



GuideMyGrades

— YOUR GOALS. OUR EXPERTISE. —

EDEXCEL INTERNATIONAL A LEVEL MATHS: QUESTIONS BY TOPIC

Hyperbolic Functions (FP3.1)

International A Level Mathematics · FP3: Further Pure Mathematics 3 (WFM03)

11 questions · 72 marks · Mark schemes included

Past papers October 2020 - June 2025 · 1 sub-topic

 guidemygrades.com

 [@Guidemygrades](https://www.instagram.com/Guidemygrades)

 [@Guidemygrades](https://twitter.com/Guidemygrades)

 [@Guidemygrades](https://www.youtube.com/Guidemygrades)

 info@guidemygrades.com

 Piccadilly Business Centre, Unit C, Aldow Enterprise Park
Blackett Street, Manchester, M12 6AE

HOW TO USE THIS BOOKLET

Questions are grouped by sub-topic and run from the lowest to the highest mark tariff, then oldest paper first.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme; < QUESTION brings you back.

A gold bar in the left margin marks exactly where the question sits on the original page.

Diagrams, graphs, grids and tables are printed exactly as they appear in the original papers.

Each question is filed under its main topic; longer questions may also test skills from other topics.

QUESTION	SOURCE	MARKS	PAGE
FP3.1a HYPERBOLIC FUNCTIONS & EQUATIONS		11 Qs · 72 marks	2
1. Q1	IAL FP3 (WFM03) June 2023	5	3
2. Q1	IAL FP3 (WFM03) June 2021	6	5
3. Q2	IAL FP3 (WFM03) October 2021	6	7
4. Q3	IAL FP3 (WFM03) January 2023	6	9
5. Q1	IAL FP3 (WFM03) January 2025	6	11
6. Q8	IAL FP3 (WFM03) January 2025	6	13
7. Q1	IAL FP3 (WFM03) June 2025	6	16
8. Q1	IAL FP3 (WFM03) October 2020	7	18
9. Q1	IAL FP3 (WFM03) June 2022	7	20
10. Q1	IAL FP3 (WFM03) January 2022	8	24
11. Q4	IAL FP3 (WFM03) June 2024	9	28



QUESTIONS

11 questions · 72 marks · 1 sub-topic

SUB-TOPIC FP3.1a

HYPERBOLIC FUNCTIONS & EQUATIONS

Sub-topic FP3.1a · 11 questions · 72 marks · lowest tariff first



1.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Solve the equation

$$7 \cosh x + 3 \sinh x = 2e^x + 7$$

Give your answers as simplified natural logarithms.

(5)



Question 1 continued

Lined area for writing the answer to Question 1.

(Total for Question 1 is 5 marks)



1. (a) Using the definitions of hyperbolic functions in terms of exponentials, show that

$$1 - \tanh^2 x \equiv \operatorname{sech}^2 x$$

(3)

(b) Solve the equation

$$2 \operatorname{sech}^2 x + 3 \tanh x = 3$$

giving your answer as an exact logarithm.

(3)

Leave
blank

Question 1 continued

Leave
blank

Handwriting practice area with horizontal lines.

(Total 6 marks)

Q1

Mark box for Q1



2. Given that

$$\cosh y = x \quad \text{and} \quad y < 0$$

use the definition of $\cosh y$ in terms of exponential functions to prove that

$$y = \ln \left(x - \sqrt{x^2 - 1} \right)$$

(6)

Leave
blank



Question 2 continued

Leave
blank

Q2

(Total 6 marks)



(6)

Question 3 continued

Lined area for writing the answer to Question 3.

(Total for Question 3 is 6 marks)



1.

In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.

Solve the equation

$$2 \sinh^2 x + 3 \cosh x = 7$$

Give your answers as simplified natural logarithms.

(6)

Question 1 continued

Lined area for writing the answer to Question 1.

(Total for Question 1 is 6 marks)



8. Given that

$$y = e^{3x} \cosh 2x$$

prove by induction that for $n \in \mathbb{N}$

$$\frac{d^n y}{dx^n} = e^{3x} \left(\frac{5^n + 1}{2} \cosh 2x + \frac{5^n - 1}{2} \sinh 2x \right)$$

(6)





Question 8 continued

Lined area for writing the answer to Question 8.

(Total for Question 8 is 6 marks)

TOTAL FOR PAPER IS 75 MARKS



1. (a) Use the definition of $\cosh x$ in terms of exponentials to show that

$$2 \cosh 5x \cosh x \equiv \cosh 6x + \cosh 4x \quad (2)$$

- (b) Hence determine the exact values of x for which

$$\cosh 6x + \cosh 4x = 8 \cosh x$$

giving your answers in terms of natural logarithms in simplest form. (4)

Question 1 continued

Lined area for writing the answer to Question 1.

(Total for Question 1 is 6 marks)



- (b) Hence determine the exact coordinates of the points of intersection of the curve with equation $y = \sinh 3x$ and the curve with equation $y = 19 \sinh x$, giving your answers as simplified logarithms where necessary.

(5)



Question 1 continued

Leave
blank

(Total 7 marks)

Q1



Question 1 continued

Lined area for writing the answer to Question 1 continued.



Question 1 continued

Lined area for writing the answer to Question 1 continued.



Question 1 continued

Lined area for writing the answer to Question 1.

(Total for Question 1 is 7 marks)



Leave
blank

- 1 (a) Use the definitions of hyperbolic functions in terms of exponentials to prove that

$$8 \cosh^4 x = \cosh 4x + p \cosh 2x + q$$

where p and q are constants to be determined.

(3)

- (b) Hence, or otherwise, solve the equation

$$\cosh 4x - 17 \cosh 2x + 9 = 0$$

giving your answers in exact simplified form in terms of natural logarithms.

(5)



Leave
blank

Question 1 continued

Leave
blank



Question 1 continued

Leave
blank

(Total 8 marks)

Q1



Question 4 continued

Lined area for writing the answer to Question 4 continued.



Question 4 continued

Lined area for writing the answer to Question 4 continued.



Question 4 continued

Lined area for writing the answer to Question 4.

