M1
	$\left[\sum_{r=2}^n \frac{2}{\sqrt{r} + \sqrt{r-2}} \right] = \sqrt{n} + \sqrt{n-1} - 1$	A1 A1
		(3)
(c)	$\left[\sum_{r=4}^{50} \frac{2}{\sqrt{r} + \sqrt{r-2}} \right] f(50) - f(3) = \sqrt{50} + \sqrt{49} - 1 - (\sqrt{3} + \sqrt{2} - 1)$	M1
	$= 5\sqrt{2} + 7 - 1 - \sqrt{3} - \sqrt{2} + 1 = 7 + 4\sqrt{2} - \sqrt{3}$	A1
		(2)
		Total 7
Notes		
(a) M1: Indicates intention to multiply either side by a correct fraction, may use $\frac{\sqrt{r-2} - \sqrt{r}}{\sqrt{r-2} - \sqrt{r}}$. The “2” may be missing. May work in reverse from right hand side to left, and this is fine. Alternatively, may multiply the initial expression through by $\sqrt{r} + \sqrt{r-2}$ and use a sequence of equivalences (though accept with $\Rightarrow$ or nothing between lines). A1: Fully correct proof. A result from rationalisation that is not the given answer must be seen. If using a sequences of equivalences there must be a minimal conclusion. (b) M1: Correct process of differences evidenced in their work, e.g. attempts any three of the 6 expression shown. There should be enough evidence of at least one pair of cancelling terms. Ignore any attempts at any of $r = 0$ or 1 if they start earlier than $r = 2$. A1: Correct algebraic terms or correct constant term(s) extracted. Accept unsimplified expressions such as “$-\sqrt{1} - \sqrt{0}$” A1: Fully correct simplified expression. Must have simplified the $\sqrt{1}$ to 1 (c) M1: Attempts $f(50) - f(3)$ using their answer to (b). Must be indication of subtraction. The ‘-1’s may be omitted, and allow if a slip is made. They may subtract the sum of their results from $r = 2$ and $r = 3$ from (b) or by using (a) again to obtain $f(3)$ but their answer to (b) must be used for $f(50)$. Allow from attempts starting at $r = 1$ in their summations. $f(50) - f(4)$ is M0. A1: Correct expression or $A = 7, B = 4, C = -1$ following a correct answer to (b). The decimal answer 10.92480344... without evidence of the M mark is 0/2		

Question	Scheme	Marks
7(a)(i)+(ii)	$\cos 5\theta + i \sin 5\theta = (\cos \theta + i \sin \theta)^5$ o.e.	M1
	$= c^5 + 5c^4si - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i$ $\Rightarrow \sin 5\theta = \dots, \cos 5\theta = \dots$	M1
	$\sin 5\theta \equiv 5\cos^4\theta \sin \theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta$	A1*
	$\cos 5\theta \equiv \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta$	A1
		(4)
(b)	$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{5\cos^4\theta \sin \theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta} \times \frac{\cos^{-5}\theta}{\cos^{-5}\theta}$	M1
	$\frac{\frac{5\sin\theta}{\cos\theta} - \frac{10\sin^3\theta}{\cos^3\theta} + \frac{\sin^5\theta}{\cos^5\theta}}{1 - \frac{10\sin^2\theta}{\cos^2\theta} + \frac{5\sin^4\theta}{\cos^4\theta}} = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$	A1*
		(2)
(c)	$x = \tan \theta, 2x^5 - 15x^4 - 20x^3 + 30x^2 + 10x - 3 = 0$ $\Rightarrow \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta} = \frac{3}{2} \Rightarrow \tan 5\theta = \frac{3}{2}$	M1
	$\theta = \frac{1}{5} \tan^{-1} \frac{3}{2} (= 0.19655 \text{ radians}, 11.2619\dots^\circ)$ Others that you may see which would also score: In radians {0.19655 ..., 0.82487..., 1.45319..., 2.0815..., 2.7098...} In degrees {11.2619 ..., 47.2619 ..., 83.2619 ..., 119.2619 ..., 155.2619 ...}	A1
	$\Rightarrow x = \tan 0.19655\dots$	M1
	Two of: $x = 0.1991\dots, 1.0822\dots, 8.464\dots, -1.7847\dots, -0.4607\dots$	A1
	All of: $x = -1.785, -0.461, 0.199, 1.082, 8.464$	A1
		(5)

(11 marks)

Notes:
(a)(i)+(ii)

M1: Demonstrates use of De Moivre such as $\cos 5\theta + i \sin 5\theta = (\cos \theta + i \sin \theta)^5$, $\sin 5\theta = \text{Im}(\cos \theta + i \sin \theta)^5$ or $\cos 5\theta =$ their real terms from the expansion of $(\cos \theta + i \sin \theta)^5$

M1: Attempts to expand, need not be simplified. Accept sign errors but should have the correct evaluated coefficients (allow terms with correct ${}^nC_r$ values if the correct coefficients are subsequently seen) the correct powers on the sin and cos terms and equate either the real or imaginary parts.

A1*: Correct expression for $\sin 5\theta$ derived from **fully correct working**. Allow notational slips with nCr etc as long as the intention is clear. To score this mark we need to see either a full expansion if working with $(\cos \theta + i \sin \theta)^5$ or the imaginary terms if working with $\text{Im}(\cos \theta + i \sin \theta)^5$



Note: Consistent omission of i's is 2nd M0, but condone the occasional slip for the M marks

A1: Correct expression for $\cos 5\theta$. Allow recovery from slips, e.g. missing arguments for this mark, and allow if seen in part (b).

(b)

M1: Applies $\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$ and shows or clearly states the intention to divide numerator and

denominator through by $\cos^5\theta$ or equivalent work (e.g. factorising $\sin \theta$ from the numerator and $\cos \theta$ from the denominator and then dividing all terms by $\cos^4\theta$ to obtain $\tan \theta$ terms)

A1*: Correct given expression reached from **fully correct work with no errors**

Note: Do not penalise slips in notation or missing arguments already penalised in (a)

(c)

Note: Must use (b) so calculator methods to solve by e.g. tracing the points or eq. solver scores 0

M1: Substitutes $x = \tan \theta$ into the equation and rearranges to identify the $\tan 5\theta$ within the equation. May be slips when rearranging, but look for the attempt to reach $\tan 5\theta$

A1: Obtains a correct value for θ either in degrees or radians, or as an expression. May be implied.

M1: Uses at least one value of θ and undoes the substitution to find at least one value for x (may have to check for their θ value(s))

A1: At least two correct values of x found correct to at least 2 s.f. (So accept 0.20, 1.1, 8.5 etc)

A1: All five solutions correct to 3 decimal places, and no extra solutions.

Question Number	Scheme	Notes	Marks
7(a)	$x = e^u \Rightarrow \frac{dx}{du} = e^u = x$ or $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$ $\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{x} \frac{dy}{du}$ or $\frac{dy}{dx} = \frac{dy}{du} \times e^{-u}$ oe	Correct form for equation in $\frac{dy}{dx}$ and $\frac{dy}{du}$	M1
	$\frac{d^2y}{dx^2} = \frac{1}{x} \frac{d^2y}{du^2} \frac{du}{dx} - \frac{1}{x^2} \frac{dy}{du}$ or $\frac{d^2y}{dx^2} = e^{-u} \frac{d^2y}{du^2} \frac{du}{dx} - e^{-2u} \frac{dy}{du}$ oe	Correct form for an equation linking $\frac{d^2y}{dx^2}$ and $\frac{d^2y}{du^2}$	M1
	$2x^2 \left(\frac{1}{x^2} \frac{d^2y}{du^2} - \frac{1}{x^2} \frac{dy}{du} \right) + 3x \left(\frac{1}{x} \frac{dy}{du} \right) - y = 27x^2$	Substitutes to eliminate $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ and $\frac{du}{dx}$	M1
	$2 \frac{d^2y}{du^2} + \frac{dy}{du} - y = 27e^{2u} *$	Fully correct proof	A1*
(4)			
(b)	$2m^2 + m - 1 = (2m - 1)(m + 1) = 0 \Rightarrow m = \frac{1}{2}, -1$	Forms auxiliary equation and solves (usual rules)	M1
	$(y =) Ae^{\frac{1}{2}u} + Be^{-u}$	Correct CF for their two real values of m	M1
	$y = \lambda e^{2u}, y' = 2\lambda e^{2u}, y'' = 4\lambda e^{2u}$ $\Rightarrow 8\lambda + 2\lambda - \lambda = 9\lambda = 27 \Rightarrow \lambda = 3$	With correct form of PI, differentiates twice and substitutes to find a value for λ	M1
	$\left(y = Ae^{\frac{1}{2}u} + Be^{-u} + 3e^{2u} \right)$ $y = Ax^{\frac{1}{2}} + Bx^{-1} + 3x^2$	Correct GS in terms of x with no exponentials. Must include the $y =$	A1
(4)			
(c)	$\frac{dy}{dx} = \frac{1}{2} Ax^{-\frac{1}{2}} - Bx^{-2} + 6x$	Correct form for $\frac{dy}{dx}$ from GS of form $y = Ax^p + Bx^q + rx^2$	M1
	$x = \frac{1}{4}, y = \frac{11}{16} \Rightarrow \frac{11}{16} = \frac{1}{2} A + 4B + \frac{3}{16}$ and $x = \frac{1}{4}, \frac{dy}{dx} = 1 \Rightarrow 1 = A - 16B + \frac{3}{2}$	Uses $x = \frac{1}{4}$ with $y = \frac{11}{16}$ and $\frac{dy}{dx} = 1$ to obtain two equations in the constants (from CF)	M1
	$\frac{1}{2} A + 4B = \frac{1}{2}, A - 16B = -\frac{1}{2}$ $\Rightarrow A = \frac{1}{2}, B = \frac{1}{16}$	Having found two equations in two CF constants from using the conditions, solves for A and B . Requires previous M mark	dM1
	$y = \frac{1}{2} \left(\frac{1}{8} \right)^{\frac{1}{2}} + \frac{1}{16} \left(\frac{1}{8} \right)^{-1} + 3 \left(\frac{1}{8} \right)^2 = \dots$	Substitutes $x = \frac{1}{8}$ into their PS and obtains a value for y . Requires previous 2 M marks	ddM1
	$= \frac{\sqrt{2}}{8} + \frac{1}{2} + \frac{3}{64} = \frac{\sqrt{2}}{8} + \frac{35}{64} = \frac{1}{64} (8\sqrt{2} + 35)$	$\frac{1}{64} (8\sqrt{2} + 35)$ only	A1
(5)			



Total 13

Additional Notes:

(a)

M1: Applies the chain rule with an attempt at $\frac{du}{dx}$ or $\frac{dx}{du}$ for form an equation linking $\frac{dy}{dx}$ and $\frac{dy}{du}$

M1: Attempts the second derivative with product and chain rules to form an equation linking

 $\frac{d^2y}{dx^2}$ and $\frac{d^2y}{du^2}$ of correct form.M1: Substitutes to eliminate $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ and $\frac{du}{dx}$ from the equation to get an equation in $\frac{d^2y}{du^2}$ and $\frac{dy}{du}$ as the only derivatives.

A1: Completely correct proof, with intermediate work and no errors.

(b)

M1: Forms and solves the auxiliary equation. For solving the auxiliary equation no method need be shown. The M can be implied by the correct values.

M1: Correct complementary function for their solutions as long as they are distinct real roots. The $y =$ is not needed and may be mislabelled as $x =$ or have x in place of u for this mark.

M1: Full process to find the particular integral, with correct form used, differentiated twice and substituted to find the value of the constant.

A1: Correct general solution in terms of x (u must be eliminated) and including the $y =$. Allow if the GS in terms of x is stated at the beginning of part (c) instead of in (b).

(c)

M1: Differentiates the general solution to the correct form.

M1: Uses the given conditions to form simultaneous equations in the constants of their equations.

dM1: Depends on previous M. Solves their two simultaneous equations (must have formed two equations from the conditions).

ddM1: Depends on previous two M's. Uses their particular solution to find the value for y when $x = \frac{1}{8}$ Condone a slip when substituting as long as it is clear they are attempting the correct x value.

A1: Correct answer only.

NB Part (c) can be done in terms of u if they do not convert back to an equation in x M1: Differentiates the GS in terms of u : $\frac{dy}{du} = \frac{A}{2}e^{\frac{1}{2}u} - Be^{-u} + 6e^{2u}$ M1: Forms simultaneous equations using $u = \ln \frac{1}{4}$ (oe) and the value for y and $\frac{dy}{du} = x \frac{dy}{dx} = \frac{1}{4}$. So

$$\frac{11}{16} = Ae^{\frac{1}{2}\ln\frac{1}{4}} + Be^{-\ln\frac{1}{4}} + 3e^{2\ln\frac{1}{4}} = \frac{A}{2} + 4B + \frac{3}{16} \quad \text{and} \quad \frac{1}{4} = \frac{A}{2}e^{\frac{1}{2}\ln\frac{1}{4}} - Be^{-\ln\frac{1}{4}} + 6e^{2\ln\frac{1}{4}} = \frac{A}{4} - 4B + \frac{3}{8}$$

dM1: Solves the equations. $\Rightarrow A = \dots, B = \dots$ ddM1: Substitutes $u = \ln \frac{1}{8}$ into their equation in terms u to find y .

A1: cao.

Note that they must be using attempts at u , not just x values, in such an approach.Note: where candidates confuse u and x 's accept they must be substituting in the correct things, so using $\frac{dy}{du} = 1$ is 2nd M0, but if they state $\frac{dy}{dx} =$ function of x following an attempt $\frac{dy}{du}$ first then replacing u , youcan allow the second M for using $\frac{dy}{dx} = 1$ in this (the first M will be lost).

