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EDEXCEL INTERNATIONAL A LEVEL MATHS: QUESTIONS BY TOPIC

## Further Coordinate Systems (FP3.2)

**International A Level Mathematics · FP3: Further Pure Mathematics 3 (WFM03)**

16 questions · 169 marks · Mark schemes included

Past papers October 2020 - June 2025 · 1 sub-topic

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**HOW TO USE THIS BOOKLET**

Questions are grouped by sub-topic and run from the lowest to the highest mark tariff, then oldest paper first.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme; < QUESTION brings you back.

A gold bar in the left margin marks exactly where the question sits on the original page.

Diagrams, graphs, grids and tables are printed exactly as they appear in the original papers.

Each question is filed under its main topic; longer questions may also test skills from other topics.

| QUESTION                                | SOURCE                       | MARKS             | PAGE      |
|-----------------------------------------|------------------------------|-------------------|-----------|
| <b>FP3.2a ELLIPSES &amp; HYPERBOLAS</b> |                              | 16 Qs · 169 marks | <b>2</b>  |
| 1. <b>Q2</b>                            | IAL FP3 (WFM03) January 2023 | 6                 | <b>3</b>  |
| 2. <b>Q1</b>                            | IAL FP3 (WFM03) June 2024    | 6                 | <b>5</b>  |
| 3. <b>Q2</b>                            | IAL FP3 (WFM03) January 2025 | 6                 | <b>7</b>  |
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| 5. <b>Q6</b>                            | IAL FP3 (WFM03) June 2024    | 9                 | <b>13</b> |
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| 9. <b>Q5</b>                            | IAL FP3 (WFM03) October 2020 | 12                | <b>30</b> |
| 10. <b>Q9</b>                           | IAL FP3 (WFM03) January 2021 | 12                | <b>34</b> |
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| 15. <b>Q6</b>                           | IAL FP3 (WFM03) June 2023    | 13                | <b>60</b> |
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# QUESTIONS

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16 questions · 169 marks · 1 sub-topic

SUB-TOPIC FP3.2a

## ELLIPSES & HYPERBOLAS

Sub-topic FP3.2a · 16 questions · 169 marks · lowest tariff first





**(Total for Question 2 is 6 marks)**





Question 1 continued

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(Total for Question 1 is 6 marks)



**(Total for Question 2 is 6 marks)**





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Question 3 continued

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(Total 9 marks)

Q3





Question 6 continued

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Question 6 continued

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Question 6 continued

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(Total for Question 6 is 9 marks)





Question 9 continued

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Question 9 continued

Lined area for writing the answer to Question 9 continued.



Question 9 continued

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Question 9 continued

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(Total for Question 9 is 10 marks)

TOTAL FOR PAPER IS 75 MARKS





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**Q7**

3. The ellipse  $E$  has equation

$$\frac{x^2}{49} + \frac{y^2}{b^2} = 1$$

where  $b$  is a constant and  $0 < b < 7$

The eccentricity of the ellipse is  $e$

(a) Write down, in terms of  $e$  only,

- (i) the coordinates of the foci of  $E$
- (ii) the equations of the directrices of  $E$

(2)

Given that

- the point  $P(x, y)$  lies on  $E$  where  $x > 0$
- the point  $S$  is the focus of  $E$  on the positive  $x$ -axis
- the line  $l$  is the directrix of  $E$  which crosses the positive  $x$ -axis
- the point  $M$  lies on  $l$  such that the line through  $P$  and  $M$  is parallel to the  $x$ -axis

(b) determine an expression for

- (i)  $PS^2$  in terms of  $e$ ,  $x$  and  $y$
- (ii)  $PM^2$  in terms of  $e$  and  $x$

(2)

(c) Hence show that

$$b^2 = 49(1 - e^2)$$

(2)

Given that  $E$  crosses the  $y$ -axis at the points with coordinates  $(0, \pm 4\sqrt{3})$

(d) determine the value of  $e$

(2)

Given that the  $x$  coordinate of  $P$  is  $\frac{7}{2}$

(e) determine the area of triangle  $OPM$ , where  $O$  is the origin.

(3)

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Question 3 continued

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Question 3 continued

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**(Total for Question 3 is 11 marks)**





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**Q5**



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**Q9**

**TOTAL FOR PAPER: 75 MARKS**



**Question 7 continued**

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Question 7 continued

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**Question 7 continued**

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**(Total for Question 7 is 12 marks)**





Question 9 continued

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Question 9 continued

Lined area for writing the answer to Question 9.



Question 9 continued

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Question 9 continued

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(Total for Question 9 is 12 marks)

TOTAL FOR PAPER IS 75 MARKS





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Question 8 continued

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Q8

(Total 13 marks)

END

TOTAL FOR PAPER: 75 MARKS





Question 9 continued

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Question 9 continued

Lined area for writing the answer to Question 9 continued.



Question 9 continued

Lined area for writing the answer to Question 9 continued.



Question 9 continued

Lined area for writing the answer to Question 9.

(Total for Question 9 is 13 marks)

TOTAL FOR PAPER IS 75 MARKS







Question 6 continued

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Question 6 continued

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(Total for Question 6 is 13 marks)





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**Q8**

**TOTAL FOR PAPER: 75 MARKS**

# MARK SCHEMES

official mark schemes, in the same order as the questions



**General Principles for Further Pure Mathematics Marking**

*(But note that specific mark schemes may sometimes override these general principles)*

**Method mark for solving 3 term quadratic:****1. Factorisation**

$(x^2 + bx + c) = (x + p)(x + q)$ , where  $|pq| = |c|$ , leading to  $x = \dots$

$(ax^2 + bx + c) = (mx + p)(nx + q)$ , where  $|pq| = |c|$  and  $|mn| = |a|$ , leading to  $x = \dots$

**2. Formula**

Attempt to use the correct formula (with values for  $a$ ,  $b$  and  $c$ ).

**3. Completing the square**

Solving  $x^2 + bx + c = 0$ :  $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$ ,  $q \neq 0$ , leading to  $x = \dots$

**Method marks for differentiation and integration:****1. Differentiation**

Power of at least one term decreased by 1. ( $x^n \rightarrow x^{n-1}$ )

**2. Integration**

Power of at least one term increased by 1. ( $x^n \rightarrow x^{n+1}$ )

**Use of a formula**

Where a method involves using a formula that has been learnt, the advice given in recent examiners' reports is that the formula should be quoted first.

Normal marking procedure is as follows:

Method mark for quoting a correct formula and attempting to use it, even if there are small errors in the substitution of values.

Where the formula is not quoted, the method mark can be gained by implication from correct working with values but may be lost if there is any mistake in the working.