(Total for Question 4 is 9 marks)



MARK SCHEMES

official mark schemes, in the same order as the questions



Question Number	Scheme	Notes	Marks
1	$7 \cosh x + 3 \sinh x = 2e^x + 7 \Rightarrow$ $7 \left(\frac{e^x + e^{-x}}{2} \right) + 3 \left(\frac{e^x - e^{-x}}{2} \right) = 2e^x + 7$ $\left\{ \frac{7}{2}e^x + \frac{7}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} = 2e^x + 7 \right\}$	Substitutes at least one correct exponential form for either of the hyperbolic terms and achieves an equation in exponentials and constants alone	M1
	$\Rightarrow 7(e^{2x} + 1) + 3(e^{2x} - 1) = 4e^{2x} + 14e^x$ $\left\{ \Rightarrow 5e^{2x} + 2 = 2e^{2x} + 7e^x \right\}$	Multiplies through by e^x to obtain any equation that would form a 3TQ in e^x if like terms were collected	M1
	$\Rightarrow 6e^{2x} - 14e^x + 4 = 0 \quad \{3e^{2x} - 7e^x + 2 = 0\}$	A correct three term quadratic in e^x . Could be implied by a correct root even if terms have not been collected.	A1
	$\Rightarrow (3e^x - 1)(e^x - 2) = 0 \Rightarrow e^x = \dots$	Solves their 3TQ - usual rules. One correct root for their quadratic if no working. Ignore labelling of the roots even if e.g., “ x ” is used.	M1
	$x = \ln 2, \ln \frac{1}{3}$	Both correct and simplified but do not isw if there are other answers . Allow $-\ln \frac{1}{2}$ for $\ln 2$ and $-\ln 3$ or $\ln 3^{-1}$ for $\ln \frac{1}{3}$	A1
	Answer only is 0/5		Total 5
	Note that it is possible to multiply through by e^{-x} to form an equation in e^{-2x} , e^{-x} and constants. Score as main scheme, e.g., $\frac{7}{2}e^x + \frac{7}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} = 2e^x + 7$ $\Rightarrow \frac{7}{2} + \frac{7}{2}e^{-2x} + \frac{3}{2} - \frac{3}{2}e^{-2x} = 2 + 7e^{-x} \quad (\text{M1})$ $\Rightarrow 2e^{-2x} - 7e^{-x} + 3 = 0 \quad (\text{A1})$ $(2e^{-x} - 1)(e^{-x} - 3) = 0 \Rightarrow e^{-x} = \frac{1}{2}, 3 \quad (\text{M1})$ $\Rightarrow e^x = 2, \frac{1}{3} \Rightarrow x = \ln 2, \ln \frac{1}{3} \quad (\text{A1})$		

General Principles for Further Pure Mathematics Marking
(But note that specific mark schemes may sometimes override these general principles).

Method mark for solving 3 term quadratic:

1. Factorisation

$(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$

$(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$

2. Formula

Attempt to use the correct formula (with values for a, b and c).

3. Completing the square

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$

Method marks for differentiation and integration:

1. Differentiation

Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)

2. Integration

Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)



General Principles for Further Pure Mathematics Marking*(But note that specific mark schemes may sometimes override these general principles)***Method mark for solving 3 term quadratic:****1. Factorisation** $(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$ $(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$ **2. Formula**Attempt to use the correct formula (with values for a , b and c).**3. Completing the square**Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$ **Method marks for differentiation and integration:****1. Differentiation**Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)**2. Integration**Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)

General Principles for Further Pure Mathematics Marking

(But note that specific mark schemes may sometimes override these general principles)

Method mark for solving 3 term quadratic:**1. Factorisation**

$(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$

$(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$

2. Formula

Attempt to use the correct formula (with values for a , b and c).

3. Completing the square

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$

Method marks for differentiation and integration:**1. Differentiation**

Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)

2. Integration

Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)

Use of a formula

Where a method involves using a formula that has been learnt, the advice given in recent examiners' reports is that the formula should be quoted first.

Normal marking procedure is as follows:

Method mark for quoting a correct formula and attempting to use it, even if there are small errors in the substitution of values.

Where the formula is not quoted, the method mark can be gained by implication from correct working with values but may be lost if there is any mistake in the working.

Exact answers

Examiners' reports have emphasised that where, for example, an exact answer is asked for, or working with surds is clearly required, marks will normally be lost if the candidate resorts to using rounded decimals.



Question Number	Scheme	Notes	Marks
1(a)	$\frac{dy}{dx} = 3\arcsin 2x + 3x \frac{1}{\sqrt{1-(2x)^2}} \times 2$ $\left(= 3\arcsin 2x + \frac{6x}{\sqrt{1-4x^2}} \right)$	M1: Obtains $p \arcsin qx + \frac{rx}{\sqrt{1-(sx)^2}} \text{ or }$ $p \arcsin qx + \frac{rx}{\sqrt{1-tx^2}}$ $p, q, r, s, t > 0$ A1: Correct derivative. Allow unsimplified and isw. Allow $\sin^{-1}$ and condone “arsin” but “arsinh” or “arcsinh” is M0	M1 A1
(b)	$x = \frac{1}{4} \Rightarrow \frac{dy}{dx} = \frac{\pi}{2} + \sqrt{3}$	$\frac{\pi}{2} + \sqrt{3}$ only but allow $\frac{1}{2}\pi$ or 0.5π. Terms as a sum in either order. Allow $a = \frac{1}{2}, b = \sqrt{3}$ Isw following a correct answer.	B1dep
	This is a “Hence” question so this mark can only be awarded following full marks in part (a)		
			Total 3

Question	Scheme	Marks
1	$2 \sinh^2 x + 3 \cosh x = 7 \Rightarrow 2(\cosh^2 x - 1) + 3 \cosh x = 7$	M1
	$\Rightarrow 2 \cosh^2 x + 3 \cosh x - 9 = 0$	A1
	$\Rightarrow (2 \cosh x - 3)(\cosh x + 3) = 0 \Rightarrow \cosh x = \frac{3}{2}, \cancel{>3}$	M1
	$\Rightarrow x = \operatorname{arcosh} \frac{3}{2} = (\pm) \ln \left(\frac{3}{2} + \sqrt{\frac{9}{4} - 1} \right)$	M1
	$x = \ln \left(\frac{3}{2} + \frac{\sqrt{5}}{2} \right), \ln \left(\frac{3}{2} - \frac{\sqrt{5}}{2} \right)$	A1A1
		(6)
Alt	$2 \sinh^2 x + 3 \cosh x = 7 \Rightarrow 2 \left(\frac{e^x - e^{-x}}{2} \right)^2 + 3 \left(\frac{e^x + e^{-x}}{2} \right) = 7$	M1
	$\Rightarrow e^{4x} + 3e^{3x} - 16e^{2x} + 3e^x + 1 = 0$	A1
	$\Rightarrow (e^{2x} + 6e^x + 1)(e^{2x} - 3e^x + 1) = 0 \Rightarrow e^x = \dots$	M1
	$e^x = \cancel{-3 \pm 2\sqrt{2}}, \frac{3}{2} \pm \frac{\sqrt{5}}{2} \Rightarrow x = \ln \left(\frac{3}{2} \pm \frac{\sqrt{5}}{2} \right)$	M1
	$x = \ln \left(\frac{3}{2} + \frac{\sqrt{5}}{2} \right), \ln \left(\frac{3}{2} - \frac{\sqrt{5}}{2} \right)$	A1A1
		(6)
(6 marks)		

Notes:

M1: Attempts to use the correct identity $\cosh^2 x - \sinh^2 x = 1$ to form an equation in $\cosh x$ only. (Allow slips rearranging a correctly stated identity.)