Question Number	Scheme	Notes	Marks
8(a)	$z^n = \cos n\theta + i \sin n\theta$ or $z^{-n} = \cos(-n\theta) + i \sin(-n\theta)$ or $z^{-n} = \cos n\theta - i \sin n\theta$	One correct use of de Moivre on either z^n or z^{-n} seen or implied.	M1
	$z^n + \frac{1}{z^n} = \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta$ $= 2 \cos n\theta^*$	Fully correct proof with LHS and RHS both seen and expressions explicitly added.	A1*
	Alternative: $z^n + \frac{1}{z^n} = e^{in\theta} + e^{-in\theta} \quad (\text{M1}) \quad = 2\left(\frac{1}{2}(e^{in\theta} + e^{-in\theta})\right) = 2 \cos n\theta \quad (\text{A1*})$		
(2)			
(b)	$\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z}\right)^2 + 4z\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$	Expands $\left(z + \frac{1}{z}\right)^4$ with correct binomial coefficients.	M1
	$\left(z + \frac{1}{z}\right)^4 = z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{1}{z^2}\right) + 6$ $\Rightarrow (2 \cos \theta)^4 = 2 \cos 4\theta + 4(2 \cos 2\theta) + 6$	Correct use of $z^n + \frac{1}{z^n} = 2 \cos n\theta$ shown for both terms	M1
	$16 \cos^4 \theta = 2 \cos 4\theta + 8 \cos 2\theta + 6$ $\Rightarrow \cos^4 \theta = \frac{1}{8}(\cos 4\theta + 4 \cos 2\theta + 3)$	M1: Achieves expression for $\cos^4 \theta$ of the correct form. A1: Correct expression.	M1 A1
(4)			
(c)	$\pi \int y^2 dx = \pi \int \cos^4 x (1 + \sin x) dx$	Uses volume of revolution formula	M1
	$= \{\pi\} \int (\cos^4 x + \sin x \cos^4 x) dx$ $= \{\pi\} \int \left(\frac{1}{8}(\cos 4x + 4 \cos 2x + 3) + \sin x \cos^4 x\right) dx$ $= \{\pi\} \left[\frac{1}{8}\left(\frac{1}{4} \sin 4x + 2 \sin 2x + 3x\right) - \frac{1}{5} \cos^5 x\right]$	M1: Uses their answer to (b) and integrates it to the correct form M1: ... $\sin x \cos^4 x \rightarrow \dots \cos^5 x$ A1: Fully correct integration • See notes	M1 M1 A1
	$= \{\pi\} \left[\left(\frac{1}{8}\left(0 + 2 + \frac{3\pi}{4}\right) - \frac{1}{5}\left(\frac{\sqrt{2}}{2}\right)^5\right) - \left(\frac{1}{8}\left(0 + 0 - \frac{3\pi}{2}\right) - 0\right)\right]$	Applies limits $\frac{\pi}{4}$ and $-\frac{\pi}{2}$ appropriately to an attempt at the integral. Depends on one of the two preceding Ms being scored.	dM1
	$= \pi \left(\frac{1}{4} + \frac{3\pi}{32} - \frac{\sqrt{2}}{40} + \frac{3\pi}{16}\right) = \pi \left(\frac{1}{4} + \frac{9\pi}{32} - \frac{\sqrt{2}}{40}\right)$ $= \frac{\pi}{160}(40 + 45\pi - 4\sqrt{2})$	Correct answer in correct form	A1
(6)			
Total 12			
PAPER TOTAL: 75			

Additional Notes:

(a)

M1: One correct use of de Moivre on either z^n or z^{-n} seen or implied. May be done separately or within working.

A1: Correctly completes the proof with no errors seen. The LHS and RHS of the identity must both be seen and expressions explicitly added and the $-i \sin n\theta$ seen. Condone missing brackets in e.g. $\cos - n\theta$ terms if intent is clear.

Minimum for sufficient working is $z^n + \frac{1}{z^n} = \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta = 2 \cos n\theta$

Alternatively accept if exponential forms are used.

M1: Applies $\cos \theta + i \sin \theta = e^{i\theta}$ (may be implied) and substitutes into the given equation, with correct index work on both terms.

A1: Correctly rearranges to make the $\cos(n\theta)$ identity clear and simplifies to the given result. The minimum requirement is all four statements seen in the MS.

(b)

M1: Attempts to expand $(z + z^{-1})^4$. Correct binomial coefficients must be used, terms need not be simplified. Condone at most one slip in powers.

M1: Gathers the relevant terms (must see the grouping) and clearly applies $z^n + \frac{1}{z^n} = 2 \cos n\theta$ to both terms to obtain each in $\cos n\theta$ terms. The “2” must be seen or implied by correct working.

M1: Sets their expression equal to $(2 \cos \theta)^4$ and makes $\cos^4 \theta$ the subject with correct form.

A1: Correct answer with no clearly incorrect steps throughout (but condone minor slips which are recovered).

Note: M1M0M1A1 is possible in this part if the groups of terms is never shown but they jump directly to the expression in $\cos n\theta$ terms.

(c) The π is only required for the first and last marks.

M1: For the correct use of the volumes of revolution formula applied to the problem, but condone a slip in squaring if the intention is clear (e.g. accept $\cos^2 x(1 + \sin x)$ as the attempt to use y^2). The π must be included at some stage but may be added later. Limits not required for this mark.

M1: Applies the answer to part (b) and integrates to obtain an integral of $\cos^4 \theta$ the correct form. Slips in coefficients may have been made. The π and limits are not required.

M1: Correct form for $\int \sin x \cos^4 x \, dx = K \cos^5 x$ The π and limits are not required.

Note an alternative here is to use part (b) again and the product to sum rule:

$$\begin{aligned} \int \cos^4 x \sin x \, dx &= \frac{1}{8} \int \cos 4x \sin x + 4 \cos 2x \sin x + 3 \sin x \, dx \\ &= \frac{1}{8} \int \frac{1}{2} (\sin 5x - \sin 3x) + 2(\sin 3x - \sin x) + 3 \sin x \, dx \\ &= \frac{1}{8} \left[\frac{1}{2} \left(-\frac{1}{5} \cos 5x + \frac{1}{3} \cos 3x \right) - \frac{2}{3} \cos 3x + \frac{1}{2} \cos x - 3 \cos x \right] \end{aligned}$$

A1: Fully correct integration of both terms. The π and limits are not required.

dM1: Depends on at least one of the two preceding Ms. Applies correct limits to the integral as long as a valid attempt at the integral has been made.

A1: Correct answer in correct form.

NB Allow SC M1M0M1A0dM1A1 for those who expand from scratch instead of using the result from (b).



Alt (b)	$\begin{aligned}[2 \cos 4\theta] &= (\cos \theta + i \sin \theta)^4 + (\cos \theta - i \sin \theta)^4 \\ &= c^4 + 4c^3is + 6c^2(is)^2 + 4c(is)^3 + (is)^4 \\ &\quad + c^4 - 4c^3is + 6c^2(is)^2 - 4c(is)^3 + (is)^4\end{aligned}$	Uses the identity, applies De Moivre and expands at least one set of brackets with correct binomial coefficients.	M1
	$\Rightarrow 2 \cos 4\theta = 2 \cos^4 \theta - 12 \cos^2 \theta \sin^2 \theta + 2 \sin^4 \theta$	Correct use of $z^4 + z^{-4} = 2 \cos 4\theta$ to get only real terms in sine and cosine.	M1
	$\begin{aligned}\Rightarrow \cos^4 \theta &= \cos 4\theta + 6\left(\frac{1}{2} + \frac{1}{2} \cos 2\theta\right)\left(\frac{1}{2} - \frac{1}{2} \cos 2\theta\right) - \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta\right)^2 \\ &= \cos 4\theta + \frac{3}{2}(1 - \cos^2 2\theta) - \frac{1}{4}(1 - 2 \cos 2\theta + \cos^2 2\theta) \\ &= \cos 4\theta - \frac{3}{2}\left(\frac{1}{2} + \frac{1}{2} \cos 4\theta\right) + \frac{1}{2} \cos 2\theta - \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2} \cos 4\theta\right) + \frac{5}{4} \\ &= \frac{1}{8}[8 \cos 4\theta - 6(1 + \cos 4\theta) + 4 \cos 2\theta - (1 + \cos 4\theta) + 10] \\ &\Rightarrow \cos^4 \theta = \frac{1}{8}(\cos 4\theta + 4 \cos 2\theta + 3)\end{aligned}$ M1: Achieves expression for $\cos^4 \theta$ of the correct form A1: Correct expression		M1 A1

Note the question says “hence” so answers without reference to (a) will score at most one mark (e.g. purely by De Moivre). But some attempts may use the result of (a) as shown above. In which case mark as indicated.

M1: For a correct binomial expansion applied to the problem – so e.g. sets up

$(\cos \theta + i \sin \theta)^4 + (\cos \theta - i \sin \theta)^4$ and expands at least one bracket with correct coefficients. This mark may be scored for expansion of $(\cos \theta + i \sin \theta)^4$ from an approach by De Moivre using $\operatorname{Re}(\cos \theta + i \sin \theta)^4$. This may come from other approaches that use a binomial expansion.

M1: Evidence of the use of both sides of part (a), cancelling imaginary terms to get a real equation in cosine and sines only. Note that they must be using part (a) for this mark to be scored.

M1: Having used part (a), uses correct trigonometric identities to eliminate the $\sin \theta$ terms and reduce the powers to single power $\cos 2\theta$ terms – ie achieves correct form for $\cos^4 \theta$. There may be alternate identities used (e.g. Pythagorean before using double angle identities)

A1: Correct answer.

Question Number	Scheme	Marks
8 (a)	$(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta$	B1
	$= \cos^5 \theta + 5 \cos^4 \theta (i \sin \theta) + \frac{5 \times 4}{2!} \cos^3 \theta (i \sin \theta)^2$ $+ \frac{5 \times 4 \times 3}{3!} \cos^2 \theta (i \sin \theta)^3 + \frac{5 \times 4 \times 3 \times 2}{4!} \cos \theta (i \sin \theta)^4 + (i \sin \theta)^5$	M1
	$= \cos^5 \theta + 5i \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta - 10i \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta$	A1
	$\sin 5\theta = 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta$ $= 5(1 - \sin^2 \theta)^2 \sin \theta - 10(1 - \sin^2 \theta) \sin^3 \theta + \sin^5 \theta \frac{dy}{dx}$ $= 5(1 - 2 \sin^2 \theta + \sin^4 \theta) \sin \theta - 10(1 - \sin^2 \theta) \sin^3 \theta + \sin^5 \theta$	M1
	$\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$ *	A1* (5)
	Alternative: Using " $z = \frac{1}{z}$ " $z^5 - \frac{1}{z^5} = 2i \sin 5\theta$ oe	B1
	Binomial expansion of $\left(z - \frac{1}{z}\right)^5$	M1
	$32 \sin^5 \theta = 2 \sin 5\theta - 10 \sin 3\theta + 20 \sin \theta$	A1
	Uses double angle formulae etc to obtain $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$ and then use it in their expansion	M1
	$\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$ *	A1* (5)
(b)	Let $x = \sin \theta$ $16x^5 - 20x^3 + 5x = -\frac{1}{5} \Rightarrow \sin 5\theta = \dots$	M1A1
	$\Rightarrow \theta = \frac{1}{5} \sin^{-1} \left(\pm \frac{1}{5} \right) = 38.306$ (or $-2.307, 69.692, 110.306, 141.693, 182.306$)	dM1
	(or in radians $-0.0402 \dots 0.6685 \dots, 1.216 \dots, 1.925 \dots, 2.473 \dots$)	
	Two of (awrt) $x = \sin \theta = -0.963, -0.555, -0.040, 0.620, 0.938$	A1
	All of (awrt) $x = \sin \theta = -0.963, -0.555, -0.040, 0.620, 0.938$	A1 (5)
(c)	$\int_0^{\frac{\pi}{4}} (4 \sin^5 \theta - 5 \sin^3 \theta - 6 \sin \theta) d\theta = \left(\int_0^{\frac{\pi}{4}} \frac{1}{4} (\sin 5\theta - 5 \sin \theta) - 6 \sin \theta d\theta \right)$	M1
	$= \left[\frac{1}{4} \left(-\frac{1}{5} \cos 5\theta + 5 \cos \theta \right) + 6 \cos \theta \right]_0^{\frac{\pi}{4}} = \left[-\frac{1}{20} \cos 5\theta + \frac{29}{4} \cos \theta \right]_0^{\frac{\pi}{4}}$	A1
	$\frac{1}{4} \left[-\frac{1}{5} \cos \frac{5\pi}{4} + 5 \cos \frac{\pi}{4} - \left(-\frac{1}{5} + 5 \right) \right] + 6 \cos \frac{\pi}{4} - 6$	
	$= \frac{1}{4} \left[\frac{1}{5} \times \frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}} - 4 \frac{4}{5} \right] + \frac{6}{\sqrt{2}} - 6$	dM1
	$= \frac{73\sqrt{2}}{20} - \frac{36}{5}$ oe	A1 (4)
		[14]
	Notes	