**Exact answers**

Examiners' reports have emphasised that where, for example, an exact answer is asked for, or working with surds is clearly required, marks will normally be lost if the candidate resorts to using rounded decimals.



| Question Number                                    | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | Notes                                                                                                                                                                                                                                                                                                                                        | Marks     |
|----------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------|
| <b>1(a)</b>                                        | $ae = \frac{13}{2}$ <b>or</b> $\frac{a}{e} = \frac{72}{13}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | One correct equation in $a$ and $e$ .<br>Allow equivalent correct equations.<br>Could include – or $\pm$ signs                                                                                                                                                                                                                               | <b>B1</b> |
|                                                    | e.g.,<br>$a = \frac{72}{13}e \Rightarrow \frac{72}{13}e^2 = \frac{13}{2} \Rightarrow e^2 = \dots \left(\frac{169}{144}\right)$<br>or $a = \frac{13}{2e} \Rightarrow \frac{13}{2e^2} = \frac{72}{13} \Rightarrow e^2 = \dots \left(\frac{169}{144}\right)$                                                                                                                                                                                                                                                                                          | Having obtained two equations in $a$ and $e$ of the correct form i.e.,<br>$ae = p$ and $\frac{a}{e} = q$ $p, q \neq 0$ ,<br>solves simultaneously to find a <u>positive</u> value for $e^2$ (no requirement for $e > 1$ ) or $e$ .<br>Condone poor algebra provided a value is obtained. May find $a$ first.                                 | <b>M1</b> |
|                                                    | $e = \frac{13}{12}$ or $1\frac{1}{12}$ or 1.08 $\dot{3}$ .<br>Not $\pm \frac{13}{12}$ unless negative value clearly rejected in this part.                                                                                                                                                                                                                                                                                                                                                                                                         |                                                                                                                                                                                                                                                                                                                                              | <b>A1</b> |
| <b>(3)</b>                                         |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                                                                                                                                                                                                                                                                                                              |           |
| <b>(b)</b>                                         | $\left\{ a = \frac{72}{13} \times \frac{13}{12} = 6 \text{ or } a = \frac{13}{2\left(\frac{13}{12}\right)} = 6 \right\}$<br>$b^2 = a^2(e^2 - 1) = \dots$<br>$\left\{ b^2 = 6^2 \left( \left( \frac{13}{12} \right)^2 - 1 \right) = \frac{25}{4} \text{ or } b = \frac{5}{2} \right\}$                                                                                                                                                                                                                                                              | With any value for $a$ , which might be seen in part (a), and their $e$ , <b>uses</b> a correct eccentricity formula with correct substitution to find a value for $b^2$ or $b$ .<br>Could be implied. May see<br>$b = a\sqrt{e^2 - 1}$ or use of e.g.,<br>$e = \sqrt{1 + \frac{b^2}{a^2}}$ or $e = \frac{c}{a}$ with $c = \sqrt{a^2 + b^2}$ | <b>M1</b> |
|                                                    | $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{36} - \frac{y^2}{\frac{25}{4}} = 1$                                                                                                                                                                                                                                                                                                                                                                                                                                                  | Applies $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ <b>correctly</b> for their values. Not dependent.<br>Could use e.g., $b^2x^2 - a^2y^2 = a^2b^2$                                                                                                                                                                                              | <b>M1</b> |
|                                                    | e.g., $25x^2 - 144y^2 = 900$<br><br>A correct <b>equation</b> in correct form. Requires all previous 5 marks but allow if 4 marks with A0 in (a) for $e = \pm \frac{13}{12}$ and negative value not rejected in part (a).<br>Any positive integer multiple. Allow equivalents provided variables on one side and constant on the other and $y^2$ term has negative coefficient.<br>Just $p = 25, q = 144, r = 900$ requires $px^2 - qy^2 = r$ to be seen.<br>Ignore wrong values for $p, q, r$ following a correct equation (e.g., “ $q = -144$ ”) |                                                                                                                                                                                                                                                                                                                                              | <b>A1</b> |
| <b>Alt</b><br><br><b>Using</b><br>$PS^2 = e^2PM^2$ | $\left(x - \frac{13}{2}\right)^2 + y^2 = \left(\frac{13}{12}\right)^2 \left(x - \frac{72}{13}\right)^2$ M1: Forms equation correct for their $ae, e$ and $\frac{a}{e}$<br>$x^2 - 13x + \frac{169}{4} + y^2 = \frac{169}{144}x^2 - 13x + 36 \Rightarrow \frac{25}{144}x^2 - y^2 = \frac{25}{4}$<br>M1: Expands and reaches $rx^2 - sy^2 = t, r, s, t \neq 0$<br>A1: e.g., $25x^2 - 144y^2 = 900$ as main scheme                                                                                                                                     |                                                                                                                                                                                                                                                                                                                                              |           |
| <b>(3)</b>                                         |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                                                                                                                                                                                                                                                                                                              |           |
| <b>Total 6</b>                                     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                                                                                                                                                                                                                                                                                                              |           |

| Question         | Scheme                                                                                                                                      | Marks       |
|------------------|---------------------------------------------------------------------------------------------------------------------------------------------|-------------|
| <b>1</b>         | $2 \sinh^2 x + 3 \cosh x = 7 \Rightarrow 2(\cosh^2 x - 1) + 3 \cosh x = 7$                                                                  | <b>M1</b>   |
|                  | $\Rightarrow 2 \cosh^2 x + 3 \cosh x - 9 = 0$                                                                                               | <b>A1</b>   |
|                  | $\Rightarrow (2 \cosh x - 3)(\cosh x + 3) = 0 \Rightarrow \cosh x = \frac{3}{2}, \cancel{\frac{3}{2}}$                                      | <b>M1</b>   |
|                  | $\Rightarrow x = \operatorname{arcosh} \frac{3}{2} = (\pm) \ln \left( \frac{3}{2} + \sqrt{\frac{9}{4} - 1} \right)$                         | <b>M1</b>   |
|                  | $x = \ln \left( \frac{3}{2} + \frac{\sqrt{5}}{2} \right), \ln \left( \frac{3}{2} - \frac{\sqrt{5}}{2} \right)$                              | <b>A1A1</b> |
|                  |                                                                                                                                             | <b>(6)</b>  |
| <b>Alt</b>       | $2 \sinh^2 x + 3 \cosh x = 7 \Rightarrow 2 \left( \frac{e^x - e^{-x}}{2} \right)^2 + 3 \left( \frac{e^x + e^{-x}}{2} \right) = 7$           | <b>M1</b>   |
|                  | $\Rightarrow e^{4x} + 3e^{3x} - 16e^{2x} + 3e^x + 1 = 0$                                                                                    | <b>A1</b>   |
|                  | $\Rightarrow (e^{2x} + 6e^x + 1)(e^{2x} - 3e^x + 1) = 0 \Rightarrow e^x = \dots$                                                            | <b>M1</b>   |
|                  | $e^x = \cancel{-3 \pm 2\sqrt{2}}, \frac{3}{2} \pm \frac{\sqrt{5}}{2} \Rightarrow x = \ln \left( \frac{3}{2} \pm \frac{\sqrt{5}}{2} \right)$ | <b>M1</b>   |
|                  | $x = \ln \left( \frac{3}{2} + \frac{\sqrt{5}}{2} \right), \ln \left( \frac{3}{2} - \frac{\sqrt{5}}{2} \right)$                              | <b>A1A1</b> |
|                  |                                                                                                                                             | <b>(6)</b>  |
| <b>(6 marks)</b> |                                                                                                                                             |             |

**Notes:**

**M1:** Attempts to use the correct identity  $\cosh^2 x - \sinh^2 x = 1$  to form an equation in  $\cosh x$  only. (Allow slips rearranging a correctly stated identity.)