A1: Correct 3 term quadratic in $\cosh x$. The “=0” may be implied by their attempt to solve.

M1: Solves the 3 term quadratic (usual rules) leading to at least one exact real value for $\cosh x$ (which may be negative). The negative value may not be seen, but need not be rejected at this stage. If using a calculator they must state at least one correct root for their equation.

M1: Forms a solution in terms of natural logarithms from a positive solution to their equation using the correct formula or via using the correct exponential form and solving a 3 term quadratic in e^x .

A1: One value correct.

A1: Both values correct, accepting equivalents, and no others. Equivalents are e.g.

$$x = \pm \ln \left(\frac{3}{2} + \frac{\sqrt{5}}{2} \right), \pm \ln \left(\frac{3}{2} - \frac{\sqrt{5}}{2} \right) \quad \text{Condone missing brackets.}$$

Alt

M1: Substitutes **correct** definitions for $\sinh x$ and $\cosh x$ in terms of exponentials.

A1: Correct simplified quartic in e^x .

M1: Solves their 5 term quartic as far as $e^x = \dots$ to achieve exact values (at least one). There should be a recognisable attempt at solving, e.g. via a product of two quadratics, with at least first and last terms



B1: States or identifies the point B is $(0, -b)$ and substitutes into the normal equation to obtain the correct equation shown or equivalent.

M1: Correct attempt at the area and sets equal to $\frac{3}{4}b^2$ to form a second equation. Note there are other possible methods for the area of the triangle e.g. “shoelace”/determinant method, or use of separate triangles. Some will essentially give the same result as in the scheme, but others will give more complicated formulae, making the next M much more challenging (and unlikely to be gained).

dm1: Solves the equations simultaneously to achieve an equation in θ only. Alternatively, eliminates $\sin \theta$ and forms a polynomial equation in a^2 and b^2 . This latter is unlikely to make further progress.

ddM1: Rearranges to an expanded (but not necessarily fully simplified) cubic in $\sin \theta$. For equivalent working in a^2 or b^2 they must proceed as far as finding a relationship between a^2 and b^2 .

A1*: Fully correct work, showing the factorised cubic to justify the given value of $\sin \theta$. No need to show there is only one solution (we allow the factorised to imply the quadratic factor does not reduce). Note substitution of $\sin \theta = \frac{1}{2}$ into the equation is A0 as it provides no evidence of no other solutions.

For working in a and b the factorisation to deduce the relationship between a^2 and b^2 must have been seen.

(c)

M1: Correct process to find b in terms of a

B1: x coordinate correct

A1: y coordinate correct, accept $\frac{a}{2\sqrt{3}}$.

(b) Working in a and b :

$$\sin^2 \theta + \cos^2 \theta = 1 \Rightarrow \frac{b^4}{(b^2 - a^2)^2} + \frac{9b^2}{4a^2} = 1 \Rightarrow 4a^2b^4 + 9b^2(b^2 - a^2)^2 = 4a^2(b^2 - a^2)^2 \quad \text{dM1}$$

$$\Rightarrow 9b^6 - 18a^2b^4 + 17a^4b^2 - 4a^6 = 0 \Rightarrow (3b^2 - a^2)(3b^4 - 5a^2b^2 + 4a^4) = 0$$

$$\Rightarrow a^2 = 3b^2 \quad \text{ddM1}$$

$$\Rightarrow \sin \theta = -\frac{b^2}{b^2 - 3b^2} = -\frac{1}{-2} = \frac{1}{2} \quad \text{A1}$$

Note: A lot of solutions are not using that the coordinate of B is $(0, -b)$ but instead use $\sin \theta = \frac{1}{2}$ to

show $\sin \theta = \frac{1}{2}$. These will score at most B0M1M0M0A0. as they have no equations to solve simultaneously.

Question	Scheme	Marks
8	$y = e^{3x} \cosh 2x$ (See notes for Alt via $n = 0$ for M1A1) $\Rightarrow \frac{dy}{dx} = 3e^{3x} \cosh 2x + e^{3x} \times 2 \sinh 2x$ or $e^{3x} (3 \cosh 2x + 2 \sinh 2x)$	M1
	$= e^{3x} \left(\frac{5+1}{2} \cosh 2x + \frac{5-1}{2} \sinh 2x \right)$ (so the result is true for $n = 1$)	A1
	(Assume the result is true for $n = k$, then) $\frac{d^{k+1}y}{dx^{k+1}} = 3e^{3x} \left(\frac{5^k+1}{2} \cosh 2x + \frac{5^k-1}{2} \sinh 2x \right) + e^{3x} ((5^k+1) \sinh 2x + (5^k-1) \cosh 2x)$	M1
	$= \left(3 \times \frac{5^k+1}{2} + 5^k - 1 \right) e^{3x} \cosh 2x + \left(3 \times \frac{5^k-1}{2} + 5^k + 1 \right) e^{3x} \sinh 2x$ $= e^{3x} \left(\frac{5 \times 5^k + 1}{2} \cosh 2x + \frac{5 \times 5^k - 1}{2} \sinh 2x \right)$	dM1
	$= e^{3x} \left(\frac{5^{k+1}+1}{2} \cosh 2x + \frac{5^{k+1}-1}{2} \sinh 2x \right)$	A1
	Hence the result is also true for $n = k + 1$, so <u>if true for $n = k$ then true for $n = k + 1$</u> , and as also <u>true for $n = 1$</u> , so the result is <u>true for all positive integers</u> .	A1
		(6)
(6 marks)		

Notes:

Note: Condone use of n instead of k throughout the inductive step. If reasoned correctly full marks are available. Also note that going from $n = k - 1$ to $n = k$ is a correct method.

Note: Apply benefit of the doubt if poor handwriting makes it difficult to distinguish between 5^k and $5k$ in some places (ie powers written too lower) if they are treated correctly in subsequent steps. This will not be counted as errors affecting the A's.

M1: Attempts the first derivative - form must be correct, but coefficients may be incorrect. Alternatively, allow for substitution of $n = 0$ into the right hand side – accept as minimum seeing

$$e^{3x} \left(\frac{1+1}{2} \cosh 2x + \frac{1-1}{2} \sinh 2x \right).$$

A1: Correct derivative and reaches appropriate form, either by factoring out the e^{3x} or expanding the RHS of result for $n = 1$ and comparing, to be able to deduce the result is true for $n = 1$ (which may be tacit for this mark). If expanding the RHS the substitution of the $n = 1$ should be clear.