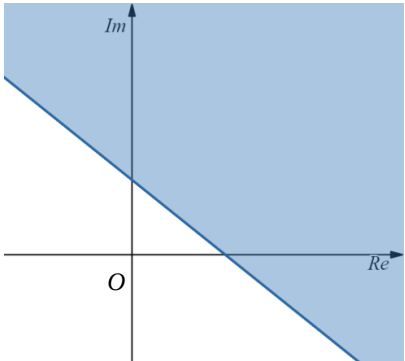


(a)	
B1	Applies de Moivre correctly. Need not see full statement, but must be correctly applied.
M1	Use binomial theorem to expand $(\cos \theta + i \sin \theta)^5$ May only show imaginary parts - ignore errors in real parts. Binomial coefficients must be evaluated.
A1	Simplify coefficients to obtain a simplified result with all imaginary terms correct
M1	Equate imaginary parts and obtain an expression for $\sin 5\theta$ in terms of powers of $\sin \theta$ No $\cos \theta$ now
A1*	Correct given result obtained from fully correct working with at least one intermediate line with the $(1 - \sin^2 \theta)^2$ expanded. Must see both sides of answer (may be split across lines). A0 if equating of imaginary terms is not clearly implied.
(b)	
	Note Answers only with no working score no marks as the “hence” has not been used. But if the first M1A1 gained then dM1 may be implied by a correct answer.
M1	Use substitution $x = \sin \theta$ and attempts to use the result from (a) to obtain a value for $\sin 5\theta$
A1	Correct value for $\sin 5\theta$
dM1	Proceeds to apply arcsin and divide by 5 to obtain at least one value for θ . Note for $\sin 5\theta = \frac{1}{5}$ the values you may see are the negatives of the true answers. FYI: ($5\theta = -11.53\dots, 191.53\dots, 348.46\dots, 551.53\dots, 708.46\dots, 911.53\dots$) (Or in radians $-0.201\dots, 3.3428\dots, 6.0819\dots, 9.6260\dots, 12.365\dots, 15.909\dots$)
A1	Proceeds to take sin and achieve at least 2 different correct values for x or $\sin \theta$
A1	For all 5 values of x or $\sin \theta$ awrt 3 d.p. (allow 0.62 and -0.04)
(c)	
M1	Use previous work to change the integrand into a function that can be integrated
A1	Correct result after integrating. Any limits shown can be ignored
dM1	Substitute given limits, subtracts and uses exact numerical values for trig functions
A1	Final answer correct (or provided in the given form)

Question Number	Scheme	Marks
8(a)	$z^n = e^{in\theta} = \cos n\theta + i \sin n\theta$ $\frac{1}{z^n} = e^{-in\theta} = \cos(-n\theta) + i \sin(-n\theta) = \cos n\theta - i \sin n\theta$ $z^n + \frac{1}{z^n} = \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta = 2 \cos n\theta^*$	M1A1cso (2)
(b)	$\left(z + \frac{1}{z}\right)^6 = z^6 + 6z^5 \times \frac{1}{z} + \frac{6 \times 5}{2!} z^4 \times \frac{1}{z^2} + \frac{6 \times 5 \times 4}{3!} z^3 \times \frac{1}{z^3}$ $+ \frac{6 \times 5 \times 4 \times 3}{4!} z^2 \times \frac{1}{z^4} + \frac{6 \times 5 \times 4 \times 3 \times 2}{5!} z \times \frac{1}{z^5} + \frac{1}{z^6}$ $(2 \cos \theta)^6 = z^6 + 6z^4 + 15z^2 + 20 + 15 \times \frac{1}{z^2} + 6 \times \frac{1}{z^4} + \frac{1}{z^6}$ $64 \cos^6 \theta = z^6 + \frac{1}{z^6} + 6 \left(z^4 + \frac{1}{z^4} \right) + 15 \left(z^2 + \frac{1}{z^2} \right) + 20$ $64 \cos^6 \theta = 2 \cos 6\theta + 6 \times 2 \cos 4\theta + 15 \times 2 \cos 2\theta + 20$ $\cos^6 \theta = \frac{1}{32} (\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10)^*$	M1A1
(c)	$\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10 = 10$ $32 \cos^6 \theta = 10$ $\cos \theta = \pm \sqrt[6]{\frac{10}{32}}$ $\theta = 0.6027..., 2.5388... \quad \theta = 0.603, 2.54$	M1A1
(d)	$\int_0^{\frac{\pi}{3}} (32 \cos^6 \theta - 4 \cos^2 \theta) d\theta$ $= \int_0^{\frac{\pi}{3}} (\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10 - 4 \cos^2 \theta) d\theta$ $= \int_0^{\frac{\pi}{3}} (\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10 - 2 - 2 \cos 2\theta) d\theta$ $= \left[\frac{1}{6} \sin 6\theta + \frac{3}{2} \sin 4\theta + \frac{13}{2} \sin 2\theta + 8\theta \right]_0^{\frac{\pi}{3}}$ $= (0) + \frac{3}{2} \left(-\frac{\sqrt{3}}{2} \right) + \frac{13}{2} \times \frac{\sqrt{3}}{2} + \frac{8\pi}{3} \quad (-0)$ $= \frac{5\sqrt{3}}{2} + \frac{8\pi}{3} \quad \text{oe}$	M1 M1A1 dM1
		A1 (5)
		[16]

Question Number	Scheme	Marks
(a)		
M1	Attempt to obtain $z^n + \frac{1}{z^n}$	
A1cso	Reach the given result with clear working and no errors Must see $\cos(-n\theta) + i\sin(-n\theta)$ changed to $\cos n\theta - i\sin n\theta$ (ie both included)	
(b)		
	<i>The first 3 marks apply to the binomial expansion only</i>	
M1	Apply the binomial expansion to $\left(z + \frac{1}{z}\right)^6$ Coefficients must be numerical (ie " C_r " is not acceptable). The expansion must have 7 terms with at least 4 correct	
A1	Correct expansion, terms need not be simplified	
M1	Simplify the coefficients and pair the appropriate terms on RHS (At least 2 pairs must be correct)	
M1	Use the result from (a) throughout. Must include 2^6 or 64 now	
A1*	Obtain the given result with no errors in the working	
(c)		
M1	Use the result from (b) to simplify the given equation	
A1	Reach $32\cos^6\theta = 10$ oe	
M1	Solve to obtain at least one correct value for θ , in radians and in the given range, 3 sf or better	
A1	2 correct values, and no extras, in radians and in the given range. Must be 3 sf here Ignore extras outside the range	
(d)		
M1	Use the result in (b) to change $\cos^6\theta$ to a sum of multiple angles ready for integration and use $\cos^2\theta = \pm\frac{1}{2}(\cos 2\theta \pm 1)$ on $\cos^2\theta$ Limits not needed, ignore any shown	
M1	Integrate their expression to obtain an expression containing terms in $\sin 6\theta, \sin 4\theta, \sin 2\theta$ and θ Limits not needed	
A1	Correct integration Limits not needed	
dM1	Substitute limit $\pi/3$. Depends second M mark	
A1	Correct, exact, answer (any equivalent to that shown). Award M1A1 for a correct final answer following fully correct working.	
	There are other ways to integrate the function in (d), eg parts on one or both of the powers of $\cos\theta$, using $\cos^6\theta = (\cos^2\theta)^3 = \frac{1}{8}(1 + \cos 2\theta)^3 = \dots$	
	If in doubt about the marking of alternative methods which are not completely correct, send to review	

General Principles for Further Pure Mathematics Marking*(But note that specific mark schemes may sometimes override these general principles)***Method mark for solving 3 term quadratic:****1. Factorisation** $(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$ $(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$ **2. Formula**Attempt to use the correct formula (with values for a , b and c).**3. Completing the square**Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$ **Method marks for differentiation and integration:****1. Differentiation**Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)**2. Integration**Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)

Question Number	Scheme	Marks
1(a)	$ z - 3 - 4i = z + 1 + i \Rightarrow x + iy - 3 - 4i = x + iy + 1 + i $ $(x - 3)^2 + (y - 4)^2 = (x + 1)^2 + (y + 1)^2$	M1
	$x^2 - 6x + 9 + y^2 - 8y + 16 = x^2 + 2x + 1 + y^2 + 2y + 1 \Rightarrow ax + by + c = 0$	dM1
	$8x + 10y - 23 = 0$	A1
		(3)
(a) ALT	Mid-point $\left(\frac{3-1}{2}, \frac{4-1}{2}\right)$ and $m = \frac{4+1}{3+1}$	M1
	$y - \frac{3}{2} = -\frac{4}{5}(x - 1) \Rightarrow ax + by + c = 0$	dM1
	$8x + 10y - 23 = 0$	A1
(b)		B1
		(1)
		Total 4

Notes:

(a)

M1: Introduces $z = x + iy$ (or with other variables) and attempts a correct modulus – the i's should be dealt with correctly but allow for e.g. minor slips in notation with them before applying Pythagoras. Allow sign slips in the i coordinates, e.g.

$(x - 3)^2 + (y \pm 4)^2 = (x + 1)^2 + (y \pm 1)^2$ Must be + between the brackets.

dM1: Expands and attempts the required form.

A1: This equation or any integer multiple.

Alternative: via perpendicular bisector

M1: Uses the points $(3, \pm 4)$ and $(-1, \pm 1)$ to find the midpoint and gradient between them.

dM1: Correct attempt at the equation of the perpendicular bisector and rearranges to the required form. Must be using negative reciprocal of the gradient between points (but watch for those who go directly to this).

A1: This equation or any integer multiple.

(b) May be scored independently of (a) – using the perpendicular bisector.

B1: Shades the area above an appropriate line, must be in quadrants 1, 2 and 4. Accept with solid, dashed or dotted line. Allow for any line with negative gradient and positive y intercept. Do not be concerned about labelling. Part (a) need not have been correct.



Use of a formula

Where a method involves using a formula that has been learnt, the advice given in recent examiners' reports is that the formula should be quoted first.

Normal marking procedure is as follows:

Method mark for quoting a correct formula and attempting to use it, even if there are small errors in the substitution of values.

Where the formula is not quoted, the method mark can be gained by implication from correct working with values but may be lost if there is any mistake in the working.

Exact answers

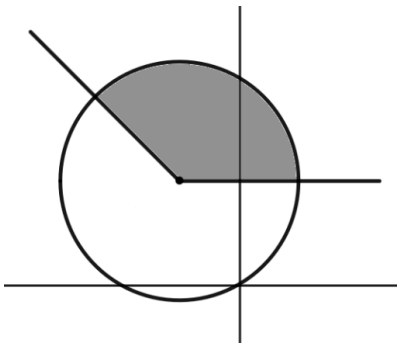
Examiners' reports have emphasised that where, for example, an exact answer is asked for, or working with surds is clearly required, marks will normally be lost if the candidate resorts to using rounded decimals.

WFM02 Further Pure Mathematics F2 Mark Scheme

Question Number	Scheme	Notes	Marks
1	$z^5 - 32i = 0 \Rightarrow r^5 = 32 \Rightarrow r = 2$	Correct value for r . May be shown explicitly or used correctly.	B1
	$5\theta = \frac{\pi}{2} + 2n\pi \Rightarrow \theta = \frac{\pi}{10} + \frac{2n\pi}{5}$	Applies a correct strategy for establishing at least 2 values of θ . This can be awarded if if the initial angle $\left(\frac{\pi}{2} \text{ or } \frac{\pi}{10}\right)$ is incorrect but otherwise their strategy is correct.	M1
	$z = 2e^{i\frac{\pi}{10}}, 2e^{i\frac{\pi}{2}}, 2e^{i\frac{9\pi}{10}}, 2e^{i\frac{13\pi}{10}}, 2e^{i\frac{17\pi}{10}}$ or $z = 2e^{i\left(\frac{\pi}{10} + \frac{2n\pi}{5}\right)}, n = 0, 1, 2, 3, 4$	At least 2 correct, follow through their r	A1ft
		All correct. Must have $r = 2$	A1
			(4)
			Total 4



Question Number	Scheme	Notes	Marks
2	$\frac{x}{2-x} \geq \frac{x+3}{x}$		
Way 1	$\frac{x}{2-x} \geq \frac{x+3}{x} \Rightarrow \frac{x}{2-x} - \frac{x+3}{x} \geq 0$	Collects to one side	M1
	$\frac{x}{2-x} - \frac{x+3}{x} \geq 0 \Rightarrow \frac{x^2 - (2-x)(x+3)}{x(2-x)} \geq 0$ M1: Attempt common denominator A1: Correct fraction		M1 A1
	$x = 0, 2$	These critical values	B1
	$x^2 - (2-x)(x+3) = 0$ $\Rightarrow 2x^2 + x - 6 = 0 \Rightarrow x = \dots$	Solves the 3TQ in the numerator	M1
	$x = \frac{3}{2}, -2$	These critical values	A1
	$x \geq -2, 0 < x < \frac{3}{2}, x > 2$ A1: Any 2 of these with strict inequalities allowed A1: All correct with inequalities as shown. Ignore what they have between their inequalities e.g. allow “or”, “and”, “,” etc. but not $\cap$		A1A1
			(8)
			Total 8
	Alternative 1: $\times x^2(2-x)^2$		
	$x^3(2-x) \geq x(x+3)(2-x)^2$	Multiplies by a positive expression	M1
	$x^3(2-x) - x(x+3)(2-x)^2 \geq 0$	Collects to one side	M1
		Correct inequality	A1
	$x = 0, 2$	These critical values	B1
	$x(2-x)[x^2 - (x+3)(2-x)] = 0$ $x^2 - (x+3)(2-x) = 0$ $\Rightarrow 2x^2 + x - 6 = 0 \Rightarrow x = \dots$	Attempts to factorise by taking out a factor of $x(2-x)$ and solves resulting 3TQ. May have quartic and apply the factor theorem.	M1
	$x = \frac{3}{2}, -2$	These critical values	A1
	$x \geq -2, 0 < x < \frac{3}{2}, x > 2$ A1: Any 2 of these with strict inequalities allowed A1: All correct with inequalities as shown. Ignore what they have between their inequalities e.g. allow “or”, “and”, “,” etc. but not $\cap$		A1A1

Question Number	Scheme	Marks	
2(a)	$\left(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12}\right)^4 = \cos \frac{20\pi}{12} + i \sin \frac{20\pi}{12} \quad \text{or/and}$ $\left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}\right)^3 = \cos(-\pi) + i \sin(-\pi)$	M1 / A1	
	$(z_1 =) \frac{\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}}{\cos(-\pi) + i \sin(-\pi)} = \cos\left(\frac{5\pi}{3} - (-\pi)\right) + i \sin\left(\frac{5\pi}{3} - (-\pi)\right)$ $\text{Alt : } (z_1 =) \frac{\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}}{-1} = -\cos \frac{5\pi}{3} - i \sin \frac{5\pi}{3}$	M1	
	$= \cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} *$ $\text{Alt: if denominator } -1 \text{ used via e.g. } \cos\left(\frac{5\pi}{3} - \pi\right) + i \sin\left(-\frac{5\pi}{3}\right)$	A1*	
		(4)	
(b)	$ z - z_1 \leq 1 \quad 0 \leq \arg(z - z_1) \leq \frac{3\pi}{4}$		
		A circle in any position (may just see the minor arc)	M1
		A pair of half -lines in correct directions from their centre, one with negative gradient and one parallel to (but not) the x-axis. <i>If full lines are used the M marks can be implied by their shading</i>	M1
		Area shaded inside their circle between the two half lines from from the parallel one anticlockwise to the negative gradient line.	M1
	Fully correct shaded sector. See notes.		A1
		(4)	
(c)	$\arctan\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \dots \quad \frac{\pi}{3} \quad (\text{or } 60^\circ)$	M1 A1	
		(2)	
		Total 10	

Notes

(a)

M1: One correct use of de Moivre in polar (or exponential) form. Allow use of $e^{i\theta}$ for $\cos \theta + i \sin \theta$ throughout until the final A.