**A1:** Correct 3 term quadratic in  $\cosh x$ . The “=0” may be implied by their attempt to solve.

**M1:** Solves the 3 term quadratic (usual rules) leading to at least one exact real value for  $\cosh x$  (which may be negative). The negative value may not be seen, but need not be rejected at this stage. If using a calculator they must state at least one correct root for their equation.

**M1:** Forms a solution in terms of natural logarithms from a positive solution to their equation using the correct formula or via using the correct exponential form and solving a 3 term quadratic in  $e^x$ .

**A1:** One value correct.

**A1:** Both values correct, accepting equivalents, and no others. Equivalents are e.g.

$$x = \pm \ln \left( \frac{3}{2} + \frac{\sqrt{5}}{2} \right), \pm \ln \left( \frac{3}{2} - \frac{\sqrt{5}}{2} \right) \quad \text{Condone missing brackets.}$$

**Alt**

**M1:** Substitutes **correct** definitions for  $\sinh x$  and  $\cosh x$  in terms of exponentials.

**A1:** Correct simplified quartic in  $e^x$ .

**M1:** Solves their 5 term quartic as far as  $e^x = \dots$  to achieve exact values (at least one). There should be a recognisable attempt at solving, e.g. via a product of two quadratics, with at least first and last terms



matching their quartic, but allow answers only from a calculator – but must be correct and exact for their equation.

**M1:** Proceeds to a solution in terms of natural logarithms from a positive solution to their equation using the correct formula. Not dependent, so this may be scored from decimal solutions to the quartic, for proceeding from  $e^x = \dots$  to  $x = \ln \dots$

**A1:** One value correct.

**A1:** Both values correct, accepting equivalents, and no others.

There may be other variations via substitutions etc.

**M1:** Attempts to use the correct identities to form an equation in one “variable”.

**A1:** Correct simplified equation.

**M1:** Full attempt to solve the equation to at least  $\cosh x = \dots$  or  $e^x = \dots$  (exact)

**M1:** Forms a solution in terms of natural logarithms from a valid solution to their equation using the correct formula.

**A1:** One value correct.

**A1:** Both values correct, accepting equivalents, and no others. Equivalents are e.g.

$$x = \pm \ln\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right), \quad \pm \ln\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right) \quad \text{Condone missing brackets.}$$

**General Principles for Further Pure Mathematics Marking***(But note that specific mark schemes may sometimes override these general principles)***Method mark for solving 3 term quadratic:****1. Factorisation** $(x^2 + bx + c) = (x + p)(x + q)$ , where  $|pq| = |c|$ , leading to  $x = \dots$  $(ax^2 + bx + c) = (mx + p)(nx + q)$ , where  $|pq| = |c|$  and  $|mn| = |a|$ , leading to  $x = \dots$ **2. Formula**Attempt to use the correct formula (with values for  $a$ ,  $b$  and  $c$ ).**3. Completing the square**Solving  $x^2 + bx + c = 0$ :  $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$ ,  $q \neq 0$ , leading to  $x = \dots$ **Method marks for differentiation and integration:****1. Differentiation**Power of at least one term decreased by 1. ( $x^n \rightarrow x^{n-1}$ )**2. Integration**Power of at least one term increased by 1. ( $x^n \rightarrow x^{n+1}$ )

**Use of a formula**

Where a method involves using a formula that has been learnt, the advice given in recent examiners' reports is that the formula should be quoted first.

Normal marking procedure is as follows:

Method mark for quoting a correct formula and attempting to use it, even if there are small errors in the substitution of values.

Where the formula is not quoted, the method mark can be gained by implication from correct working with values but may be lost if there is any mistake in the working.

**Exact answers**

Examiners' reports have emphasised that where, for example, an exact answer is asked for, or working with surds is clearly required, marks will normally be lost if the candidate resorts to using rounded decimals.

| Question  | Scheme                                                                                                                                                                     | Marks |
|-----------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------|
| 1         | $\frac{dy}{dx} = \frac{1}{2} \times \frac{2}{\sqrt{(2x)^2 - 1}}$                                                                                                           | M1    |
|           | $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4x^2 - 1} = \frac{4x^2}{4x^2 - 1}$                                                                                        | M1    |
|           | $\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int \sqrt{\frac{4x^2}{4x^2 - 1}} dx = 2 \int \frac{x}{\sqrt{4x^2 - 1}} dx$                                             | A1    |
|           | $= \frac{2(4x^2 - 1)^{\frac{1}{2}}}{8 \times \frac{1}{2}}$                                                                                                                 | M1    |
|           | $s = \left[ \frac{(4x^2 - 1)^{\frac{1}{2}}}{2} \right]_{\frac{7}{2}}^{13} = \frac{1}{2} \left( \sqrt{4 \times 169 - 1} - \sqrt{4 \times \frac{49}{4} - 1} \right) = \dots$ | dM1   |
|           | $= \frac{1}{2} (15\sqrt{3} - 4\sqrt{3}) = \frac{11}{2} \sqrt{3}$                                                                                                           | A1    |
|           |                                                                                                                                                                            | (6)   |
| (6 marks) |                                                                                                                                                                            |       |

## Notes:

**M1:** Attempts  $\frac{dy}{dx}$ , accept the form  $\frac{A}{\sqrt{(2x)^2 - 1}}$ . Allow  $\frac{A}{\sqrt{2x^2 - 1}}$  (condone missing brackets)

**Alternative 1:**

Writes  $\frac{1}{2} \operatorname{ar} \cosh 2x$  as  $\frac{1}{2} \ln(2x + \sqrt{4x^2 - 1})$  leading to

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{2x + \sqrt{4x^2 - 1}} \times \left( 2 + \frac{4x}{\sqrt{4x^2 - 1}} \right) = \frac{2x + \sqrt{4x^2 - 1}}{\sqrt{4x^2 - 1} (2x + \sqrt{4x^2 - 1})} = \frac{1}{\sqrt{4x^2 - 1}}$$

**Alternative 2:**

$$y = \frac{1}{2} \operatorname{ar} \cosh 2x \Rightarrow 2y = \operatorname{ar} \cosh 2x \Rightarrow \cosh 2y = 2x \rightarrow 4 \sinh 2y \frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{1}{\sinh 2y} = \frac{1}{\sqrt{4x^2 - 1}}$$

If either approach is taken then the same condition for the form of the derivative applies.

Note that this differentiation may be seen in an attempt by parts of  $\int y dx$

**M1:** Attempts to find  $1 + \left(\frac{dy}{dx}\right)^2$  using their  $\frac{dy}{dx}$  and attempts common denominator.