Either way a minimum of $e^{3x} \left(\frac{5+1}{2} \cosh 2x + \frac{5-1}{2} \sinh 2x \right)$ must be seen in their working in order to award this mark.

Alternatively, substituting $n = 0$, proceeds from the minimum shown to reach $e^{3x} \cosh 2x = y$ to establish true for $n = 0$.

M1: (Makes the inductive assumption and) attempts the $(k + 1)$ -th derivative from the k -th derivative (but see first note above). Allow slips in coefficients.

dM1: Gathers the $\sinh 2x$ and $\cosh 2x$ terms and factors out these terms to unsimplified coefficients. The e^{3x} may or may not be factored out, or missing, or incorrect at this stage. Accept either form shown (or equivalent unsimplified coefficients). There must be a step with unsimplified gathered coefficients between the differentiation and final line shown in the scheme to score this mark.

A1: Reaches the correct form from correct work. The e^{3x} must be factored out, or expansion of the RHS to demonstrate equivalents seen. Must have the “ $k+1$ ” showing. Depends on the previous two method marks.

A1: Makes appropriate concluding sentence covering the points indicated in scheme at some stage in the proof. Depends on all method marks having been scored and all work being correct. Must have reached at least the second line shown in the dM mark, and made an attempt at checking $n=1$ (though the first A mark need not have been scored if insufficient detail shown). Note M1A0M1dM1A0A1 is possible if the previous A marks have only been withheld due to the lack of sufficient detail shown but the working and intent is clear (e.g only reaching the final line of the dM box).

8 Alt via exponentials	$y = e^{3x} \cosh 2x = \frac{e^{5x} + e^x}{2} \Rightarrow \frac{dy}{dx} = \frac{5e^{5x} + e^x}{2}$	M1
	$= e^{3x} \left(\frac{5e^{2x} + e^{-2x}}{2} \right) = e^{3x} \left(\frac{(5+1)(e^{2x} + e^{-2x}) + (5-1)(e^{2x} - e^{-2x})}{4} \right)$ $= e^{3x} \left(\frac{5+1}{2} \cosh 2x + \frac{5-1}{2} \sinh 2x \right)$ so the result is true for $n = 1$	A1
	(Assume the result is true for $n = k$, so $\frac{d^k y}{dx^k} = e^{3x} \left(\frac{5^k + 1}{2} \frac{e^{2x} + e^{-2x}}{2} + \frac{5^k - 1}{2} \frac{e^{2x} - e^{-2x}}{2} \right) = \frac{5^k}{2} e^{5x} + \frac{1}{2} e^x \text{ then}$ $\frac{d^{k+1} y}{dx^{k+1}} = \frac{5^k}{2} \times 5e^{5x} + \frac{1}{2} e^x$	M1
	$= \frac{5^{k+1}}{2} e^{5x} + \frac{1}{2} e^x = \frac{e^{3x}}{4} (2 \times 5^{k+1} e^{2x} + 2e^{-2x})$ $= \frac{e^{3x}}{4} (5^{k+1} e^{2x} + 5^{k+1} e^{-2x} + 5^{k+1} e^{2x} - 5^{k+1} e^{-2x} + e^{2x} + e^{-2x} + e^{-2x} - e^{2x})$ $= \frac{e^{3x}}{2} (5^{k+1} \cosh 2x + 5^{k+1} \sinh 2x + \cosh 2x - \sinh 2x)$	dM1
	$= e^{3x} \left(\frac{5^{k+1} + 1}{2} \cosh 2x + \frac{5^{k+1} - 1}{2} \sinh 2x \right)$	A1
	Hence the result is also true for $n = k + 1$, so <u>if true for $n = k$ then true for $n = k + 1$</u> , and as also <u>true for $n = 1$</u> , so the result is <u>true for all positive integers</u> .	A1
		(6)
(6 marks)		
Notes:		

M1: Replaces by the correct exponential forms and differentiates - form must be correct, but coefficients may be incorrect.

A1: Correct derivative and reaches appropriate form to deduce the result is true for $n = 1$ (which may be tacit for this mark). If expanding the RHS the substitution of the $n = 1$ should be clear. The same minimum as main scheme is expected except that the $\cosh 2x$ and $\sinh 2x$ may have been replaced first.

M1: (Makes the inductive assumption and) attempts the $(k+1)$ -th derivative from the k -th derivative. Allow slips in coefficients.

dM1: Attempts to introduce the extra terms to establish the $\cosh 2x$ and $\sinh 2x$ in the expression. Allow slips, but the intention must be clear. Alternatively, considers the right hand side for $k+1$, substitutes exponential forms and expands and simplifies.

A1: Fully correct work to establish the inductive step. The final expression must be reached (with the intermediate dM1 gained) from correct work, or have been used to work back to matching expressions.

A1: Makes appropriate concluding sentence covering the points indicated in scheme at some stage in the proof. Depends on all method marks having been scored and all work being correct. Must have reached at least the final line shown in the dM mark or equivalent work in reverse, and made an attempt at checking $n=1$ (though the first A mark need not have been scored if insufficient detail shown).

Note M1A0M1dM1A0A1 is possible if the previous A marks have only been withheld due to the lack of sufficient detail shown but the working and intent is clear (e.g only reaching the final line of the dM box).

Note: Some attempts may try to use integration to work backwards from $\frac{d^{k+1}y}{dx^{k+1}}$ to $\frac{d^k y}{dx^k}$. Such methods may use hyperbolics or exponentials and the scheme will apply similarly.

M1A1: as per main scheme.

M1: Considers result for $n = k+1$ and attempts to integrate twice by parts, achieving correct form.

Let $A = \frac{5^{k+1}+1}{2} \cosh 2x + \frac{5^{k+1}-1}{2} \sinh 2x$, $B = (5^{k+1}+1) \sinh 2x + (5^{k+1}-1) \cosh 2x$

$$I = \int \frac{d^{k+1}y}{dx^{k+1}} dx = \frac{1}{3} e^{3x} (A) - \frac{1}{3} \int e^{3x} B dx = \frac{1}{3} e^{3x} (A) - \frac{1}{9} e^{3x} B + \frac{4}{9} \int e^{3x} A dx$$

Or in exponentials

$$\int \frac{d^{k+1}y}{dx^{k+1}} dx = \int \left(\frac{5^{k+1}+1}{2} \frac{e^{5x}+e^x}{2} + \frac{5^{k+1}-1}{2} \frac{e^{5x}-e^x}{2} \right) dx = \frac{5^{k+1}+1}{2} \frac{1}{5} \frac{e^{5x}+e^x}{2} + \frac{5^{k+1}-1}{2} \frac{1}{5} \frac{e^{5x}-e^x}{2}$$

dM1: Makes I the subject

$$\Rightarrow 9I = 3e^{3x} (A) - e^{3x} B + 4I \Rightarrow I = \frac{1}{5} e^{3x} (3A - B)$$

$$\text{Or: } = \frac{e^{3x}}{5} \left(\frac{5^{k+1}+1}{2} \frac{e^{2x}+5e^{-2x}}{2} + \frac{5^{k+1}-1}{2} \frac{e^{2x}-5e^{-2x}}{2} \right) = \frac{e^{3x}}{5} \left(\frac{2 \times 5^{k+1} e^{2x} + 10e^{-2x}}{4} \right) \text{ leading to identifying} \\ \dots (\text{reverse work of exponentials in main scheme})$$

exponential forms for the hyperbolics.