A1: Both correct (unsimplified) in polar form. Accept for both marks use of

$$\left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}\right)^3 \rightarrow \cos \pi - i \sin \pi \quad \text{but denominator directly to } \cos \pi + i \sin \pi \quad \text{with no evidence of}$$

dealing with the negative between terms is A0.

M1: A correct method shown for the division of the two complex numbers. No need to simplify for this mark, a difference of argument in the trig terms is fine. Look for the subtraction of the arguments. Sight of the $\frac{8\pi}{3}$ can imply the mark if no incorrect work is seen. Note it is M0 if the arguments are added (so



$\cos\left(\frac{5\pi}{3} - \pi\right) + i \sin\left(\frac{5\pi}{3} - \pi\right)$ is M0 unless the denominator has clearly been written as $\cos \pi + i \sin \pi$ first.)

May write the denominator as -1 first, which is correct, score for $-\cos \frac{5\pi}{3} - i \sin \frac{5\pi}{3}$.

Accept methods that convert both numbers into exact Cartesian form, apply a correct process to realise the denominator and convert back to polar form.

Do not allow going straight to $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$ for this mark as it is a given answer. Justification is required and an incorrect method is M0.

A1cso: Obtains $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$ with suitable intermediate step shown and no errors in the work. Must have

scored the preceding 3 marks. This will usually be via $\cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3}$ unless equivalent suitable working has been shown to justify the correct modulus (e.g. proceeding via -1 in the denominator).

(b)

Do not be concerned about lines being dashed or dotted in this part. Accept either.

M1: A circle in any position. You may see just the minor arc of a circle, which is acceptable as long as it is clearly an arc of a circle (e.g. implied by their shading) and not just a angle demarcation.

M1: Draws or indicates a pair of half-lines from their centre (which need not be in quadrant 2) with one with negative gradient proceeding up and left and one parallel to the x -axis proceeding right but not the x -axis itself. If full lines are used this can be implied by the shading of the correct region between lines.

M1: Shades the area between their pair of rays (the second ray may have positive gradient for this mark) and inside their circle, anticlockwise from the horizontal line. The half line need not stem from the centre of the circle for this mark, and accept the x -axis as the horizontal line for this mark.

A1: Fully correct shaded sector. Must

- be in quadrants 1 and 2
- have approximately correct gradients for the half-lines (-1 and 0)
- have circle with centre in quadrant 2 and (if whole circle shown) passing roughly through the origin. If only an arc is shown apply bod as long as the position is reasonable.

Ignore any centre coordinates and axes intersections of any major sector. If extra regions are shaded they must make clear which their region R is in order to access the mark.

(c)

M1: Identifies the correct point for their sector, which must be from a circle with centre in quadrant 1 or 2 (above the real axis) and ray parallel to the real axis, and attempts the relevant angle. Look for selecting the point (c, d) at the “3 o’clock” position having identified a suitable sector and proceeding to find a relevant positive angle (e.g., allow if $\pi - \theta$ is found) (accept $\arctan \pm \frac{c}{d}$ or $\arctan \pm \frac{d}{c}$ as an attempt at the angle).

Their point must be in quadrants 1 or 2. If no shading was shown in (b) allow for attempts at the relevant point of horizontal ray and circle intersection. Could use other trig.

A1 : Either correct value. Mark final answer.

Question Number	Scheme	Marks
3	$\frac{x+2}{x+4} \leq \frac{x}{k(x-1)}$	
(a)	$\left[\times k(x+4)^2(x-1)^2 \Rightarrow \right] \quad k(x+4)(x-1)^2(x+2) \leq x(x+4)^2(x-1)$ $\Rightarrow k(x+4)(x-1)^2(x+2) - x(x+4)^2(x-1) \{ \leq 0 \}$ OR $\frac{x+2}{x+4} - \frac{x}{k(x-1)} \{ \leq 0 \} \Rightarrow \frac{k(x+2)(x-1) - x(x+4)}{k(x+4)(x-1)} \{ \leq 0 \}$	M1
	$(x+4)(x-1)[kx^2 + kx - 2k - x^2 - 4x] \{ \leq 0 \}$	dM1
	$(x+4)(x-1)[(k-1)x^2 + (k-4)x - 2k] \leq 0 \text{ (oe in corect form)}$	A1cso
		(3)
(b)	$k = 3 \Rightarrow 2x^2 - x - 6 = 0$ $[(2x+3)(x-2) = 0] \quad \Rightarrow x = -\frac{3}{2}, 2$	M1
	$-4 < x \leq -\frac{3}{2} \quad 1 < x \leq 2$	dM1 A1 A1
		(4)
		Total 7

Notes

(a)

M1: Attempts to multiply both sides by e.g. $k(x+4)^2(x-1)^2$ or $k^2(x+4)^2(x-1)^2$ and bring terms together
 OR bring terms together and attempt a common denominator. There may be slips but the intention must be clear - allow if e.g. one term is missing.

dM1: Factorises out $(x+4)(x-1)$ and/or multiplies by $(x+4)^2(x-1)^2$ (which could be implied) and expands remaining terms to get unsimplified 3 term quadratic (terms need not be collected but it must be equivalent to $px^2 + qx + r$ where at least one of p , q and r is a function of k and none are zero). **Dependent on previous M mark.**

A1cso: Correct statement or $p = k - 1$, $q = k - 4$, $r = -2k$ with no algebraic errors (but condone e.g. recovery of missing brackets if work is clear). It is A0 if any incorrect statement is seen e.g., use of an incorrect inequality sign. Accept with any positive multiples of the coefficients (including k or $1/k$ etc).

(b)

M1: Uses $k = 3$ in their quadratic from (a) (implied by 2 out of three terms correct) and solves to obtain a value. Apply usual rules (may be by calculator - may need to check). If no working, obtains one consistent

solution which must be real. **Alternatively** restarts from $\frac{x+2}{x+4} \leq \frac{x}{3(x-1)}$ and uses a valid method to

obtain and solve a quadratic (see above). Marks for (a) cannot be scored in (b).

dM1: With critical values -4 , 1 and two different real solutions from their quadratic ($\neq -4$ or 1), chooses the region between the two smaller values and the region between the two larger values and no other region. Could be non-strict inequalities. **Dependent on previous M mark.**

A1: Both regions correct but condone e.g. incorrect strict non-strict inequalities.

A1: Both completely correct regions. Allow equivalent notations. Ignore any word between the regions but withhold this last mark if $\cap$ used with set notation.



Question Number	Scheme	Marks
4	$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 48x^2 - 34$	
(a)	(AE:) $m^2 - 8m + 16 = 0$ $[\Rightarrow (m - 4)^2 = 0]$ $\Rightarrow m = 4$	M1
	(CF: $y =$) $(A + Bx)e^{4x}$	A1
	(PI: $y =$) $\lambda x^2 + \mu x + \nu$	B1
	$y' = 2\lambda x + \mu$ $y'' = 2\lambda$ $2\lambda - 8(2\lambda x + \mu) + 16(\lambda x^2 + \mu x + \nu) = 48x^2 - 34$ $16\lambda x^2 = 48x^2$ $(-16\lambda + 16\mu)x = 0$ $2\lambda - 8\mu + 16\nu = -34$ $\lambda = 3$ $\mu = 3$ $\nu = -1$	M1
	$y = "(A + Bx)e^{4x}" + 3x^2 + 3x - 1$	A1ft
		(5)
(b)	$(0, 4) \Rightarrow 4 = A - 1$ $(A = 5)$	M1
	$\frac{dy}{dx} = 4(A + Bx)e^{4x} + Be^{4x} + 6x + 3$	M1
	$21 = 4A + B + 3 \Rightarrow B = -2$ $A = 5$ $[y] = (5 - 2x)e^{4x} + 3x^2 + 3x - 1$	M1A1
		(4)
(c)	$(x = -2 \Rightarrow y =) 9e^{-8} + 5$	M1 A1
		(2)
		Total 11

Notes**(a)**

M1: Forms the auxiliary equation (condone one slip/copying error) and solves 3TQ. Usual rules. One consistent solution if no working (could be complex). Implied by a correct CF if no incorrect working shown.

A1: Correct complementary function $y = \dots$ not required. May only be seen in final answer.B1: Correct form for particular integral $y = \dots$ not required.

M1: Correct method to obtain value for constants (or constant - but PI must be a quadratic - but could have 1 or 2 terms) - so differentiates twice (powers reduced) and substitutes, equates terms and solves equations.

Allow if there are minor slips in the process if the holistic approach is correct.

A1ft: A correct general solution following through on their CF only - the PI must be correct. Must have " $y =$ " e.g., not " $GS =$ "**(b)**

M1: Uses (0, 4) in their answer to (a) and forms an equation in one or both of their constants.

M1: Differentiates their GS, which must contain a term " Bxe^{4x} ", to obtain an expression of the correct form for their GS - product rule must be used, powers reduced, but may have errors in coefficients.M1: Substitutes $x = 0$, $\frac{dy}{dx} = 21$ into their equation where their derivative is a changed function and finds values for their B (and their A if not found already).A1: Any correct equation Condone e.g., " $PS = \dots$ "**(c)**M1: Substitutes $x = -2$ into their particular solution and obtains an expression of the right form or non-zero values for p, q and r, which need not be integers for the M but should be gathered terms. Implied by a correct answer as long as it fits their answer to (b) if no method shown.

A1: Correct expression or values



Question Number	Scheme	Marks
5(a)	$w = \frac{z+1}{z-3} \Rightarrow wz - 3w = z + 1 \Rightarrow wz - z = 3w + 1 \Rightarrow z = \frac{3w+1}{w-1}$	M1 A1
	$(z =) \frac{(3u+1)+3iv}{(u-1)+iv} \times \frac{(u-1)-iv}{(u-1)-iv} = \dots$	dM1
	$\Rightarrow x = \frac{3u^2 - 2u + 3v^2 - 1}{(u-1)^2 + v^2} \quad y = \frac{-4v}{(u-1)^2 + v^2}$ $(y = 4x \Rightarrow) -4v = 12u^2 - 8u + 12v^2 - 4$	ddM1
	$\Rightarrow 3u^2 + 3v^2 - 2u + v - 1 = 0^*$	A1cso*
		(5)
(b)	$u^2 - \frac{2}{3}u + v^2 + \frac{1}{3}v - \frac{1}{3} = 0 \Rightarrow \left(u - \frac{1}{3}\right)^2 - \frac{1}{9} + \left(v + \frac{1}{6}\right)^2 - \frac{1}{36} - \frac{1}{3} = 0$ $\Rightarrow \left(u - \frac{1}{3}\right)^2 + \left(v + \frac{1}{6}\right)^2 = \frac{17}{36}$ $\Rightarrow \text{centre: } \left(\frac{1}{3}, -\frac{1}{6}\right) \left[\text{allow } x = \frac{1}{3}, y = -\frac{1}{6} \right] \quad \text{radius: } \sqrt{\frac{17}{36}} \text{ or } \frac{\sqrt{17}}{6}$	B1 B1
		(2)
		Total 7

Notes

(a)

M1: Completes an attempt to make z the subject.

A1: Correct expression.

dM1: **Dependent on previous M mark.** Replaces w with $u + iv$ and indicates an appropriate attempt to rationalise the denominator. Accept if $x + iy$ is used instead for this mark.ddM1: **Dependent on both previous M marks.** Equates real and imaginary parts to obtain expressions for x and y in u and/or v only and uses $y = 4x$ to form an equation in u and v only. Condone intermediate incorrect statements that are recovered for this mark, e.g. condone, missing or incorrect (real) denominator or slips with initial inclusion of i in the y if recovered. E.g. any of

$$x = 3u^2 - 2u + 3v^2 - 1, \quad x = \frac{3u^2 - 2u + 3v^2 - 1}{(u-1)^2 - v^2}, \quad y = \frac{-4vi}{(u-1)^2 + v^2} \quad \text{or similar may be recovered for the M's}$$

A1cso*: obtains the equation (must be equal to 0) or states $k = -1$ following correct work. Do not allow recovery from incorrect statements as shown above for the final mark - all may lead to the correct answer but are A0. But if there are no incorrect statements e.g. cancelling of denominators may be implied for the A. Watch outfor an incorrect $z = \frac{3w+1}{1-w}$ which also leads to a correct expression but loses both A marks.**Note:** Alternatively methods are possible. Two are shown on the next page, but are not exhaustive.

(b)

Allow both marks from cases where $k = -1$ is guessed or achieved from fortuitous work.B1: Correct centre **or** radius. Accept unsimplified fractions for this mark.B1: Correct centre **and** radius - fractions must be simplified.