**A1:** Reaches a correct simplified integral with  $\sqrt{x^2}$  replaced with  $x$  as shown in the scheme.

Allow equivalent forms e.g.  $2 \int x \sqrt{\frac{1}{4x^2 - 1}} dx$ ,  $\frac{1}{2} \int \frac{4x}{\sqrt{(2x)^2 - 1}} dx$

This may be implied by subsequent work.



| Question Number | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | Notes | Marks                            |
|-----------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------|----------------------------------|
| 4(b)            | Condone the use of e.g., $B$ for $\alpha$ or $k$ for $R$ for the first three marks but allow the A mark if recovered which may be via a correct expression which might be in (c)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |       |                                  |
|                 | $10 \sinh x + 8 \cosh x = R \sinh x \cosh \alpha + R \cosh x \sinh \alpha$ $\Rightarrow R \sinh \alpha = 8, \quad R \cosh \alpha = 10$ <p>Equates coefficients to obtain the two correct equations. This mark could be implied by <u>either</u> correct elimination, i.e.,</p> $R^2 = 10^2 - 8^2 \text{ or } \tanh \alpha = \frac{8}{10} \text{ provided incorrect equations are not seen.}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |       | <b>B1</b><br><b>(M1 on ePen)</b> |
|                 | <p>A complete attempt at finding a <u>positive</u> value for <math>R</math>:</p> <p><b>By elimination:</b></p> $R^2 (\cosh^2 \alpha - \sinh^2 \alpha) = 10^2 - 8^2 \Rightarrow R^2 = 36 \Rightarrow R = 6$ <p>Allow this mark for <math>R = \sqrt{10^2 + 8^2} = 2\sqrt{41}</math> or <math>\sqrt{164}</math>. May just see e.g., <math>R = 2\sqrt{41}</math></p> <p><b>Following a positive value obtained for <math>\alpha</math> where <math>\alpha = k \ln p</math>, <math>k &gt; 0</math>, <math>p &gt; 1</math>:</b></p> $\alpha = \frac{1}{2} \ln 9 = \ln 3 \Rightarrow R \cosh(\ln 3) = 10 \Rightarrow R \left( \frac{e^{\ln 3} + e^{-\ln 3}}{2} \right) = 10 \Rightarrow R = \dots \left\{ \frac{5}{3} R = 10 \Rightarrow R = 6 \right\}$ $\text{or } R \sinh(\ln 3) = 8 \Rightarrow R \left( \frac{e^{\ln 3} - e^{-\ln 3}}{2} \right) = 8 \Rightarrow R = \dots \left\{ \frac{4}{3} R = 8 \Rightarrow R = 6 \right\}$ <p>Correct exponential definitions must be used but can be implied by correct work.<br/>Allow if the 10 and 8 are mixed up and allow slips in solving</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |       | <b>1<sup>st</sup> M1</b>         |
|                 | <p>A complete attempt at finding a <u>positive</u> value for <math>\alpha</math> where <math>\alpha = k \ln p</math>, <math>k &gt; 0</math>, <math>p &gt; 1</math>:</p> <p><b>By elimination:</b></p> $\tanh \alpha = \frac{8}{10} \Rightarrow \alpha = \operatorname{artanh} \left( \frac{4}{5} \right) = \frac{1}{2} \ln \left( \frac{1 + \frac{4}{5}}{1 - \frac{4}{5}} \right) = \dots \left\{ = \frac{1}{2} \ln 9 = \ln 3 \right\}$ <p>A correct logarithmic form must be used with a valid value for <math>\operatorname{artanh} (&lt;1)</math></p> <p><b>Following a positive value obtained for <math>R</math>:</b></p> $\sinh \alpha = \frac{8}{\text{"6"}} \Rightarrow \alpha = \operatorname{arsinh} \left( \frac{8}{\text{"6"}} \right) = \ln \left( \frac{8}{\text{"6"}} + \sqrt{\left( \frac{8}{\text{"6"}} \right)^2 + 1} \right) \left\{ = \ln 3 \right\}$ $\cosh \alpha = \frac{10}{\text{"6"}} \Rightarrow \alpha = \operatorname{arcosh} \left( \frac{10}{\text{"6"}} \right) = \ln \left( \frac{10}{\text{"6"}} + \sqrt{\left( \frac{10}{\text{"6"}} \right)^2 - 1} \right) \left\{ = \ln 3 \right\}$ <p>A correct logarithmic form must be used with a valid value if using <math>\operatorname{arcosh} (&gt;1)</math></p> <p>The appropriate logarithmic forms could be implied by correct values.<br/>Allow this mark if e.g., <math>\frac{8}{10}</math> is erroneously simplified but the value must be valid for the inverse hyperbolic function.</p> <p>If an exponential form is used to evaluate an inverse hyperbolic the form must be correct and the solving of any resulting 3TQ (most likely in <math>e^\alpha</math> or <math>e^x</math>) must satisfy usual rules with one root correct if no working. Note that using <math>\tanh</math> leads to a 2TQ which they must get one correct root for.<br/>They must also proceed to <math>\alpha = k \ln p</math>, <math>k &gt; 0</math>, <math>p &gt; 1</math></p> |       | <b>2<sup>nd</sup> M1</b>         |
|                 | $6 \sinh(x + \ln 3) \text{ or } R = 6 \text{ and } \alpha = \ln 3 \text{ (or } p = 3)$ <p>Correct expression but allow values for <math>R</math> and <math>\alpha</math> (or <math>p</math>).</p> <p>If all the values are not seen in (b) then allow if they are seen in (c) and they could be seen embedded in a correct expression.</p> <p>A0 for additional solutions e.g., <math>6 \sinh(x \pm \ln 3)</math></p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |       | <b>A1</b>                        |