A1: Achieves the correct result and gives due consideration to the constant of integration.

$$\Rightarrow \frac{d^k y}{dx^k} = I = \frac{1}{5} e^{3x} \left(\frac{5^{k+1}+5}{2} \cosh 2x + \frac{5^{k+1}-5}{2} \sinh 2x \right) = e^{3x} \left(\frac{5^k+1}{2} \cosh 2x + \frac{5^k-1}{2} \sinh 2x \right)$$

A1: Per main scheme (except allow if constant of integration is not considered).

If you are unsure in a particular case use the review system.

Question Number	Scheme	Marks
	There is no credit for attempts that do not use exponential definitions	
1(a) Working from LHS to RHS	$(2 \cosh 5x \cosh x) = 2 \frac{e^{5x} + e^{-5x}}{2} \times \frac{e^x + e^{-x}}{2} = \frac{e^{6x} + e^{4x} + e^{-4x} + e^{-6x}}{2}$	M1
	$= \frac{e^{6x} + e^{-6x}}{2} + \frac{e^{4x} + e^{-4x}}{2} = \cosh 6x + \cosh 4x^*$	A1*
		(2)
ALT Working from RHS to LHS	$\cosh 6x + \cosh 4x = \frac{e^{6x} + e^{-6x}}{2} + \frac{e^{4x} + e^{-4x}}{2} = \frac{e^{6x} + e^{4x} + e^{-4x} + e^{-6x}}{2} = 2 \frac{e^{5x} + e^{-5x}}{2} \times \frac{e^x + e^{-x}}{2}$	M1
	$= 2 \frac{e^{5x} + e^{-5x}}{2} \times \frac{e^x + e^{-x}}{2} = 2 \cosh 5x \cosh x$	A1*
(b)	$\cosh 6x + \cosh 4x = 8 \cosh x \Rightarrow 2 \cosh 5x \cosh x = 8 \cosh x$ $\Rightarrow 2 \cosh x (\cosh 5x - 4) = 0 \Rightarrow \cosh 5x = \dots \text{ or } 2 \cosh 5x = \dots$	M1
	$\cosh 5x = 4 \text{ or } 2 \cosh 5x = 8$	A1
	$\cosh 5x = 4 \Rightarrow 5x = \ln(4 + \sqrt{4^2 - 1})$ or $\cosh 5x = 4 \Rightarrow \frac{e^{5x} + e^{-5x}}{2} = 4 \Rightarrow e^{10x} - 8e^{5x} + 1 = 0 \Rightarrow e^{5x} = \frac{8 \pm \sqrt{60}}{2}$	M1
	$x = \pm \frac{1}{5} \ln(4 + \sqrt{15}) \text{ or } x = \frac{1}{5} \ln(4 \pm \sqrt{15})$	A1
		(4)
		Total 6

General Principles for Further Pure Mathematics Marking
(But note that specific mark schemes may sometimes override these general principles).

Method mark for solving 3 term quadratic:

1. Factorisation

$(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$

$(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$

2. Formula

Attempt to use the correct formula (with values for a, b and c).

3. Completing the square

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$

Method marks for differentiation and integration:

1. Differentiation

Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)

2. Integration

Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)

General Principles for Further Pure Mathematics Marking

(But note that specific mark schemes may sometimes override these general principles)

Method mark for solving 3 term quadratic:**1. Factorisation**

$(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$

$(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$

2. Formula

Attempt to use the correct formula (with values for a , b and c).

3. Completing the square

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$

Method marks for differentiation and integration:**1. Differentiation**

Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)

2. Integration

Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)

Use of a formula

Where a method involves using a formula that has been learnt, the advice given in recent examiners' reports is that the formula should be quoted first.

Normal marking procedure is as follows:

Method mark for quoting a correct formula and attempting to use it, even if there are small errors in the substitution of values.

Where the formula is not quoted, the method mark can be gained by implication from correct working with values but may be lost if there is any mistake in the working.

Exact answers

Examiners' reports have emphasised that where, for example, an exact answer is asked for, or working with surds is clearly required, marks will normally be lost if the candidate resorts to using rounded decimals.

Question Number	Scheme	Notes	Marks
1(a)	$8 \cosh^4 x = 8 \left(\frac{e^x + e^{-x}}{2} \right)^4 = \frac{8}{16} (e^{4x} + 4e^{2x} + 6 + 4e^{-2x} + e^{-4x})$ Applies $\cosh x = \frac{e^x + e^{-x}}{2}$ and attempts to expand the bracket to at least 4 different and no more than 5 different terms of the correct form but they may be “uncollected” depending on how they do the expansion. Allow unsimplified terms e.g. $(e^x)^3 e^{-x}$. May see $8 \left(\frac{e^x + e^{-x}}{2} \right)^2 \left(\frac{e^x + e^{-x}}{2} \right)^2$ but must attempt to expand as above		M1
	$= \frac{1}{2} (e^{4x} + e^{-4x}) + 4 \left(\frac{e^{2x} + e^{-2x}}{2} \right) + 3 = \dots$	Collects appropriate terms and reaches the form $\cosh 4x + p \cosh 2x + q$ or obtains values of p and q .	M1
	$= \cosh 4x + 4 \cosh 2x + 3$	Correct expression or values e.g. $p = 4$ and $q = 3$	A1
	No marks are available in (a) if exponentials are not used but note that they may appear in combination with the use of hyperbolic identities e.g.:		
	$\begin{aligned} 8 \cosh^4 x &= 8 (\cosh^2 x)^2 = 8 \left(\frac{\cosh 2x + 1}{2} \right)^2 = 2 \left(\frac{e^{2x} + e^{-2x}}{2} + 1 \right)^2 \\ &= 2 \left(\frac{e^{4x} + 2 + e^{-4x}}{4} + e^{2x} + e^{-2x} + 1 \right) = \frac{e^{4x} + e^{-4x}}{2} + 4 \left(\frac{e^{2x} + e^{-2x}}{2} \right) + 2 \\ &= \cosh 4x + 4 \cosh 2x + 3 \end{aligned}$ Allow to “meet in the middle” e.g. expands as above and compares with $\frac{1}{2} (e^{4x} + e^{-4x}) + p \left(\frac{e^{2x} + e^{-2x}}{2} \right) + q \Rightarrow p = \dots, q = \dots$ but to score any marks the expansion must be attempted.		
			(3)