Question Number	Scheme	Marks
2(a)	$x \frac{dy}{dx} - y^3 = 4 \Rightarrow \frac{dy}{dx} + x \frac{d^2y}{dx^2} - 3y^2 \frac{dy}{dx} = 0$ $\text{Alt: } \frac{dy}{dx} = \frac{y^3}{x} + \frac{4}{x} \Rightarrow \frac{d^2y}{dx^2} = -\frac{y^3}{x^2} + \frac{3y^2}{x} \frac{dy}{dx} - \frac{4}{x^2}$	M1A1
	$\Rightarrow \frac{d^2y}{dx^2} + x \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 6y \left(\frac{dy}{dx} \right)^2 - 3y^2 \frac{d^2y}{dx^2} = 0$ $\text{Alt: } \Rightarrow \frac{d^3y}{dx^3} = \frac{2y^3}{x^3} - \frac{3y^2}{x^2} \frac{dy}{dx} - \frac{3y^2}{x^2} \frac{dy}{dx} + \frac{1}{x} \left(6y \left(\frac{dy}{dx} \right)^2 + 3y^2 \frac{d^2y}{dx^2} \right) + \frac{8}{x^3}$	dM1
	$\Rightarrow x \frac{d^3y}{dx^3} = 6y \left(\frac{dy}{dx} \right)^2 + (3y^2 - 2) \frac{d^2y}{dx^2}$	A1
		(4)
(b)	$x = 2, y = 1 \Rightarrow \left(\frac{dy}{dx} \right)_{x=2} = \frac{4+1}{2} = \frac{5}{2}, \left(\frac{d^2y}{dx^2} \right)_{x=2} = \frac{3(1) \left(\frac{5}{2} \right) - \frac{5}{2}}{2} = \frac{5}{2}$ $\left(\frac{d^3y}{dx^3} \right)_{x=2} = \frac{6(1) \left(\frac{25}{4} \right) + \frac{5}{2}}{2} = 20$	M1
	$(y =) f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2} f''(2) + \frac{(x-2)^3}{6} f'''(2) + \dots$ $= 1 + \frac{5}{2}(x-2) + \frac{5}{2} \frac{(x-2)^2}{2} + 20 \frac{(x-2)^3}{6} + \dots$	M1
	$(y =) 1 + \frac{5}{2}(x-2) + \frac{5}{4}(x-2)^2 + \frac{10}{3}(x-2)^3 + \dots$	A1
		(3)
		Total 7

Notes: Accept other derivative notations, e.g. y' , throughout.

(a)

M1: $x \frac{dy}{dx} \rightarrow \frac{dy}{dx} + x \frac{d^2y}{dx^2}$ or $y^3 \rightarrow ky^2 \frac{dy}{dx}$ Alternatively for $\frac{y^3}{x} \rightarrow \pm \alpha \frac{y^3}{x^2} + \frac{ky^2}{x} \frac{dy}{dx}$ (oe via quotient rule), or any correct attempt at implicit differentiation on the equation (coefficient slips condoned).

A1: Any fully correct second derivative expression. The two most common are shown in the scheme.

dM1: Differentiates again using the product rule correctly at least once. Depends on the first method mark. Again, look for correct forms, but allow if coefficients are incorrect.

There may be alternatives where they substitute for $\frac{dy}{dx}$ before differentiating again.

The scheme follows the same principle. If unsure, use the Review system.

A1: Correct answer achieved from correct work. In the Alt they need to substitute the second derivative expression before reaching the final answer.

(b)

M1: Attempts to find values for the first 3 derivatives at $x = 2$. Working need not be seen, in which case accept if all are attempted with at least y' correct providing expressions involving the other derivatives were actually found in part (a).

M1: Correct application of Taylor's theorem with their values up to at least the $(x-2)^2$ term.

A1: Correct simplified expression. May be called y or $f(x)$ or unlabelled for this part.



Question Number	Scheme	Marks
3(a)	$\frac{1}{(n+3)(n+5)} \equiv \frac{A}{(n+3)} + \frac{B}{(n+5)}$ $\Rightarrow A = \dots, B = \dots$	M1
	$\frac{1}{2(n+3)} - \frac{1}{2(n+5)} \text{ (oe)}$	A1
		(2)
(b)	$\left(\frac{1}{2}\right) \sum_{r=1}^n \left(\frac{1}{r+3} - \frac{1}{r+5} \right) =$ $\left(\frac{1}{2}\right) \left(\frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8} + \dots + \frac{1}{n+2} - \frac{1}{n+4} + \frac{1}{n+3} - \frac{1}{n+5} \right)$	M1
	$= \left(\frac{1}{2}\right) \left(\frac{1}{4} + \frac{1}{5} - \frac{1}{n+4} - \frac{1}{n+5} \right) \text{ (oe)}$	A1
	$= \left(\frac{1}{2}\right) \left(\frac{9(n+4)(n+5) - 20(n+5) - 20(n+4)}{20(n+4)(n+5)} \right) = \dots$	dM1
	$= \frac{n(9n+41)}{40(n+4)(n+5)}$	A1
		(4)
(c)	$\frac{1}{9 \times 11} + \frac{1}{10 \times 12} + \dots + \frac{1}{24 \times 26} = \frac{21(9 \times 21 + 41)}{40 \times 25 \times 26} - \frac{5(9 \times 5 + 41)}{40 \times 9 \times 10}$	M1
	$= \frac{194}{2925}$	A1
		(2)
		Total 8

Notes:

(a)

M1: Correct partial fraction attempt to obtain values for A and B . May be implied by correct values.A1: Correct expression as shown or any equivalent. May be $\frac{1}{2}$ in each numerator, or the $\frac{1}{2}$ may be factored out. The expression must be given in (a) not just embedded in the summation in (b).

(b)

M1: Uses the method of differences to identify non-cancelling terms. Expect to see at least first two pairs and last pair of terms listed, but this may be implied by the correct first two or last two non-cancelling terms being stated with no cancelling explicitly shown. Allow if the $\frac{1}{2}$ is missing for this mark.A1: Identifies the correct non-cancelling terms. Allow if the $\frac{1}{2}$ is missing for this mark.dM1: Puts over a common denominator $k(n+4)(n+5)$, where $k \neq 1$, and makes progress towards the required form. The $\frac{1}{2}$ may be missing for this mark. Allow slips in the numerator coefficients but should have correct terms in n .

A1: Achieves the correct answer.

(c)

M1: Attempts to use the formula from (b) with $\sum_{r=1}^{21} - \sum_{r=1}^k$ where $k = 5$ or 6 .

Substitution into the formula must be seen at least once.

A1: Correct answer achieved (the M must have been scored). Must be simplified.



General Principles for Further Pure Mathematics Marking

(But note that specific mark schemes may sometimes override these general principles).

Method mark for solving 3 term quadratic:**1. Factorisation**

$(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$

$(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$

2. Formula

Attempt to use the correct formula (with values for a, b and c).

3. Completing the square

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$

Method marks for differentiation and integration:**1. Differentiation**

Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)

2. Integration

Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)



Question	Scheme	Marks
7(a)	$z = (0+)iy \Rightarrow w = \frac{(1+i)iy + 2(1-i)}{iy-i} = \frac{-y+2+i(y-2)}{i(y-1)} = \frac{y-2+i(y-2)}{y-1}$	M1
	$\Rightarrow u = v$ or $\text{Im } w = \text{Re } w$	A1
		(2)
(b)	$w = \frac{(1+i)z + 2(1-i)}{z-i} \Rightarrow z = \frac{2(1-i)+iw}{w-1-i} = \frac{2-v+i(u-2)}{u-1+i(v-1)}$	M1
	$\frac{2-v+i(u-2)}{u-1+i(v-1)} \times \frac{u-1-i(v-1)}{u-1-i(v-1)}$ $= \frac{(2-v)(u-1) + (u-2)(v-1) + i((u-1)(u-2) - (2-v)(v-1))}{\dots}$ $\text{Im } z = 0 \Rightarrow (u-1)(u-2) - (2-v)(v-1) = 0$	M1
	$\Rightarrow (u-1)(u-2) - (2-v)(v-1) = 0 \Rightarrow u^2 - 3u + 2 + v^2 - 3v + 2 = 0$	A1
	$\Rightarrow \left(u - \frac{3}{2}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{1}{2}$	M1
	Centre is $\frac{3}{2} + \frac{3}{2}i$ and radius is $\frac{\sqrt{2}}{2}$	A1A1
		(6)
(8 marks)		

Notes:**(a)**

M1: Correct method to find the equation of the image line – e.g. substitutes in $z = iy$ and rearranges to Cartesian form. May use $x + iy$ and later set $x = 0$. Alternatively, may start as in (b) and then set $(2-v)(u-1) + (u-2)(v-1) = 0 \Rightarrow 2u - v - uv - 2 + uv + 2 - 2v - u = 0$ etc.

Another alternative is to find the image points of two points on the imaginary axis and to find the line from these.

A1: For $u = v$ or equation. Accept $\text{Im } w = \text{Re } w$, or $x = y$ if they have set $w = x + iy$.

(b)

Note: Accept work done in part (a) that is relevant to part (b) for credit if appropriate.

M1: Makes z the subject, substitutes $w = u + iv$ into the equation.

M1: Multiplies the numerator by the complex conjugate of denominator **and** extracts the imaginary part and sets it equal to zero to form an equation in u and v . Do not be concerned about the denominator.

A1: Correct equation in u and v for the circle.

M1: Completes the square on their equation to extract centre and radius. Not dependent, so allow as long as a suitable equation in u and v has been reached.

A1: Correct centre or correct radius. Accept either $\frac{3}{2} + \frac{3}{2}i$ or $\left(\frac{3}{2}, \frac{3}{2}\right)$ for the centre.

A1: Correct centre and correct radius. As above. Accept equivalent forms (need not be simplified)

Allow the final two A marks if all that is wrong is an error in the denominator. (M1M0A0M1A1A1 is possible.)



7(b) Alt1	Real axis is $z = x(+0i)$, so $u + iv = \frac{(1+i)x + 2(1-i)}{x-i} = \frac{(1+i)x + 2(1-i)}{x-i} \times \frac{x+i}{x+i} =$ $\frac{(1+i)x^2 + 2x(1-i) + (i-1)x + 2(i+1)}{x^2+1} = \frac{x^2 + x + 2 + i(x^2 - x + 2)}{x^2+1}$	M1
	$u = \frac{x^2 + x + 2}{x^2 + 1} = 1 + \frac{x+1}{x^2+1}; v = \frac{x^2 - x + 2}{x^2 + 1} = 1 - \frac{x-1}{x^2+1} \Rightarrow u + v = 2 + \frac{2}{x^2+1}$ $\Rightarrow (u-1)^2 + (v-1)^2 = \frac{(x+1)^2 + (x-1)^2}{(x^2+1)^2} = \frac{2x^2+2}{(x^2+1)^2} = \frac{2}{x^2+1} = u + v - 2$	M1 A1
	$\Rightarrow \left(u - \frac{3}{2}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{1}{2}$	M1
	Centre is $\frac{3}{2} + \frac{3}{2}i$ and radius is $\frac{\sqrt{2}}{2}$	A1A1
		(6)
Notes M1: Sets $z = x$ in the equation (or uses $x + iy$ and later sets $y = 0$) and multiplies by complex conjugate. M1: Eliminates x from the equations (one suitable method is shown, others are possible). A1: Correct equation in u and v for the circle. M1: Completes the square on their equation to extract centre and radius A1: Correct centre or correct radius. Accept either $\frac{3}{2} + \frac{3}{2}i$ or $\left(\frac{3}{2}, \frac{3}{2}\right)$ for the centre. A1: Correct centre and correct radius. As above.		
7(b) Alt 2	Unlikely to be seen As i and $-i$ are inverse points of the line, so their images are inverse points of the circle. $i \rightarrow \infty, -i \rightarrow \frac{-i+1+2-2i}{-2i} = \frac{3}{2} + \frac{3}{2}i$ Hence (as ∞ is the other point) the centre is $\frac{3}{2} + \frac{3}{2}i$ $0 \rightarrow \frac{2-2i}{-i} = 2+2i \text{ So radius is } \left \frac{3}{2} + \frac{3}{2}i - 2 - 2i \right = \dots$ $= \frac{\sqrt{2}}{2}$	M1 M1 A1 M1 A1 A1
(b) Alt 3	M1: Attempt to find images of three different points on the real axis. M1: Correct method to find centre from three points – e.g. intersection of two perpendicular bisectors. A1: Correct equation for the centre. M1: Uses centre and one point to find radius. A1: Correct centre A1: Correct radius	

General Principles for Further Pure Mathematics Marking

(But note that specific mark schemes may sometimes override these general principles)

Method mark for solving 3 term quadratic:

1. Factorisation

$(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$

$(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$

2. Formula

Attempt to use the correct formula (with values for a , b and c).

3. Completing the square

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$

Method marks for differentiation and integration:

1. Differentiation

Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)

2. Integration

Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)

Use of a formula

Where a method involves using a formula that has been learnt, the advice given in recent examiners' reports is that the formula should be quoted first.

Normal marking procedure is as follows:

Method mark for quoting a correct formula and attempting to use it, even if there are small errors in the substitution of values.

Where the formula is not quoted, the method mark can be gained by implication from correct working with values but may be lost if there is any mistake in the working.

Exact answers

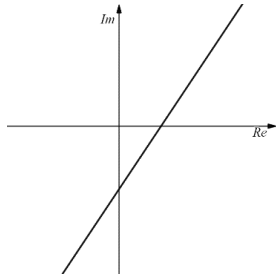
Examiners' reports have emphasised that where, for example, an exact answer is asked for, or working with surds is clearly required, marks will normally be lost if the candidate resorts to using rounded decimals.