| Question Number | Scheme/Notes                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         | Marks          |
|-----------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------|
| 4(c)            | <p>There is no credit for attempts that do not use part (b) so e.g., do not award marks for attempts that apply exponential definitions to <math>10\sinh x + 8\cosh x = 18\sqrt{7}</math> but note that it is acceptable to use exponential definitions with <math>6\sinh(x + \ln 3) = 18\sqrt{7}</math>.<br/>Allow work with “made up” values for <math>R</math> and <math>p</math> provided <math>R &gt; 0</math>, <math>p \in \mathbb{Z}</math>, <math>p &gt; 1</math></p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |                |
|                 | $6\sinh(x + \ln 3) = 18\sqrt{7}$ $\Rightarrow x = \operatorname{arsinh}(3\sqrt{7}) - \ln 3$ $\Rightarrow x = \ln\left(3\sqrt{7} + \sqrt{(3\sqrt{7})^2 + 1}\right) - \ln 3$ <p>Obtains <math>x = \operatorname{arsinh}\left(\frac{18\sqrt{7}}{6}\right) \pm \ln 3</math> or <math>x \pm \ln 3 = \operatorname{arsinh}\left(\frac{18\sqrt{7}}{6}\right)</math> from “6”<math>\sinh(x \pm \ln 3) = 18\sqrt{7}</math><br/> <b>and</b> uses the correct logarithmic form to obtain an expression for, or equation in <math>x</math> in “ln”s only but condone loss of the <math>-\ln 3</math> or <math>+\ln 3</math> after it has been seen.<br/>                     If the <math>-\ln 3</math> or <math>+\ln 3</math> is immediately incorporated to make a single logarithm the subtraction/addition law must be applied correctly.<br/>                     Work must be exact and not in decimals.</p> <p>If e.g., <math>C = \operatorname{arsinh}(3\sqrt{7})</math> is found using <math>\frac{e^C - e^{-C}}{2} = 3\sqrt{7}</math>, the exponential definition must be correct and they must solve a 3TQ in <math>e^C</math> satisfying usual rules (or one root correct if no working) and proceed to a valid <math>C = \dots</math> (e.g., not <math>\ln(\text{negative})</math>). This also applies to attempts via</p> $6 \frac{e^{x+\ln 3} - e^{-x-\ln 3}}{2} = 18\sqrt{7} \quad \left\{ \Rightarrow 3e^x - \frac{1}{3}e^{-x} = 6\sqrt{7} \Rightarrow 9e^{2x} - 18\sqrt{7}e^x - 1 = 0 \Rightarrow x = \ln\left(\frac{8+3\sqrt{7}}{3}\right) \right\}$ <p>Note that <math>e^{2(x+\ln 3)} - 6\sqrt{7}e^{x+\ln 3} - 1 = 0 \Rightarrow e^{x+\ln 3} = 8+3\sqrt{7} \Rightarrow x = \ln\left(\frac{8+3\sqrt{7}}{3}\right)</math> is also possible<br/>                     and in such cases the <math>x + \ln 3</math> must be handled correctly</p> | M1             |
|                 | $\left\{ x = \ln\left(\frac{3\sqrt{7}+8}{3}\right) \right\} = \ln\left(\sqrt{7} + \frac{8}{3}\right)$ <p>Correct answer in correct form. Accept e.g., <math>\ln\left(2\frac{2}{3} + \sqrt{7}\right)</math>. Must be fully bracketed correctly. Accept <math>q = \frac{8}{3}</math> if <math>\ln(\sqrt{7} + q)</math> is seen. No additional answers.</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | A1             |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      | (2)            |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      | <b>Total 9</b> |

|     |                                                                                                                                                                                    |                                                                                                                           |                 |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------|-----------------|
| (c) | Reflection of $P$ in $l_1$ is<br>$\begin{pmatrix} -4 \\ -5 \\ 3 \end{pmatrix} + 2 \times \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \\ -1 \end{pmatrix}$ | Correct strategy for another point on $l_3$                                                                               | M1              |
|     | $\begin{pmatrix} 8 \\ 11 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ -5 \end{pmatrix}$                                            | Attempts direction of $l_3$ . <b>Depends on the first mark.</b>                                                           | dM1             |
|     | $\mathbf{r} = \begin{pmatrix} -1 \\ 6 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ 5 \\ -5 \end{pmatrix}$                                                                     | Forms the equation of $l_3$ using $R$ (or their reflected $P$ ) and their direction. <b>Depends on the previous mark.</b> | ddM1            |
|     |                                                                                                                                                                                    | Any correct equation in vector form                                                                                       | A1 (4)          |
|     |                                                                                                                                                                                    |                                                                                                                           | <b>Total 12</b> |

| Question Number | Scheme                                                                                                                                                                                                                                                                                                                        | Notes                                                                                                                                                                                      | Marks      |
|-----------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|
| <b>9</b>        | $\frac{x^2}{9} + \frac{y^2}{4} = 1, \quad y = kx - 3$                                                                                                                                                                                                                                                                         |                                                                                                                                                                                            |            |
| <b>(a)</b>      | $\frac{x^2}{9} + \frac{(kx-3)^2}{4} = 1 \left( \text{or } \frac{x^2}{9} + \frac{k^2x^2 - 6kx + 9}{4} = 1 \right) \Rightarrow 4x^2 + 9(k^2x^2 - 6kx + 9) = 36$<br>Substitutes to obtain a quadratic in $x$ and eliminates fractions                                                                                            |                                                                                                                                                                                            | M1         |
|                 | $(9k^2 + 4)x^2 - 54kx + 45 = 0^*$                                                                                                                                                                                                                                                                                             | Correct proof with no errors                                                                                                                                                               | A1*        |
|                 |                                                                                                                                                                                                                                                                                                                               |                                                                                                                                                                                            | <b>(2)</b> |
| <b>(b)</b>      | $x = \frac{1}{2} \left( \frac{54k}{9k^2 + 4} \right) = \frac{27k}{9k^2 + 4}$<br>OR $x = \frac{54k \pm \sqrt{\text{discriminant}}}{2(9k^2 + 4)}$                                                                                                                                                                               | Uses $\frac{1}{2}$ sum of roots for the $x$ coordinate<br>OR Solve the equation (by formula), add the 2 roots and halve the result.<br>Must reach $x_m$ . Allow errors in the discriminant | M1         |
|                 | $y = k \left( \frac{27k}{9k^2 + 4} \right) - 3$<br>$y = \frac{27k^2 - 27k^2 - 12}{9k^2 + 4} = -\frac{12}{9k^2 + 4}$                                                                                                                                                                                                           | Uses the straight line equation to find $y$ as a single fraction, can be unsimplified<br>Depends on first M mark of (b)                                                                    | dM1        |
|                 | $x = \frac{27k}{9k^2 + 4}, \quad y = -\frac{12}{9k^2 + 4}$                                                                                                                                                                                                                                                                    | Fully correct work                                                                                                                                                                         | A1         |
|                 |                                                                                                                                                                                                                                                                                                                               |                                                                                                                                                                                            | <b>(3)</b> |
| <b>(c)</b>      | $x^2 = \frac{729k^2}{(9k^2 + 4)^2} \Rightarrow x^2 + py^2 = \frac{729k^2 + 144p}{(9k^2 + 4)^2}$<br>Obtains an expression for $x^2 + py^2$ using their coordinates obtained in (b) and obtains a common denominator                                                                                                            |                                                                                                                                                                                            | M1         |
|                 | $\frac{729k^2 + 144p}{(9k^2 + 4)^2} = -\frac{12q}{(9k^2 + 4)} \Rightarrow 729k^2 + 144p = -12q(9k^2 + 4)$<br>$729k^2 + 144p = 81 \left( 9k^2 + \frac{16}{9}p \right)$<br>$\Rightarrow \frac{16}{9}p = 4 \Rightarrow p = \dots$<br>Correct strategy to obtain a value for $p$ or for $q$<br>Depends on the first M mark of (c) |                                                                                                                                                                                            | dM1        |
|                 | $p = \frac{9}{4} \text{ or } q = -\frac{27}{4} \text{ oe}$                                                                                                                                                                                                                                                                    | Correct value (or for $q$ if found first)                                                                                                                                                  | A1         |
|                 | $-12q = 81 \Rightarrow q = \dots$                                                                                                                                                                                                                                                                                             | Correct strategy to obtain a value for the second variable<br>Depends on both previous M marks                                                                                             | ddM1       |
|                 | $\Rightarrow x^2 + \frac{9}{4}y^2 = -\frac{27}{4}y$<br>$p = \frac{9}{4} \text{ and } q = -\frac{27}{4} \text{ oe}$                                                                                                                                                                                                            | Both values correct – can be embedded in the equation                                                                                                                                      | A1         |
|                 |                                                                                                                                                                                                                                                                                                                               |                                                                                                                                                                                            | <b>(5)</b> |