(b) Way 1	$\cosh 4x - 17 \cosh 2x + 9 = 0 \Rightarrow 8 \cosh^4 x - 4 \cosh 2x - 3 - 17 \cosh 2x + 9 = 0$ $\Rightarrow 8 \cosh^4 x - 21 \cosh 2x + 6 = 0 \Rightarrow 8 \cosh^4 x - 21(2 \cosh^2 x - 1) + 6 = 0$ Uses their result from part (a) and $\cosh 2x = \pm 2 \cosh^2 x \pm 1$ to obtain a quadratic equation in $\cosh^2 x$ or $\cosh 4x - 17 \cosh 2x + 9 = 0 \Rightarrow 2(2 \cosh^2 x - 1)^2 - 1 - 17(2 \cosh^2 x - 1) + 9 = 0$ Uses $\cosh 4x = \pm 2 \cosh^2 2x \pm 1$ and $\cosh 2x = \pm 2 \cosh^2 x \pm 1$ to obtain a quadratic equation in $\cosh^2 x$	M1
	$\Rightarrow 8 \cosh^4 x - 42 \cosh^2 x + 27 = 0$	A1
	$\Rightarrow 8 \cosh^4 x - 42 \cosh^2 x + 27 = 0$ $\Rightarrow \cosh^2 x = \frac{9}{2} \left(\frac{3}{4} \right)$	M1
	$\cosh^2 x = \frac{9}{2} \Rightarrow \cosh x = \frac{3}{\sqrt{2}} \Rightarrow x = \pm \ln \left(\frac{3}{\sqrt{2}} + \sqrt{\frac{9}{2} - 1} \right)$ or $\cosh x = \frac{3}{\sqrt{2}} \Rightarrow \frac{e^x + e^{-x}}{2} = \frac{3}{\sqrt{2}} \Rightarrow \sqrt{2}e^{2x} - 6e^x + \sqrt{2} = 0 \Rightarrow e^x = \dots \Rightarrow x = \dots$ or $\cosh^2 x = \frac{9}{2} \Rightarrow \left(\frac{e^x + e^{-x}}{2} \right)^2 = \frac{9}{2} \Rightarrow e^{4x} - 16e^{2x} + 1 = 0 \Rightarrow e^{2x} = \dots \Rightarrow x = \dots$ Takes square root to obtain $\cosh x = k$ ($k > 1$) and applies the correct logarithmic form for arcosh or uses the correct exponential form for $\cosh x$ to obtain at least one value for x The root(s) must be real to score this mark.	M1
	$x = \pm \ln \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{14}}{2} \right)$ Both correct and exact including brackets. Accept simplified equivalents e.g. $x = \ln \left(\frac{3}{\sqrt{2}} \pm \frac{\sqrt{7}}{\sqrt{2}} \right)$ but withhold this mark if additional answers are given unless they are the same e.g. allow $x = \pm \ln \left(\frac{3\sqrt{2}}{2} \pm \frac{\sqrt{14}}{2} \right)$	A1
		(5)

(b) Way 2	$\cosh 4x - 17 \cosh 2x + 9 = 0 \Rightarrow 2 \cosh^2 2x - 1 - 17 \cosh 2x + 9 = 0$ Applies $\cosh 4x = \pm 2 \cosh^2 2x \pm 1$ to obtain a quadratic equation in $\cosh 2x$	M1
	$2 \cosh^2 2x - 17 \cosh 2x + 8 = 0$	A1
	$2 \cosh^2 2x - 17 \cosh 2x + 8 = 0$ $\Rightarrow \cosh 2x = 8 \left(\frac{1}{2} \right)$	M1
	$\cosh 2x = 8 \Rightarrow 2x = \pm \ln(8 + \sqrt{8^2 - 1})$ or $\cosh 2x = 8 \Rightarrow \frac{e^{2x} + e^{-2x}}{2} = 8 \Rightarrow e^{4x} - 16e^{2x} + 1 = 0 \Rightarrow e^{2x} = \dots \Rightarrow 2x = \dots$ Applies the correct logarithmic form for arcosh from $\cosh 2x = k$ ($k > 1$) or uses the correct exponential form for $\cosh 2x$ to obtain at least one value for $2x$ The root(s) must be real to score this mark.	M1
	$x = \pm \frac{1}{2} \ln(8 + 3\sqrt{7})$ or e.g. $x = \pm \ln(8 + 3\sqrt{7})^{\frac{1}{2}}$	A1
(b) Way 3	$\cosh 4x - 17 \cosh 2x + 9 = 0 \Rightarrow \frac{e^{4x} + e^{-4x}}{2} - \frac{17}{2}(e^{2x} + e^{-2x}) + 9 = 0$ $\Rightarrow e^{8x} - 17e^{6x} + 18e^{4x} - 17e^{2x} + 1 = 0$ M1: Applies the correct exponential forms and attempts a quartic equation in e^{2x} A1: Correct equation	M1A1
	$e^{8x} - 17e^{6x} + 18e^{4x} - 17e^{2x} + 1 = 0$ $\Rightarrow e^{2x} = 8 \pm 3\sqrt{7}, \dots$	M1
	$\Rightarrow e^{2x} = 8 \pm 3\sqrt{7} \Rightarrow 2x = \ln(8 \pm 3\sqrt{7})$	M1
	$x = \frac{1}{2} \ln(8 \pm 3\sqrt{7})$ or e.g. $x = \ln(8 \pm 3\sqrt{7})^{\frac{1}{2}}$	A1
		Total 8