Question Number	Scheme	Marks
1		
(a)	$2n+1 = A(n+1)^2 + Bn^2 \Rightarrow 2n+1 = An^2 + 2An + 1 + Bn^2$	
	$A=1 \quad B=-1 \text{ or } \frac{1}{n^2} - \frac{1}{(n+1)^2}$	B1 (1)
(b)	$\sum_{r=5}^n \frac{2r+1}{r^2(r+1)^2} = \sum_{r=5}^n \left(\frac{1}{r^2} - \frac{1}{(r+1)^2} \right)$ $= \left(\frac{1}{5^2} - \frac{1}{6^2} \right) + \left(\frac{1}{6^2} - \frac{1}{7^2} \right) + \dots + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right)$	M1
	$\sum_{r=5}^n \frac{2r+1}{r^2(r+1)^2} = \frac{1}{5^2} - \frac{1}{(n+1)^2}$	A1
	$= \frac{n^2 + 2n + 1 - 25}{25(n+1)^2} = \frac{n^2 + 2n - 24}{25(n+1)^2}$	M1A1 (4)
		[5]
	Notes	
(a) B1	Both values correct with or without working seen, may be in the expression. Ignore incorrect working.	
(b) M1	Show sufficient terms to demonstrate the cancelling. Require at least one cancelling term seen. Must start at $r = 5$ - M0 if starting at e.g. $r = 1$ unless there is a full process to complete the difference method (same condition) and apply $f(n) - f(4)$	
A1	Extract the two correct terms, or in the Alt obtains a correct overall expression.	
M1	Write the terms with a (non-zero) common denominator with at least numerator correct for their terms. Not dependent - may be scored following M0 if no cancelling terms were shown, but must have had exactly two terms to combine from differences.	
A1	Correct answer in the required form or accept correct values stated following an unsimplified form. (Allow as long as correct terms were extracted, even if no cancelling terms were shown.) Note: this means M0A0M1A1 can be scored for answers which show only the last two lines of the scheme with no cancelling process shown. Note: if e.g. r is used in place of n allow full marks if recovered, but A0 if left in terms of r .	
Alt (b) for first two marks	$\sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^n \left(\frac{1}{r^2} - \frac{1}{(r+1)^2} \right)$ $= \left(\frac{1}{1^2} - \frac{1}{2^2} \right) + \left(\frac{1}{2^2} - \frac{1}{3^2} \right) + \dots + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) = 1 - \frac{1}{(n+1)^2}$ $\rightarrow \sum_{r=5}^n \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2} - \left(1 - \frac{1}{25} \right) = \frac{n^2 + 2n}{(n+1)^2} - \frac{24}{25}$	M1A1

Question Number	Scheme	Marks
2 (a)	E.g. $(x+3)(x-5)=9 \Rightarrow x^2-2x-24=0 \Rightarrow x=...$ OR $(x-5)(x+3)^2-9(x+3)=0 \Rightarrow (x+3)(x-6)(x+4)=0 \Rightarrow x=...$ OR $\frac{(x+3)(x-5)-9}{x+3} < 0 \Rightarrow x^2-2x-24=0 \Rightarrow x=...$	M1
	CVs: 6, -4; -3	A1;B1
	$x < -4, -3 < x < 6$	dM1A1A1
	OR: $x \in (-3, 6) \cup (-\infty, -4)$ or any equivalent notation.	(6)
(b)	$x < 6, x \neq -3$ or any equivalent notation.	B1ftB1 (2)
		[8]
	Notes	
(a) M1	For a correct algebraic method to find the intersection points of $y = x - 5$ and $y = \frac{9}{x+3}$. May set these equal and form a quadratic and solve. May multiply through by $(x+3)^2$ and collect on one side or use any other valid method Eg work from $\frac{(x+3)(x+2)-12}{x+3} > 0$ Answers only from a calculator score M0. Must reach at least a quadratic or cubic before answers given. Do not be concerned with the equality or inequality for this mark.	
A1	For 6, -4 obtained via a valid algebraic method.	
B1	for the CV -3 seen anywhere	
dM1	Obtaining (any) inequalities using all of their critical values and no other numbers.	
A1	For at least one correct interval allowing for ... or ,, used instead of < and >	
A1cso	Both correct ranges and no extras. Use of ... or ,, scores A0. May be written in set notation, and all work should have been correct so penalise if incorrect inequalities method was used at the start. Accept $x < -4$ and/or $-3 < x < 6$ with “and” or “or”	
	For candidates who draw a sketch graph and follow with the cvs without any algebra shown only the B mark is available. Those who use some algebra after their graph may gain marks as earned (possibly all)	
(b) B1ft	For the " $x < 6$ " in some form with the possible exception of the CVs from (a). Allow $x, 6$ if already penalised in (a). It is essentially for realising all the extra (valid) values less than -3 are solutions while retaining all their given solutions. If only the CVs themselves are excluded allow B1. Follow through their answer to (a).	
B1	Fully correct answer. May give as intervals $x < -3, -3 < x < 6$	



Question Number	Scheme	Notes	Marks
6(a)		A straight line anywhere that is not vertical or horizontal which does not pass through the origin. It may be solid or dotted. Clear “V” shapes score M0.	M1
		A straight line in the correct position. Must have a positive gradient and lie in quadrants 1, 3 and 4. Ignore any intercepts correct or incorrect. If there are other lines that are clearly “construction” lines e.g. a line from $2i$ to 3 they can be ignored. The line may be solid or dotted. However, if there are clearly several lines then score A0.	A1
			(2)

Part (b)

The approaches below are the ones that have been seen most often.

Apply the mark scheme to the overall method the candidate has chosen.

There may be several attempts:

- If none are crossed out, mark all attempts and score the best single complete attempt
- If some attempts are crossed out, mark the uncrossed out work
- If everything is crossed out, mark all the work and score the best single complete attempt

Note that the question does not specify the variables the candidates should work in so they may use: e.g. $z = x + iy$ and $w = u + iv$ or $w = x + iy$ and $z = u + iv$ or any other letters so please check the work carefully.

Note that the M marks are all dependent on each other.



(b) Way 1	$w = \frac{iz}{z-2i} \Rightarrow z = \frac{2wi}{w-i}$	Attempts to make z the subject. Must obtain the form $\frac{awi}{bw+ci}$, a, b, c real and non-zero.	M1
	$z = \frac{2(u+iv)i}{u+iv-i} \text{ or e.g. } z = \frac{2(x+iy)i}{x+iy-i}$ $z = \frac{2(u+iv)i}{u+(v-1)i} \times \frac{u-(v-1)i}{u-(v-1)i} \text{ or equivalent}$ Introduces $w = u + iv$ or e.g. $w = x + iy$ and attempts to multiply numerator and denominator by the complex conjugate of the denominator. The above statement would be sufficient e.g. no expansion is needed for this mark.		dM1
	$z = \frac{-2u}{u^2+(v-1)^2} + \frac{2u^2+2v(v-1)}{u^2+(v-1)^2}i \text{ or e.g. } z = \frac{-2x}{x^2+(y-1)^2} + \frac{2x^2+2y(y-1)}{x^2+(y-1)^2}i$ or $z = \frac{-2uv+2u(v-1)+(2u^2+2v(v-1))i}{u^2+(v-1)^2} \text{ or e.g. } z = \frac{-2xy+2x(y-1)+(2x^2+2y(y-1))i}{x^2+(y-1)^2}$ Correct expression for z in terms of their variables with real and imaginary parts identified. May be embedded as above or stated explicitly.		A1
	$ z-2i = z-3 \Rightarrow y-1 = \frac{3}{2}\left(x-\frac{3}{2}\right)\left(y=\frac{3}{2}x-\frac{5}{4}, 6x-4y=5\right)$ $\Rightarrow \frac{2u^2+2v(v-1)}{u^2+(v-1)^2}-1 = \frac{3}{2}\left(\frac{-2u}{u^2+(v-1)^2}-\frac{3}{2}\right)$ Attempts the Cartesian equation of the locus of z and substitutes for x and y or equivalent using their variables to obtain an equation in u and v (or their variables). Condone slips with the locus of z but must be a linear equation in any form but with a non-zero constant term. or $ z-2i = z-3 \Rightarrow \left \frac{-2u}{u^2+(v-1)^2} + \frac{2u^2+2v(v-1)}{u^2+(v-1)^2}i - 2i\right = \left \frac{-2u}{u^2+(v-1)^2} + \frac{2u^2+2v(v-1)}{u^2+(v-1)^2}i - 3\right $ $\Rightarrow \left(\frac{-2u}{u^2+(v-1)^2}\right)^2 + \left(\frac{2u^2+2v(v-1)}{u^2+(v-1)^2}-2\right)^2 = \left(\frac{-2u}{u^2+(v-1)^2}-3\right)^2 + \left(\frac{2u^2+2v(v-1)}{u^2+(v-1)^2}\right)^2$ Substitutes their z into the locus of z and applies Pythagoras correctly to obtain an equation in u and v (or their variables). Note that here, further progress is unlikely.		ddM1
	$13u^2+13v^2+12u-18v+5=0 \Rightarrow u^2+v^2+\frac{12}{13}u-\frac{18}{13}v+\frac{5}{13}=\frac{4}{13}$ $\Rightarrow \left(u+\frac{6}{13}\right)^2 + \left(v-\frac{9}{13}\right)^2 = \frac{4}{13}$ Attempts to complete the square on their equation in u and v where u^2 and v^2 have the same coefficient. Award for e.g. $u^2+v^2+\alpha u+\beta v+\dots = \left(u+\frac{\alpha}{2}\right)^2 + \left(v+\frac{\beta}{2}\right)^2 + \dots = \dots$		dddM1

	Attempts using the form $u^2 + v^2 + 2gu + 2fv + c = 0$ send to review.	
	$\left w - \left(-\frac{6}{13} + \frac{9}{13}i \right) \right = \frac{2}{\sqrt{13}}$	Correct equation in the required form
		A1
		Total 8

(b) Way 2	$w = \frac{iz}{z-2i} \Rightarrow z = \frac{2wi}{w-i}$	Attempts to make z the subject. Must obtain the form $\frac{awi}{bw+ci}$, a, b, c real and non-zero.	M1
	$ z-2i = z-3 \Rightarrow \left \frac{2wi}{w-i} - 2i \right = \left \frac{2wi}{w-i} - 3 \right $ $\Rightarrow \left \frac{2wi-2wi-2}{w-i} \right = \left \frac{2wi-3w+3i}{w-i} \right $ Introduces z in terms of w into the given locus and attempts to combine terms		dM1
	$\left \frac{-2}{w-i} \right = \left \frac{2wi-3w+3i}{w-i} \right \Rightarrow -2 = 2wi-3w+3i $ Correct equation with fractions removed		A1
	$ 2(u+iv)i-3(u+iv)+3i = 2 \Rightarrow (3u+2v)^2 + (3v-2u-3)^2 = 4$ Introduces e.g. $w = u + iv$ and applies Pythagoras correctly		ddM1
	$13u^2 + 13v^2 + 12u - 18v + 9 = 4 \Rightarrow u^2 + v^2 + \frac{12}{13}u - \frac{18}{13}v + \frac{9}{13} = \frac{4}{13}$ $\Rightarrow \left(u + \frac{6}{13} \right)^2 + \left(v - \frac{9}{13} \right)^2 = \frac{4}{13}$ Attempts to complete the square on their equation in u and v where u^2 and v^2 have the same coefficient. Award for e.g. $u^2 + v^2 + \alpha u + \beta v + \dots = \left(u + \frac{\alpha}{2} \right)^2 + \left(v + \frac{\beta}{2} \right)^2 + \dots = \dots$ Attempts using the form $u^2 + v^2 + 2gu + 2fv + c = 0$ send to review.		dddM1
	$\left w - \left(-\frac{6}{13} + \frac{9}{13}i \right) \right = \frac{2}{\sqrt{13}}$	Correct equation in the required form	A1
			Total 8

(b) Way 3	$w = \frac{iz}{z-2i} \Rightarrow z = \frac{2wi}{w-i}$	Attempts to make z the subject. Must obtain the form $\frac{awi}{bw+ci}$, a, b, c real and non-zero.	M1
	$ z-2i = z-3 \Rightarrow \left \frac{2wi}{w-i} - 2i \right = \left \frac{2wi}{w-i} - 3 \right $ $\Rightarrow \left \frac{2wi - 2wi - 2}{w-i} \right = \left \frac{2wi - 3w + 3i}{w-i} \right $ Introduces z and attempts to combine terms		dM1
	$\left \frac{-2}{w-i} \right = \left \frac{2wi - 3w + 3i}{w-i} \right \Rightarrow -2 = 2wi - 3w + 3i $ Correct equation with fractions removed		A1
	$ w(2i-3) + 3i = \left (2i-3) \left(w + \frac{3i}{2i-3} \right) \right = 2i-3 \left w + \frac{6-9i}{13} \right = 2$ Attempts to isolate w and rationalise denominator of other term		ddM1
	$\sqrt{13} \left w - \left(-\frac{6}{13} + \frac{9}{13}i \right) \right = 2 \Rightarrow \left w - \left(-\frac{6}{13} + \frac{9}{13}i \right) \right = \frac{2}{\sqrt{13}}$ M1: Completes the process by dividing by their $ 2i-3 $ A1: Correct equation in the required form		dddM1A1
			(6)