|              |                                                                                                                                                                                                                                                                                                                                                                             |                                                                                    |                 |
|--------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------|-----------------|
| (c)<br>Way 2 | $x = \frac{27k}{9k^2 + 4}, \quad y = -\frac{12}{9k^2 + 4} \Rightarrow \frac{x}{y} = -\frac{27k}{12} \Rightarrow k = -\frac{4x}{9y}$ <p>Obtains <math>k</math> in terms of <math>x</math> and <math>y</math> using their coordinates found in (b)</p>                                                                                                                        |                                                                                    | M1              |
|              | $k = -\frac{4x}{9y} \Rightarrow y = -\frac{12}{9\left(\frac{16x^2}{81y^2}\right) + 4} \text{ or } x = \frac{27\left(-\frac{4x}{9y}\right)}{9\left(\frac{16x^2}{81y^2}\right) + 4}$ <p>dM1:Substitutes <math>k</math> into <math>y</math> or <math>x</math> to obtain a Cartesian equation<br/>A1: Any correct Cartesian equation<br/>Depends on the first M mark of (c)</p> |                                                                                    | dM1A1           |
|              | $\Rightarrow x^2 + \frac{9}{4}y^2 = -\frac{27}{4}y$                                                                                                                                                                                                                                                                                                                         | <p>Rearranges to the form required<br/>Depends on both previous M marks of (c)</p> | ddM1            |
|              |                                                                                                                                                                                                                                                                                                                                                                             | <p>Correct equation or correct values stated</p>                                   | A1              |
|              |                                                                                                                                                                                                                                                                                                                                                                             |                                                                                    | <b>Total 10</b> |



| Question                       | Scheme                                                                                                                                                                                                                                                             | Marks                            |
|--------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------|
| 7(a)                           | $b^2 = a^2(e^2 - 1) \Rightarrow e^2 = \frac{25}{a^2} + 1 = \frac{25 + a^2}{a^2}$ oe                                                                                                                                                                                | <b>B1</b>                        |
|                                |                                                                                                                                                                                                                                                                    | <b>(1)</b>                       |
| (b)                            | $x = (\pm)\frac{a}{e} \quad \frac{x}{a} = (\pm)\frac{y}{5}$                                                                                                                                                                                                        | <b>B1</b>                        |
|                                | $\frac{a}{e} \times \frac{1}{a} = \pm \frac{y}{5} \Rightarrow y = \pm \frac{5}{e} \Rightarrow AA' = 2 \times \frac{5}{e}$ or $\frac{5}{e} - \left(-\frac{5}{e}\right)$                                                                                             | <b>M1</b>                        |
|                                | $= \frac{10}{e}$                                                                                                                                                                                                                                                   | <b>A1</b>                        |
|                                |                                                                                                                                                                                                                                                                    | <b>(3)</b>                       |
| (c)                            | $\frac{1}{2} \times \frac{10}{e} \times \left(ae + \frac{a}{e}\right)$ or e.g. $\frac{1}{2} \times \frac{10a}{\sqrt{25+a^2}} \times \left(\sqrt{25+a^2} + \frac{a^2}{\sqrt{25+a^2}}\right)$                                                                        | <b>M1</b>                        |
|                                | $\frac{1}{2} \frac{10}{e} \left(ae + \frac{a}{e}\right) = \frac{164}{3} \Rightarrow 15 \left(a + \frac{a}{e^2}\right) = 164$<br>or<br>$\frac{1}{2} \times \frac{10a}{\sqrt{25+a^2}} \times \left(\sqrt{25+a^2} + \frac{a^2}{\sqrt{25+a^2}}\right) = \frac{164}{3}$ | <b>M1</b>                        |
|                                | $\Rightarrow 15a \left(1 + \frac{a^2}{25+a^2}\right) = 164$                                                                                                                                                                                                        | <b>A1</b><br><b>(M1 on EPEN)</b> |
|                                | $\Rightarrow 15a \left(\frac{25+2a^2}{25+a^2}\right) = 164 \Rightarrow 375a + 30a^3 = 164(25+a^2)$<br>$\Rightarrow 30a^3 - 164a^2 + 375a - 4100 = 0^*$                                                                                                             | <b>A1*</b>                       |
|                                |                                                                                                                                                                                                                                                                    | <b>(4)</b>                       |
|                                |                                                                                                                                                                                                                                                                    |                                  |
| (d)                            | $30a^3 - 164a^2 + 375a - 4100 = (3a - 20)(10a^2 + 12a + 205)$                                                                                                                                                                                                      | <b>B1</b><br><b>(M1 on EPEN)</b> |
|                                | $12^2 - 4(10)(205) = \dots$<br>$10a^2 + 12a + 205 = 10 \left[ \left(a + \frac{12}{20}\right)^2 - \frac{144}{400} \right] + 205$                                                                                                                                    | <b>M1</b>                        |
|                                | E.g. $12^2 - 4(10)(205) < 0$ so there are no other roots of the equation.<br>Hence $a = \frac{20}{3}$ is only possible value.                                                                                                                                      | <b>A1</b>                        |
|                                |                                                                                                                                                                                                                                                                    | <b>(3)</b>                       |
| <b>(11 marks)</b>              |                                                                                                                                                                                                                                                                    |                                  |
| <b>Notes:</b>                  |                                                                                                                                                                                                                                                                    |                                  |
| <b>(a)</b>                     |                                                                                                                                                                                                                                                                    |                                  |
| <b>B1:</b> Correct expression. |                                                                                                                                                                                                                                                                    |                                  |



**(b)****B1:** Identifies at least one correct equation for a directrix and at least one asymptote, stated or used – including the  $b = 5$ .**M1:** Solves to find  $y$  coordinates of  $A$  and  $A'$  or just one of these and doubles to get length. Allow if  $b$  is used rather than 5.**A1:** Correct length (from subtracting or doubling). Must be positive.**(c)****M1:** Uses focus  $(-ae, 0)$  and directrix  $x = \frac{a}{e}$  (allow if the alternative pair is used) with their length from (b), to form a **correct or correct ft** expression for the area of triangle  $AF A'$ .**M1:** Sets their area equation equal to  $\frac{164}{3}$  to obtain an equation in  $e^2$  and  $a$ .Their attempt at the area must be of the form  $\frac{1}{2} \times \frac{10}{e} \times \pm \left( ae \pm \frac{a}{e} \right)$ Alternatively, allow an equation in just  $a^2$  if  $e = \sqrt{\frac{25+a^2}{a^2}}$  is substituted first.**A1(M1 on EPEN):** Correct equation in terms of  $a$  only. Allow any correct form.**A1\*:** Correct result achieved with no errors seen and sufficient working shown.**(d)****B1(M1 on EPEN):** A correct method for showing that  $a = \frac{20}{3}$  is a solution of the equation.