Question Number	Scheme	Notes	Marks
2(a)	$\det(\mathbf{M} - \lambda \mathbf{I}) = \begin{vmatrix} 2 - \lambda & 0 & 3 \\ 0 & -4 - \lambda & -3 \\ 0 & -4 & -\lambda \end{vmatrix}$ $= \text{e.g., } (2 - \lambda)[(-4 - \lambda)(-\lambda) - (-4)(-3)] - 0 + 3(0)$ $\text{or } (2 - \lambda)[(-4 - \lambda)(-\lambda) - (-4)(-3)] - 0 + 0$ $\text{Sarrus} \Rightarrow (2 - \lambda)(-4 - \lambda)(-\lambda) - (2 - \lambda)(-3)(-4)$	Obtains an unsimplified cubic expression for $\det(\mathbf{M} - \lambda \mathbf{I})$ condoning sign/copying slips only. Allow poor bracketing if intention clear.	M1
	Note: It is possible to just use $\mathbf{M}\mathbf{x} = \lambda\mathbf{x}$ e.g., $-4y = \lambda z \Rightarrow y = -\frac{\lambda z}{4} \text{ and } -4y - 3z = \lambda y \Rightarrow \lambda z - 3z = -\frac{\lambda^2 z}{4} \Rightarrow \lambda^2 + 4\lambda - 12 = 0 \Rightarrow \dots$ Score the M1 for achieving a 3TQ in λ from appropriate work condoning copying/sign slips only		
	$(2 - \lambda)(\lambda^2 + 4\lambda - 12) = 0 \text{ or } \lambda^3 + 2\lambda^2 - 20\lambda + 24 = 0 \text{ or } -\lambda^3 - 2\lambda^2 + 20\lambda - 24 = 0$ $(2 - \lambda)(\lambda - 2)(\lambda + 6) = 0 \text{ or } (\lambda + 6)(\lambda - 2)(\lambda - 2) = 0$ $\lambda_1 = -6 \quad (\lambda_2 = 2)$ dM1: Solves $\det(\mathbf{M} - \lambda \mathbf{I}) = 0$ to obtain any value for λ including 2. Not usual rules – award for any value seen that is consistent with their equation. The “=0” can be implied by a solution. Note that they may disregard the $(2 - \lambda)$ and solve a quadratic. A1: –6 from a correct equation. Accept both solutions e.g., “–6, 2” and allow if mislabelled and/or –6 rejected. No incorrect solutions.		dM1 A1
	$2x + 3z = -6x$ $\mathbf{M}\mathbf{x} = -6\mathbf{x} \Rightarrow -4y - 3z = -6y \Rightarrow x = \dots, y = \dots, z = \dots$ $-4y = -6z$ $8x + 3z = 0$ $(\mathbf{M} + 6\mathbf{I})\mathbf{x} = \mathbf{0} \Rightarrow 2y - 3z = 0 \Rightarrow x = \dots, y = \dots, z = \dots$ $-4y + 6z = 0$	Uses $\mathbf{M}\mathbf{x} = \lambda\mathbf{x}$ or $(\mathbf{M} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$ with any of their non-zero eigenvalues (however obtained) to form simultaneous equations and solves. No requirement for a vector for this mark. There is no need to check their values but award M0 for a zero solution.	M1
	Note: Could find vector product of first 2 rows of $\mathbf{M} - \lambda \mathbf{I}$ i.e., $(8\mathbf{i} + 3\mathbf{k}) \times (2\mathbf{j} - 3\mathbf{k}) = (-6\mathbf{i} + 24\mathbf{j} + 16\mathbf{k}) \quad (\text{two correct components})$		
	$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \\ 8 \end{pmatrix} \Rightarrow \frac{1}{\sqrt{3^2 + 12^2 + 8^2}} \begin{pmatrix} -3 \\ 12 \\ 8 \end{pmatrix}$	Correct method to normalise their eigenvector no matter how this vector is obtained provided it has at least 2 non-zero components. Only allow slips if there is working.	M1
	$\text{e.g., } \frac{1}{\sqrt{217}} \begin{pmatrix} -3 \\ 12 \\ 8 \end{pmatrix} \text{ or } \begin{pmatrix} -\frac{3\sqrt{217}}{217} \\ \frac{12\sqrt{217}}{217} \\ \frac{8\sqrt{217}}{217} \end{pmatrix} \text{ or } \begin{pmatrix} -\frac{3}{\sqrt{217}} \\ \frac{12}{\sqrt{217}} \\ \frac{8}{\sqrt{217}} \end{pmatrix} \text{ or } \frac{1}{2\sqrt{217}} \begin{pmatrix} -6 \\ 24 \\ 16 \end{pmatrix}$	A correct normalised eigenvector in any form. Note direction may be reversed. May use i, j, k notation	A1

(6)



Question Number	Scheme	Notes	Marks
2(b)	May use i, j, k notation		
	Multiplies position and direction by M (not e.g., M – $\lambda \mathbf{I}$) In parametric form: $\begin{pmatrix} 2 & 0 & 3 \\ 0 & -4 & -3 \\ 0 & -4 & 0 \end{pmatrix} \begin{pmatrix} 4+2\mu \\ -1 \\ -\mu \end{pmatrix} = \dots \quad \left\{ \begin{pmatrix} 8+4\mu-3\mu \\ 4+3\mu \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}$ There is no requirement to extract the vectors if parametric form is used. Allow this mark if e.g., $8+4\mu-3\mu$ written as $2(4+2\mu)-3\mu$ Allow this work without a parameter i.e., $\begin{pmatrix} 2 & 0 & 3 \\ 0 & -4 & -3 \\ 0 & -4 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} = \dots \quad \left\{ \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix} \right\} \text{ and } \begin{pmatrix} 2 & 0 & 3 \\ 0 & -4 & -3 \\ 0 & -4 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \dots \quad \left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}$ or $\begin{pmatrix} 2 & 0 & 3 \\ 0 & -4 & -3 \\ 0 & -4 & 0 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} = \dots \quad \left\{ \begin{pmatrix} 8 & 1 \\ 4 & 3 \\ 4 & 0 \end{pmatrix} \right\}$ Alternatively: Could find 2 points on l_1, transform them both and subtract to find direction. Allow slips and condone the matrix product written the wrong way round provided they have attempted to multiply the elements appropriately and they obtain a vector (or 3×2 matrix) with the resulting values correctly placed. Condone if they proceed to confuse which is the position and which is the direction.		M1
	$\mathbf{r} \times \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$	Forms: $\mathbf{r} \times \text{direction} = \text{position} \times \text{direction}$ Must not clearly confuse their vectors. Allow if RHS = direction $\times$ position. Requires previous M mark. No requirement to calculate vector product but the RHS could be implied by 2 correct components (or the negative version if the product is reversed)	dM1
	$\mathbf{r} \times \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -12 \\ 4 \\ 20 \end{pmatrix}$	Any correct equation in the correct form. Not $\mathbf{b} = \dots$, $\mathbf{c} = \dots$ unless $\mathbf{r} \times \mathbf{b} = \mathbf{c}$ seen. Isw once a correct answer is seen.	A1
(3)			
Total 9			