Question Number	Scheme	Notes	Marks
2(a)	$(x+2)\frac{dy}{dx} + y = 3x^2 + 6x \Rightarrow$ $\frac{dy}{dx} + \frac{y}{x+2} = \frac{3x(x+2)}{x+2}$ $\Rightarrow \frac{dy}{dx} + \frac{y}{x+2} = 3x$	M1: Reaches $\frac{dy}{dx} + f(x)y = g(x)$ A1: Correct equation in required form. Correct answer implies both marks.	M1 A1
(2)			
(b)	$\text{IF} = \left\{ e^{\int \frac{1}{x+2} (dx)} \right\} = e^{\ln(x+2)} = x+2$	Correct attempt at the IF for their $f(x)$ – must attempt the integration but need not be fully simplified (may be implied by working)	M1
	$"(x+2)" y = \int "3"x"(x+2)"(dx)$	Applies their IF correctly with their kx	M1
	$(x+2)y = x^3 + 3x^2 (+c)$	Correct equation after integration condoning omission of the constant	A1
	$x = 4, y = 18 \Rightarrow c = 108 - 64 - 48 = -4$	Substitutes correct values to find a value for c in their integrated equation	M1
	$y = \frac{x^3 + 3x^2 - 4}{x+2} = \frac{(x+2)(x+2)(x-1)}{x+2}$ $\Rightarrow y = (x+2)(x-1) \text{ or } x^2 + x - 2$	A1: Correct particular solution with $g(x)$ in any form. Condone missing “$y = \dots$” A1: y or $g(x) = (x+2)(x-1)$ or $x^2 + x - 2$	A1 A1
(6)			
Total 8			
Note : The question can be completed via a complementary function and particular integral approach, as below. If the CF is never considered a maximum of 011000 is possible.			
(b) Alt	$(x+2)\frac{dy}{dx} + y = 0 \Rightarrow \frac{1}{y} \frac{dy}{dx} = -\frac{1}{x+2}$ $\Rightarrow \ln y = -\ln(x+2) + c$ $\Rightarrow \ln y = \ln \frac{A}{x+2} \Rightarrow y = \frac{A}{x+2}$	Considers the homogeneous equation and makes progress integrating to an equation $y = \dots$ with constant of integration.	M1
	$y = ax^2 + bx + c \Rightarrow \frac{dy}{dx} = 2ax + b$ $\Rightarrow (x+2)(2ax+b) + ax^2 + bx + c = 3x^2 + 6x$ $x^2 : 3a = 3 \quad x : 4a + 2b = 6 \quad x^0 : 2b + c = 0$ $\Rightarrow a = \dots, b = \dots, c = \dots$	Tries the PI $y = ax^2 + bx + c$, differentiates and substitutes into equation to find a, b and c	M1
	$y = x^2 + x - 2$	Correct particular integral	A1
	$y = x^2 + x - 2 + \frac{A}{x+2} \Rightarrow 18 = 16 + 4 - 2 + \frac{A}{6}$ $\Rightarrow A = \dots ; A = 0$	Uses (4, 18) in their general solution to find the constant ; Correctly shows $A = 0$	M1 A1
	$\Rightarrow y = (x+2)(x-1) \text{ or } x^2 + x - 2$	Correct answer following attempt to combine the CF and PI. Accept y or $g(x)$ as before.	A1

Additional Notes:

No need to see method for solving the cubic – if the correct answer follows directly from the unfactorized cubic that is acceptable for the final A.



Question Number	Scheme	Notes	Marks
3	$y = \arcsin 2x \quad \frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}} \left\{ = 2(1-4x^2)^{-\frac{1}{2}} \right\}$		
(a)	$\frac{d^2y}{dx^2} = -(1-4x^2)^{-\frac{3}{2}}(-8x) \left\{ = 8x(1-4x^2)^{-\frac{3}{2}} \right\}$	Obtains correct form for $\frac{d^2y}{dx^2}$	M1
	$\frac{d^3y}{dx^3} = (8x) \left(-\frac{3}{2} \right) (1-4x^2)^{-\frac{5}{2}}(-8x) + 8(1-4x^2)^{-\frac{3}{2}}$ or $\frac{8(1-4x^2)^{\frac{3}{2}} + 96x^2(1-4x^2)^{\frac{1}{2}}}{(1-4x^2)^3}$	Uses product/quotient rule to obtain correct form for $\frac{d^3y}{dx^3}$	dM1
	$= \frac{96x^2 + 8(1-4x^2)}{(1-4x^2)^{\frac{5}{2}}} = \frac{64x^2 + 8}{(1-4x^2)^{\frac{5}{2}}}$	Correct answer in the correct form	A1
(3)			
Allow full access to the remaining marks with an incorrect value of A			
(b)	$y(0)=0, y'(0)=2, y''(0)=0, y'''(0)=8$ $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$ $(\Rightarrow \arcsin 2x) \approx 2x + \frac{4}{3}x^3$	M1: Finds values for $y(0), y'(0), y''(0)$ and $y'''(0)$ and obtains a consistent Maclaurin expansion A1: Correct expression	M1 A1
(2)			
(c)	$e^{3x} \arcsin 2x = (1+3x+\dots) \left(2x + \frac{4}{3}x^3 + \dots \right) = \dots$	Attempts to find the expansion of e^{3x} and multiplies by their expression for $\arcsin 2x$	M1
	$= \left(1+3x+\frac{(3x)^2}{2}+\dots \right) \left(2x + \frac{4}{3}x^3 + \dots \right) = \dots$ $= 2x + 6x^2 + 9x^3 + \frac{4}{3}x^3 + \dots$	For a correct expansion of e^{3x} up to the x^2 term multiplied by their expression for $\arcsin$ with at least one correct term obtained.	A1
	$\Rightarrow e^{3x} \arcsin 2x \approx 2x + 6x^2 + \frac{31}{3}x^3$	Fully correct expression	A1
(3)			
Total 8			

Additional Notes:

(a)

 M1: Attains the form $Kx(1-4x^2)^{-\frac{3}{2}}$, $K \neq 0$ or equivalent form, e.g. by quotient rule $\frac{0-Kx(1-4x^2)^{-\frac{1}{2}}}{1-4x^2}$

dM1: Differentiates again, any correct unsimplified form is fine (chain rule must have been applied).

A1: Reaches the correct simplified answer.

Note: Full marks are available in (b) and (c) if an incorrect A was found in part (a).

(b)

M1: Attempts all the values for the derivatives and finds the Maclaurin expansion. If the formula is quoted correctly allow any values appearing in the formula as their values if they are not separately stated. If no formula is given and values not stated, the denominators in their expansion must be seen to imply a correct attempt at the formula.



A1: The $\arcsin 2x$ need not be stated, the A can be awarded for a correct expression. May have come from an incorrect A in (a).

(c)

M1: Attempts the series for e^{3x} and multiplies by their answer to part (b) – an attempt at the multiplication must be carried out but need not be complete or correct. Attempts via differentiation from scratch or

expanding the series for $(e^x)^3 = \left(1 + x + \frac{x^2}{2}\right)^3 = \dots$ will need to successfully obtain a correct unsimplified series up to the x term, ie $1 + 3x + \dots$ oe.

A1: Must have a **correct expansion of e^{3x} up to the x^2 term** (ie $1 + 3x + \frac{(3x)^2}{2}$) and multiply by their expression for $\arcsin 2x$ with at least one correct term obtained for the final series attained.

A1: Correct series.

Question Number	Scheme	Notes	Marks
4(a)	$a = 8, \quad b = \frac{1}{4}, \quad c = -18$	B1: Correct value for a B1: Correct value for b B1: Correct value for c	B1 B1 B1
(3)			
(b)	$ w _{\min} = \sqrt{8^2 + 7^2} \quad w _{\max} = \sqrt{8^2 + 15^2} + 8$	M1: A correct method for either $ w _{\min}$ or $ w _{\max}$ A1: $ w _{\min} = \sqrt{113}$ and $ w _{\max} = 25$	M1 A1
	$\sqrt{113} \leq w \leq 25$	Correct region	A1
(3)			
(c)	$\arg w_{\min} = \arctan \frac{8}{23} = 0.335$	M1: Correct method for a relevant angle. A1: awrt 0.335 only	M1 A1
(2)			
Total 8			

Additional Notes:

(a)

B1B1B1: Values may be given on the diagram, but they must be clearly labelled/associated as the correct values. Do not accept e.g. 8 marked as a radius on the diagram without being associated as a .

Accept if the values are embedded in the relevant formula as long as they do not subsequently state incorrect values for the constants – if they state values explicitly these take precedence.

(b)

M1: For a correct method for either end of the region, they may show working on the diagram. Note for the upper limit, they may find the equation of the line through O and the centre of the circle, find the intersections of this line with the circle then attempt the distance for the furthest of these points from O .

A1: Both correct extreme values.

A1: Correct answer – must be the whole range, including $|w|$, not just state the maximum and minimum.

(c)

M1: For a correct method for either the angle directly or for another angle in a triangle on the diagram which includes the required angle or its complement. May Implied by awrt 0.33 or 0.335 or 1.2 or (degrees) awrt 19 or 71

A1: Accept anything which rounds to 0.335 and no others.



Question Number	Scheme	Notes	Marks
6(a)	$ z+1-13i =3 z-7-5i \Rightarrow (x+1)^2 + (y-13)^2 = 9\{(x-7)^2 + (y-5)^2\}$ Correct application of Pythagoras Accept 3 or 9 on RHS		M1
	$\Rightarrow x^2 + y^2 - 16x - 8y + 62 = 0$	Correct equation in any form with terms collected	A1
	Centre (8, 4)	Correct centre. i included scores A0	A1
	$r^2 = 64 + 16 - 62 = \dots$	Correct method for r or r^2	M1
	$r = \sqrt{18}$ or $3\sqrt{2}$	Correct radius. Must be exact.	A1
			(5)
(b)	$\arg(z-8-6i) = -\frac{3\pi}{4} \Rightarrow y-6 = x-8$	Converts the given locus to the correct Cartesian form	B1
	$\Rightarrow x^2 + y^2 - 16x - 8y + 62 = 0$ $\Rightarrow x^2 + (x-2)^2 - 16x - 8(x-2) + 62 = 0 \Rightarrow x = \dots$ or $\Rightarrow (y+2)^2 + y^2 - 16x - 8(y+2) + 62 = 0 \Rightarrow y = \dots$	Uses both Cartesian equations to obtain an equation in one variable and attempts to solve	M1
	$x = 7 - 2\sqrt{2}$ or $y = 5 - 2\sqrt{2}$	One correct “coordinate”	A1
	R is $7 - 2\sqrt{2} + (5 - 2\sqrt{2})i$ or $x = 7 - 2\sqrt{2}$ and $y = 5 - 2\sqrt{2}$	Correct complex number or coordinates and no others. Must be exact	A1
			(4)
			Total 9

Question Number	Scheme	Notes	Marks
7(a) Way 1	$w = \frac{z-3}{2i-z} \Rightarrow 2iw - wz = z-3 \Rightarrow z = \dots$	Attempts to make z the subject and obtains any $f(w)$	M1
	$z = \frac{3+2iw}{w+1}$ or $\frac{-3-2iw}{-w-1}$	Any correct expression for z in terms of w	A1
	$= \frac{3+2iu-2v}{u+iv+1} \times \frac{u+1-iv}{u+1-iv}$ Applies $w = u + iv$ and a correct multiplier for their z seen or implied by a correct result from their z . Denominator must have had a “ w ”. Note alternative route below.		M1
	$x+iy = \frac{3+2iu-2v}{u+iv+1} \times \frac{u+1-iv}{u+1-iv} = \frac{(3-2v)(u+1)+2uv+2u(u+1)i-(3-2v)vi}{(u+1)^2+v^2}$ $y = x+3$ oe $\Rightarrow \frac{2u(u+1)-(3-2v)v}{(u+1)^2+v^2} = \frac{(3-2v)(u+1)+2uv}{(u+1)^2+v^2} + 3$ Multiplies, extracts real and imaginary parts and uses them in the equation $y = x + 3$ (oe) to produce an equation in u and v only – no “ i ”s. Condone $y = \dots i$ if recovered. Can follow slips with multiplier but denominator of z must have had a “ w ” Note: Just $2u(u+1)-(3-2v)v = (3-2v)(u+1)+2uv+3$ is M0 (lost denominators)		M1
	$2u(u+1)-(3-2v)v = (3-2v)(u+1)+2uv+3(u+1)^2+3v^2$ $\Rightarrow u^2+7u+v^2+v+6=0$	Expands and simplifies to obtain an equation of a circle with 4 or 5 real unlike terms. All previous Ms required.	dddM1
	Alternative for the above 3 marks (note this could be done by equating expressions for y) $x+iy = \frac{3+2iu-2v}{u+iv+1} \Rightarrow (x+i(x+3))(u+1+iv) = 3+2ui-2v$ M1: Applies $z = x + iy$, uses $y = x + 3$ and cross multiplies		
	$x(u+1)-v(x+3)+(x+3)(u+1)i+xvi = 3-2v+2ui$ $\Rightarrow ux+x-vx-3v = 3-2v, \quad ux+x+3u+3+3v = 2u$ $\Rightarrow x = \frac{3+v}{u+1-v}, \quad x = \frac{-u-3}{u+1+v}$ M1: Equates real and imaginary parts and makes x the subject twice $(3+v)(u+1+v) = -(u+3)(u+1-v) \Rightarrow 3u+3+3v+uv+v+v^2 = -u^2-u+uv-3u-3+3v$ $\Rightarrow u^2+v^2+7u+v+6=0$ M1: Equates expressions for x to obtain a circle equation with 4 or 5 real unlike terms		
	$\Rightarrow \left(u + \frac{7}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{49}{4} + \frac{1}{4} - 6 = \frac{13}{2} \Rightarrow \text{centre: } \left(-\frac{7}{2}, -\frac{1}{2}\right) \text{ radius: } \frac{\sqrt{26}}{2} \text{ or } \sqrt{\frac{13}{2}}$ M1: Extracts the centre and/or radius from their circle equation, however obtained, with 4 or 5 real unlike terms. Circle equation must not be in terms of z or w . They must get one correct coordinate (but condone wrong sign) or the correct radius for their circle. May use $u^2+v^2+2gu+2fv+c=0 \Rightarrow \text{centre: } (-g, -f), \text{ radius} = \sqrt{g^2+f^2-c}$ A1: For a correct centre or radius from a correct circle equation A1: For correct centre and radius from a correct circle equation Centre as coordinates, $x/u = \dots, y/v = \dots$ or as $-\frac{7}{2} - \frac{1}{2}i$ and allow $(-\frac{7}{2}, -\frac{1}{2}i)$ Allow exact equivalents for coordinates/radius		
			(8)