Examples:

$$30a^3 - 164a^2 + 375a - 4100 = (3a - 20)(10a^2 + 12a + 205)$$

$$30a^3 - 164a^2 + 375a - 4100 = \left(a - \frac{20}{3}\right)(30a^2 + 36a + 615)$$

$$f\left(\frac{20}{3}\right) = \frac{80000}{9} - \frac{65600}{9} + 2500 - 4100 = 0$$

Or e.g. long division and obtains correct quotient and no remainder

**M1:** A correct method for showing there are no other roots. May use completing the square (as in scheme) or attempt discriminant or differentiation,e.g.  $\frac{d}{da}(eqn) = 90a^2 - 328a + 375 = 90\left(a - \frac{82}{45}\right)^2 + \frac{3427}{45} > 0$  so strictly increasing hence only one solution.If using discriminant then values must be used i.e. not just  $b^2 - 4ac < 0$ 

An attempt at the discriminant may be seen as part of the quadratic formula e.g.

$$a = \frac{-12 \pm \sqrt{12^2 - 4(10)(205)}}{2(10)}$$

**A1:** All work correct with **reason** and **conclusion** made that  $a = \frac{20}{3}$  is the only possible value. If the discriminant is evaluated then it must be correct. For reference  $12^2 - 4(10)(205) = -8056$  and  $36^2 - 4(30)(615) = -72504$  but note that e.g.  $12^2 - 4(10)(205) < 0$  with a conclusion is acceptable.**Note that just using a calculator to solve the cubic generally scores no marks.**

| Question Number | Scheme                                                                                                                                                                                                                                                                               | Notes                                                                                                                                                                                                                     | Marks |
|-----------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------|
| 3(a)(i)         | $(\pm 7e, 0)$                                                                                                                                                                                                                                                                        | Correct <b>coordinates</b> or $x = \pm 7e, y = 0$                                                                                                                                                                         | B1    |
| (ii)            | $x = \pm \frac{7}{e}$                                                                                                                                                                                                                                                                | Correct <b>equations</b>                                                                                                                                                                                                  | B1    |
|                 | SC: If “49” used for “7” consistently in (i) and (ii) score B0 B1                                                                                                                                                                                                                    |                                                                                                                                                                                                                           |       |
|                 |                                                                                                                                                                                                                                                                                      |                                                                                                                                                                                                                           | (2)   |
| (b)(i)          | $(PS^2 =)(x - '7e')^2 + y^2$<br>oe e.g.<br>$(PS^2 =)('7e' - x)^2 + y^2$                                                                                                                                                                                                              | Correct expression or equivalent with their $7e$ . Must be in terms of $e, x$ and $y$ only. Apply isw once a correct expression is seen.                                                                                  | B1ft  |
| (ii)            | $(PM^2 =)\left(\frac{7}{e} - x\right)^2$ oe e.g. $\left(x - \frac{7}{e}\right)^2$                                                                                                                                                                                                    | Correct expression or equivalent with their $\frac{7}{e}$ . Must be in terms of $e$ and $x$ only. Apply isw once a correct expression is seen.                                                                            | B1ft  |
|                 |                                                                                                                                                                                                                                                                                      |                                                                                                                                                                                                                           | (2)   |
| (c)             | $\left(\frac{PS}{PM} = e \Rightarrow\right) PS^2 = e^2 PM^2 \Rightarrow (x - '7e')^2 + y^2 = e^2 \left(\frac{7}{e} - x\right)^2$ $\Rightarrow x^2 - 14ex + 49e^2 + y^2 = 49 - 14ex + e^2 x^2$ Applies $PS^2 = e^2 PM^2$ with their $PS$ and $PM$ and expands (condone poor squaring) |                                                                                                                                                                                                                           | M1    |
|                 | $x^2(1 - e^2) + y^2 = 49(1 - e^2)$ $\Rightarrow \frac{x^2}{49} + \frac{y^2}{49(1 - e^2)} = 1 \Rightarrow b^2 = 49(1 - e^2) *$                                                                                                                                                        | Reaches given answer with fully correct proof. All shown steps required. Note that it is possible to obtain this result even if the B marks are not scored in (b) e.g. correct expressions but not in the forms required. | A1*   |
|                 |                                                                                                                                                                                                                                                                                      |                                                                                                                                                                                                                           | (2)   |
| (d)             | $(4\sqrt{3})^2 = 49(1 - e^2) \Rightarrow e^2 \dots$ or $e = \dots$                                                                                                                                                                                                                   | Replaces $b^2$ with $(4\sqrt{3})^2$ and solves for $e^2$ or $e$ .                                                                                                                                                         | M1    |
|                 | $e = \frac{1}{7}$                                                                                                                                                                                                                                                                    | Correct exact value for $e$ (Not $\pm$ )                                                                                                                                                                                  | A1    |
|                 |                                                                                                                                                                                                                                                                                      |                                                                                                                                                                                                                           | (2)   |

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |                                                                   |                 |
|-----|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------|-----------------|
| (e) | $x = \frac{7}{2} \Rightarrow \frac{\left(\frac{7}{2}\right)^2}{49} + \frac{y^2}{48} = 1 \Rightarrow y = \dots [(\pm)6]$                                                                                                                                                                                                                                                                                                                                                                    | Substitutes into their ellipse equation and obtains a value for y | M1              |
|     | $\text{Area } \triangle OPM = \left(\frac{1}{2}\right) \left( \frac{7}{\left(\frac{1}{7}\right)} - \frac{7}{2} \right) (6) = \dots$ <p>Correct method for area of triangle <math>OPM</math> with their <math>\frac{7}{e}</math> and their 6</p> <p>May see other approaches, e.g., “shoelace” method</p> <p>e.g. <math>\frac{1}{2} \begin{vmatrix} 3.5 &amp; 0 &amp; 49 &amp; 3.5 \\ 6 &amp; 0 &amp; 6 &amp; 6 \end{vmatrix} = \frac{1}{2} (49 \times 6 - 6 \times 3.5) = \dots</math></p> |                                                                   | dM1             |
|     | $\frac{273}{2}$ or $136\frac{1}{2}$ or 136.5                                                                                                                                                                                                                                                                                                                                                                                                                                               | Any correct exact value                                           | A1              |
|     | <p style="text-align: center;"><b>Special Case:</b></p> $x = \frac{7}{2} \Rightarrow \frac{\left(\frac{7}{2}\right)^2}{49} + \frac{y^2}{48} = 1 \Rightarrow y = 36 \Rightarrow \text{Area } \triangle OPM = \left(\frac{1}{2}\right) \left( \frac{7}{\left(\frac{1}{7}\right)} - \frac{7}{2} \right) (36) = \dots (819)$ <p style="text-align: center;"><b>Scores M0M1A0</b></p>                                                                                                           |                                                                   |                 |
|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |                                                                   | (3)             |
|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |                                                                   | <b>Total 11</b> |