Question Number	Scheme	Notes	Marks
3(a)	$y = \operatorname{arsinh}(\sqrt{x^2 - 1})$		
	For all Ways allow the final answer to be written as $\frac{1}{(x^2 - 1)^{\frac{1}{2}}}$ or $(x^2 - 1)^{-\frac{1}{2}}$		
Way 1	$\frac{dy}{dx} = \frac{1}{\sqrt{1 + (\sqrt{x^2 - 1})^2}} \times \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}}(2x)$		M1 A1
	M1: Obtains $\frac{1}{\sqrt{1 + (\sqrt{x^2 - 1})^2}} \times f(x)$ or e.g., $\frac{1}{x} \times f(x)$ $f(x) \neq k$ A1: Fully correct unsimplified expression		
	$= \frac{1}{\sqrt{1 + x^2 - 1}} \times \frac{x}{\sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}} *$ or e.g., $= \frac{1}{x} \times \frac{x}{\sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}} *$	Correct completion with intermediate line of working and no errors	A1*
(3)			
Way 2 Takes sinh of both sides	$y = \operatorname{arsinh}(\sqrt{x^2 - 1}) \Rightarrow \sinh y = \sqrt{x^2 - 1} \Rightarrow \cosh y \frac{dy}{dx} = \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}}(2x)$		M1 A1
	M1: Takes sinh of both sides and differentiates to obtain $\cosh y \frac{dy}{dx} = f(x)$ $f(x) \neq k$ A1: Fully correct unsimplified equation		
	$\cosh y = \sqrt{1 + \sinh^2 y}$ or $\sqrt{1 + x^2 - 1} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}} *$	Correct completion with clear use of identity (must see more than just $\cosh y = x$) and no errors	A1*
(3)			
Way 3 Takes sinh & squares	$y = \operatorname{arsinh}(\sqrt{x^2 - 1}) \Rightarrow \sinh y = \sqrt{x^2 - 1} \Rightarrow \sinh^2 y = x^2 - 1 \Rightarrow 2 \sinh y \cosh y \frac{dy}{dx} = 2x$		M1 A1
	M1: Takes sinh of both sides, squares and differentiates to obtain $\cosh y \frac{dy}{dx} = f(x)$ $f(x) \neq k$ A1: Fully correct unsimplified expression or equation		
	$\cosh y = \sqrt{1 + \sinh^2 y}$ or $\sqrt{1 + x^2 - 1} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$	Correct completion with clear use of identity (must see more than just $\cosh y = x$) and no errors	A1*
(3)			
Way 4 Takes sinh & squares & uses identity	$\Rightarrow \sinh y = \sqrt{x^2 - 1} \Rightarrow \sinh^2 y = x^2 - 1 \Rightarrow \cosh^2 y = 1 + (x^2 - 1) \Rightarrow \cosh^2 y = x^2 \Rightarrow 2 \sinh y \cosh y \frac{dy}{dx} = 2x$		M1 A1
	M1: Takes sinh of both sides, squares, uses identity and differentiates to obtain $\cosh y \frac{dy}{dx} = f(x)$ $f(x) \neq k$ Allow sign errors with identity for the M mark. A1: Fully correct unsimplified expression or equation		
	$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$	Correct completion with clear use of identity and no errors	A1*
(3)			

Question Number	Scheme	Notes	Marks
3(a) Way 5 Takes sinh & squares & uses identity & roots	$\Rightarrow \sinh y = \sqrt{x^2 - 1} \Rightarrow \sinh^2 y = x^2 - 1 \Rightarrow \cosh^2 y = 1 + (x^2 - 1) \Rightarrow \cosh y = x \Rightarrow \sinh y \frac{dy}{dx} = 1$ M1: Takes sinh of both sides, squares, uses identity, roots and differentiates to obtain $\cosh y \frac{dy}{dx} = f(x)$ or k Allow sign errors with identity. A1: Fully correct unsimplified expression or equation		M1 A1
	$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$	Correct completion with clear use of identity and no errors	A1*
(3)			
Way 6 Uses log form of arsinh first	$y = \operatorname{arsinh}(\sqrt{x^2 - 1}) \Rightarrow y = \ln(\sqrt{x^2 - 1} + \sqrt{x^2 - 1 + 1}) = \ln(\sqrt{x^2 - 1} + x) \Rightarrow \frac{dy}{dx} = \frac{\frac{1}{2}(x^2 - 1)^{-\frac{1}{2}}(2x) + 1}{\sqrt{x^2 - 1} + x}$ M1: Use log form of arsinh correctly and differentiates to obtain $\frac{f(x) \neq k}{\sqrt{x^2 - 1} + x}$ A1: Fully correct unsimplified expression		M1 A1
	$= \frac{\frac{x}{\sqrt{x^2 - 1}} + 1}{\sqrt{x^2 - 1} + x} \text{ or } \frac{x + \sqrt{x^2 - 1}}{\sqrt{x^2 - 1}} \times \frac{1}{\sqrt{x^2 - 1} + x} = \frac{1}{\sqrt{x^2 - 1}} *$	Correct completion with intermediate line of working and no errors	A1*
(3)			
You may see other variations e.g., using exponential definitions, attempts via dx/dy. The M mark is for differentiating to obtain correct forms and the first A is awarded if it is correct. The final A is for correct completion.			

Question Number	Scheme	Notes	Marks
3(b)	$f(x) = \frac{1}{3} \operatorname{arsinh}(\sqrt{x^2-1}) - \arctan x$		
	$f'(x) = \frac{1}{3\sqrt{x^2-1}} - \frac{1}{1+x^2}$	$f'(x) = \frac{A}{\sqrt{x^2-1}} \pm \frac{1}{1 \pm x^2} \quad A = \frac{1}{3}, 3 \text{ or } 1$	M1 (B1 on ePen)
	$1+x^2 = 3\sqrt{x^2-1}$ $1+2x^2+x^4 = 9x^2-9$	Sets $\frac{A}{\sqrt{x^2-1}} \pm \frac{1}{1+x^2} = 0 \quad A = \frac{1}{3}, 3 \text{ or } 1$ Denominator of derivative of arctan x must now be $1+x^2$ Cross multiplies and squares to obtain the correct form for both sides so do not condone e.g., $(1+x^2)^2 = 1+x^4$ May see the quartic obtained through equivalent work.	M1
	$x^4 - 7x^2 + 10 = 0 \Rightarrow (x^2-2)(x^2-5) = 0 \Rightarrow x^2 = 2, 5$ Solves a 3TQ in x^2 (usual rules and one correct root if no working). No requirement to see the terms collected. Ignore labelling of solutions so allow e.g., " $x = 2, 5$ ". One correct value for their equation if no working, which may be for x or x^2 , so just look for the values. May change the variable. Allow for a correct solution with no working from solving a three term quartic of the correct form on a calculator. Allow if value for x^2 is negative or if roots are complex. Requires previous M marks.		ddM1
	$x = \sqrt{2}, \sqrt{5}$	Both exact and no other solutions e.g., $\pm$ is A0 unless negatives rejected. Must not reject either correct solution.	A1
			(4)
			Total 7