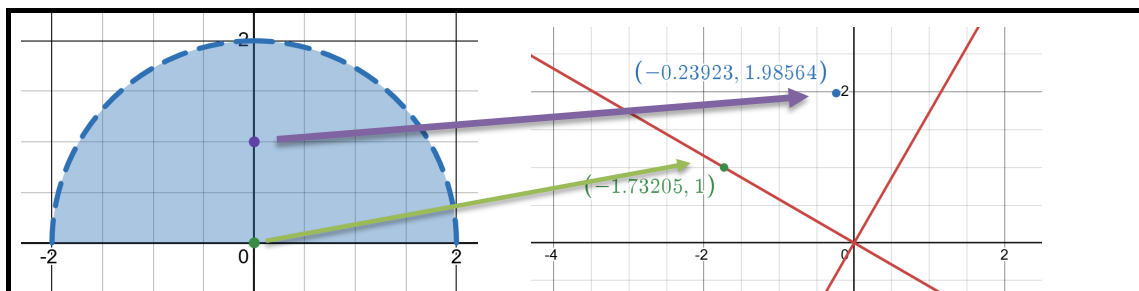
Question Number	Scheme	Notes	Marks
7(a) Way 2	$w = \frac{z-3}{2i-z} = \frac{x+iy-3}{2i-x-iy} = \frac{x-3+i(x+3)}{2i-x-i(x+3)}$ [Note that it is possible to replace x with $y-3$]	M1: Uses $z = x + iy$ and $y = x + 3$ in the given transformation A1: Correct expression for w in terms of x	M1 A1
	$\frac{x-3+i(x+3)}{-x-i(x+1)} = u + iv \Rightarrow x-3+i(x+3) = -xu + v(x+1) - iu(x+1) - ivx$	Applies $w = u + iv$ and multiplies	M1
	$x-3 = -ux + vx + v, \quad x+3 = -ux - u - vx$ $x = \frac{3+v}{1+u-v}, \quad x = \frac{-3-u}{1+u+v}$	Equates real and imaginary parts and makes x the subject twice	M1
	$3+3u+3v+v+uv+v^2 = -3-3u+3v-u-u^2+uv$ $\Rightarrow u^2 + v^2 + 7u + v + 6 = 0$	Equates expressions for x to obtain a circle equation with 4 or 5 real unlike terms.	dddM1
	$\Rightarrow \left(u + \frac{7}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{49}{4} + \frac{1}{4} - 6 = \frac{13}{2} \Rightarrow \text{centre: } \left(-\frac{7}{2}, -\frac{1}{2}\right) \text{ radius: } \frac{\sqrt{26}}{2} \text{ or } \sqrt{\frac{13}{2}}$ M1: Applies a correct process to extract the centre and/or radius from a circle equation, however obtained, with 4 or 5 real unlike terms. One correct coordinate (but condone wrong sign) or radius correct for their circle. May use $u^2 + v^2 + 2gu + 2fv + c = 0 \Rightarrow \text{centre: } (-g, -f), \text{ radius} = \sqrt{g^2 + f^2 - c}$ A1: For correct centre or radius from a correct circle equation A1: For correct centre and radius from a correct circle equation Centre as coordinates, $x/u = \dots, y/v = \dots$ or as $-\frac{7}{2} - \frac{1}{2}i$ and allow $(-\frac{7}{2}, -\frac{1}{2}i)$ (8)		M1 A1 A1
Way 3	e.g., 3 points on line are (0,3), (1,4) and (2,5) or $z_1 = 3i, z_2 = 1+4i, z_3 = 2+5i$	Attempts three points/complex numbers on $y = x + 3$ with 2 correct	M1
	$w = \frac{z-3}{2i-z} \Rightarrow w_1 = \frac{3i-3}{-i} \quad w_2 = \frac{-2+4i}{-1-2i} \quad w_3 = \frac{-1+5i}{-2-3i}$	Correct transformed complex numbers	A1
	$w_1 = \frac{3i-3}{-i} \times \frac{i}{i} \quad w_2 = \frac{-2+4i}{-1-2i} \times \frac{-1+2i}{-1+2i} \quad w_3 = \frac{-1+5i}{-2-3i} \times \frac{-2+3i}{-2+3i}$ At least two correct multipliers to remove "i" from denominator seen or implied (one if $(-1, 2)$ used). Requires 2 correct points/complex numbers on line		M1
	$w_1 = -3-3i \quad w_2 = -\frac{6}{5} - \frac{8}{5}i \quad w_3 = -1-i$	Two correct complex numbers in $a + ib$ form or as points	M1
	$\text{e.g., } x^2 + y^2 + 2gx + 2fy + c = 0 \Rightarrow \frac{12}{5}g + \frac{16}{5}f - c = 0$ $2g + 2f - c = 0$	Uses a correct general equation of a circle to form three simultaneous equations.	dddM1
	$\Rightarrow g = \frac{7}{2}, f = \frac{1}{2}, c = 6 \Rightarrow \text{centre } (-g, -f): \left(-\frac{7}{2}, -\frac{1}{2}\right) \text{ radius} = \sqrt{g^2 + f^2 - c} = \frac{\sqrt{26}}{2} \text{ or } \sqrt{\frac{13}{2}}$ M1: Solves and obtains at least one correct coordinate (but condone wrong sign) or radius for their constants A1: Correct centre or radius from correct work A1: Correct centre and radius from correct work (8)		M1 A1 A1

Question	Scheme	Marks
8(a)	Real line is $y = 0$, so have $u + iv = \frac{(\sqrt{3} - i)(x - 2)}{x + 2} = \frac{\sqrt{3}(x - 2)}{x + 2} - \frac{(x - 2)}{x + 2}i$	M1
	$\Rightarrow \frac{u}{v} = \frac{\sqrt{3}}{-1} \Rightarrow v = -\frac{1}{\sqrt{3}}u^*$	M1A1*
		(3)
For any attempts at solutions which use image points under the transformation T for parts (a) or (b) please send to review		
(a) Alt (i)	$w = \frac{(\sqrt{3} - i)(z - 2)}{z + 2} \Rightarrow wz + 2w = \sqrt{3}z - 2\sqrt{3} - iz + 2i$	M1
	$\Rightarrow z = \frac{2(i - w - \sqrt{3})}{i + w - \sqrt{3}} = \frac{-2(u + \sqrt{3}) - 2(v - 1)i}{(u - \sqrt{3}) + (v + 1)i} \times \frac{(u - \sqrt{3}) - (v + 1)i}{(u - \sqrt{3}) - (v + 1)i}$	
	$= \frac{-2(u^2 - 3) + 2(v + 1)(u + \sqrt{3})i - 2i(uv - \sqrt{3}v - u + \sqrt{3}) - 2(v^2 - 1)}{(u - \sqrt{3})^2 + (v + 1)^2}$	M1
	$y = 0 \Rightarrow 2(v + 1)(u + \sqrt{3}) - 2(uv - \sqrt{3}v - u + \sqrt{3}) = 0$	
	$2(uv + \sqrt{3}v + u + \sqrt{3}) - 2uv + 2\sqrt{3}v + 2u - 2\sqrt{3} = 0$	
	$\Rightarrow 4\sqrt{3}v + 4u = 0 \Rightarrow v = -\frac{1}{\sqrt{3}}u^*$	A1*
Alt (ii)	$w = \frac{(\sqrt{3} - i)(z - 2)}{z + 2} \Rightarrow ux + 2u + ivx + 2iv = \sqrt{3}x - 2\sqrt{3} - ix + 2i$	M1
	$\Rightarrow ux + 2u = \sqrt{3}x - 2\sqrt{3} \text{ and } vx + 2v = -x + 2$	
	$\Rightarrow x = \frac{-2u - 2\sqrt{3}}{u - \sqrt{3}} \text{ and } x = \frac{-2v + 2}{v + 1}$	
	$\Rightarrow \frac{-2u - 2\sqrt{3}}{u - \sqrt{3}} = \frac{-2v + 2}{v + 1}$	M1
	$\Rightarrow -2uv - 2u - 2\sqrt{3}v - 2\sqrt{3} = -2uv + 2\sqrt{3}v + 2u - 2\sqrt{3}$	
	$4\sqrt{3}v = -4u \Rightarrow v = -\frac{1}{\sqrt{3}}u^*$	A1*

	Note: If a candidate finds an expression for z in (a) then this may be used/seen in (b) and can score M1A1 in part (b) <u>if used in (b)</u>	
(b)	$w = \frac{(\sqrt{3}-i)(z-2)}{z+2} \Rightarrow w(z+2) = (\sqrt{3}-i)z - 2(\sqrt{3}-i) \Rightarrow z = \dots \text{ or } kz = \dots$	M1
	$z = \frac{-2w-2(\sqrt{3}-i)}{w-\sqrt{3}+i} \text{ or } z(w-\sqrt{3}+i) = -2w-2(\sqrt{3}-i)$	A1
	$ z = 2 \Rightarrow 2 u+iv-\sqrt{3}+i = -2u-2iv-2\sqrt{3}+2i $	M1
	$\Rightarrow 4(u-\sqrt{3})^2 + 4(v+1)^2 = (2u+2\sqrt{3})^2 + (2v-2)^2$ $\Rightarrow \cancel{4u^2} - 8\sqrt{3}u + 12 + \cancel{4v^2} + 8v + 4 = \cancel{4u^2} + 8\sqrt{3}u + 12 + \cancel{4v^2} - 8v + 4$	M1
	$\Rightarrow 16v = 16\sqrt{3}u \Rightarrow v = \sqrt{3}u \text{ o.e.}$	A1
		(5)
(b) Alt (i)	$w = \frac{(\sqrt{3}-i)(z-2)}{z+2} \Rightarrow w(z+2) = (\sqrt{3}-i)z - 2(\sqrt{3}-i) \Rightarrow z = \dots \text{ or } kz = \dots$	M1
	$z = \frac{-2w-2(\sqrt{3}-i)}{w-\sqrt{3}+i} \text{ or } z(w-\sqrt{3}+i) = -2w-2(\sqrt{3}-i)$	A1
	$ z = 2 \Rightarrow w-\sqrt{3}+i = -w-\sqrt{3}+i $ $ w-(\dots) = w-(\dots) $	M1
	$ w-(\sqrt{3}-i) = w-(-\sqrt{3}+i) $ So perp bisector of $(\sqrt{3}, -1)$ and $(-\sqrt{3}, 1) \Rightarrow v = \sqrt{3}u \text{ o.e.}$	M1A1
(c)	$R = \{z \in \mathbb{C}: z < 2\} \cap \{z \in \mathbb{C}: \operatorname{Im} z > 0\}$	
	$v = \sqrt{3}u \Rightarrow \arg w = \arctan \sqrt{3} = \dots \left(\frac{\pi}{3}\right)$	M1
	$v = -\frac{1}{\sqrt{3}}u \Rightarrow \arg w = \arctan -\frac{1}{\sqrt{3}}; = -\frac{\pi}{6} \text{ or } \frac{5\pi}{6}$	M1A1
	$z = 0 \Rightarrow w = i - \sqrt{3}$ so need $\beta = \frac{5\pi}{6}$	M1
	Image region is $\{w \in \mathbb{C}: \frac{\pi}{3} < \arg w < \frac{5\pi}{6}\}$	A1*
		(5)
(13 marks)		



Notes:**(a)****M1:** Realises real line is $y = 0$ and substitutes into the equation for w **M1:** Equates real and imaginary parts and solves simultaneously to eliminate x or equivalent work**A1*:** Correct work leading to the given equation, no errors seen. At least one intermediate step required after identifying- which may be implied- the real and imaginary parts (u and v in terms of x)**Alt (i):****M1:** Makes z the subject and multiplies by a valid conjugate. Accept any suitable attempt to make z the subject**M1:** Extracts the imaginary part and sets equal to 0 to achieve an unsimplified line equation in u and v **A1*:** Achieves the correct line equation with no errors seen**Alt (ii)****M1:** Replaces $w = u + iv$ and $z = x$, then rearranges the given transformation and equates the real and imaginary parts**M1:** Finds two expressions for x and equates**A1*:** Achieves the correct equation with no errors seen**(b)****M1:** Rearranges the transformation to make z the subject, or at least as far as isolating all the z terms together.**A1:** Correct expression for z or with z isolated within the expression.**M1:** Applies the circle equation and substitutes $w = u + iv$ **M1:** Expands the moduli and cancels the square terms to arrive at an equation for a line.**A1:** Correct equation (any equivalent form acceptable)**(b) Alt (i)****M1:** Rearranges the transformation to make z the subject, or at least as far as isolating all the z terms together.**A1:** Correct expression for z or with z isolated within the expression.**M1:** Processes to an expression of the form $|z - a| = |z - b|$ **M1:** Recognises that the locus is a perpendicular bisector of $(\sqrt{3}, -1)$ and $(-\sqrt{3}, 1)$ **A1:** Achieves the correct equation**(c)****M1:** Attempts the argument for the line from the image of the circle from their line equation from part (b).**M1:** Attempts the argument for the image of the real axis as given in (a)**A1:** Correct possible arguments for the image of real axis identified. Need not select at this stage, so accept if just $-\frac{\pi}{6}$ is given.**M1:** Uses a point on the real line segment of the boundary, or in the region R or elsewhere to determine which arguments are needed. May see image of i used (which is $\frac{4-3\sqrt{3}}{5} + \frac{3+4\sqrt{3}}{5}i$) to identify the correct quadrant, which is fine.**Alt:** Uses the condition that $\text{Im } z > 0$ in their imaginary part for z to **show that** $v > -\frac{1}{\sqrt{3}}u$ and $|z| < 2$ in their expression for z to **show that** $v > \sqrt{3}u$ **A1:** Correct answer obtained with full justification. Condone the omission of $w \in \mathbb{C}$ etc



In (c) candidates are effectively finding which side of the lines found in (a), (b) the points inside the semicircle with radius 2 and positive imaginary part, are mapped to under T .

The origin is mapped to point $(-\sqrt{3}, 1)$ which is $(-1.73 \dots, 1)$ shown above, which is on the half line with argument $\frac{5\pi}{6}$

The point i which is $(0, 1)$ is mapped to $\frac{4-3\sqrt{3}}{5} + \frac{3+4\sqrt{3}}{5}i$ which is $(-0.239\dots, 1.985\dots)$ shown above