| Question Number                               | Scheme                                                                                                                                                                                                                                                                                                                                                                                        | Notes                                                                                       | Marks          |
|-----------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------|----------------|
| <b>1(a)</b>                                   | $4\sinh^3 x + 3\sinh x = 4\left(\frac{e^x - e^{-x}}{2}\right)^3 + 3\left(\frac{e^x - e^{-x}}{2}\right)$ $= 4\left(\frac{e^{3x} - 3e^x + 3e^{-x} - e^{-3x}}{8}\right) + 3\left(\frac{e^x - e^{-x}}{2}\right)$ <p>Uses <math>\sinh x = \frac{e^x - e^{-x}}{2}</math> on both <math>\sinh</math> terms and attempts to cube the bracket<br/>(min accepted is a linear x a quadratic bracket)</p> |                                                                                             | M1             |
|                                               | $= \frac{1}{2}e^{3x} - \frac{3}{2}e^x + \frac{3}{2}e^{-x} - \frac{1}{2}e^{-3x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x}$ $= \frac{e^{3x} - e^{-3x}}{2} = \sinh 3x^*$                                                                                                                                                                                                                              |                                                                                             | A1*            |
|                                               |                                                                                                                                                                                                                                                                                                                                                                                               |                                                                                             | <b>(2)</b>     |
| <b>(b)</b>                                    | $\sinh 3x = 19\sinh x \Rightarrow 4\sinh^3 x + 3\sinh x = 19\sinh x$ $\Rightarrow 4\sinh^3 x - 16\sinh x = 0$ <p>Uses the result from (a) and combines terms</p>                                                                                                                                                                                                                              |                                                                                             | M1             |
|                                               | $(\sinh x = 0 \text{ or } \sinh^2 x = 4)$                                                                                                                                                                                                                                                                                                                                                     | $\sinh^2 x = 4 \text{ or } \sinh x = (\pm)2$                                                | A1             |
|                                               | $(0, 0)$                                                                                                                                                                                                                                                                                                                                                                                      | States the origin as one intersection                                                       | B1             |
|                                               | $\ln(2 + \sqrt{5})$ and $-\ln(2 + \sqrt{5})$                                                                                                                                                                                                                                                                                                                                                  | Two correct non-zero $x$ values (allow e.g. $\ln(-2 + \sqrt{5})$ for $-\ln(2 + \sqrt{5})$ ) | A1             |
|                                               | $(\ln(2 + \sqrt{5}), 38)$ and $(-\ln(2 + \sqrt{5}), -38)$                                                                                                                                                                                                                                                                                                                                     | Two correct <b>points</b> (allow e.g. $\ln(-2 + \sqrt{5})$ for $-\ln(2 + \sqrt{5})$ )       | A1             |
|                                               |                                                                                                                                                                                                                                                                                                                                                                                               |                                                                                             | <b>(5)</b>     |
| <b>Alternative for (b) using exponentials</b> |                                                                                                                                                                                                                                                                                                                                                                                               |                                                                                             |                |
|                                               | $\sinh 3x = 19\sinh x \Rightarrow \frac{e^{3x} - e^{-3x}}{2} = \frac{19(e^x - e^{-x})}{2} \Rightarrow \dots$ <p>Substitutes the correct exponential forms and collects terms to one side</p>                                                                                                                                                                                                  |                                                                                             | M1             |
|                                               | $\Rightarrow e^{6x} - 19e^{4x} + 19e^{2x} - 1 = 0$                                                                                                                                                                                                                                                                                                                                            | Correct equation (or equivalent)                                                            | A1             |
|                                               | $(0, 0)$                                                                                                                                                                                                                                                                                                                                                                                      | States the origin as one intersection                                                       | B1             |
|                                               | $\frac{1}{2}\ln(9 + 4\sqrt{5})$ or $\frac{1}{2}\ln(9 - 4\sqrt{5})$                                                                                                                                                                                                                                                                                                                            | Two correct non-zero $x$ values (oe)                                                        | A1             |
|                                               | $\left(\frac{1}{2}\ln(9 + 4\sqrt{5}), 38\right)$ and $\left(\frac{1}{2}\ln(9 - 4\sqrt{5}), -38\right)$                                                                                                                                                                                                                                                                                        | Two correct <b>points</b> (oe)                                                              | A1             |
|                                               |                                                                                                                                                                                                                                                                                                                                                                                               |                                                                                             | <b>Total 7</b> |

| Question Number | Scheme                                                                                                                                                                                                                                                                                                   | Notes                                                                                | Marks          |
|-----------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------|----------------|
| <b>2(i)</b>     | $3x^2 + 12x + 24 = 3(x^2 + 4x + 8)$ $= 3((x+2)^2 + 4)$                                                                                                                                                                                                                                                   | Obtains $3((x+2)^2 + \dots)$ or<br>$3(x+2)^2 + \dots$<br>Must include 3 now or later | M1             |
|                 | $3((x+2)^2 + 4)$ or $3(x+2)^2 + 12$                                                                                                                                                                                                                                                                      |                                                                                      | A1             |
|                 | $\int \frac{1}{3x^2 + 12x + 24} dx = \frac{1}{3} \int \frac{1}{(x+2)^2 + 4} dx = \frac{1}{6} \arctan \frac{x+2}{2} (+c)$ <p>M1: Use of arctan<br/>A1: Fully correct expression (condone omission of + c)</p>                                                                                             |                                                                                      | M1A1           |
|                 |                                                                                                                                                                                                                                                                                                          |                                                                                      | <b>(4)</b>     |
| <b>(ii)</b>     | $27 - 6x - x^2 = -(x^2 + 6x - 27)$ $= -((x+3)^2 - 36)$                                                                                                                                                                                                                                                   | Obtains $-((x+3)^2 + \dots)$ or<br>$-(x+3)^2 + \dots$                                | M1             |
|                 | $-((x+3)^2 - 36)$ or $36 - (x+3)^2$                                                                                                                                                                                                                                                                      |                                                                                      | A1             |
|                 | $\int \frac{1}{\sqrt{27 - 6x - x^2}} dx = \int \frac{1}{\sqrt{36 - (x+3)^2}} dx = \arcsin\left(\frac{x+3}{6}\right) (+c)$ <p>(Or <math>= -\arccos\left(\frac{x+3}{6}\right) (+c)</math>)<br/>M1: Use of arcsin (or <math>-\arccos</math>)<br/>A1: Fully correct expression (condone omission of + c)</p> |                                                                                      | M1A1           |
|                 |                                                                                                                                                                                                                                                                                                          |                                                                                      | <b>(4)</b>     |
|                 |                                                                                                                                                                                                                                                                                                          |                                                                                      | <b>Total 8</b> |

|              |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                                                                                                                |          |
|--------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------|----------|
| (b)<br>Way 1 | $d = \frac{ 6(2) - 2k + 3(-7) - 5 }{\sqrt{6^2 + 2^2 + 3^2}}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          | Correct method for the shortest distance                                                                       | M1       |
|              | $= \frac{1}{7} -2k - 14  = \frac{2}{7} k + 7 *$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | Correct completion                                                                                             | A1*      |
|              | (2)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |                                                                                                                |          |
| (b)<br>Way 2 | Distance $O$ to $\Pi_1$ is $\frac{5}{\sqrt{6^2 + 2^2 + 3^2}}$ .<br>Distance $O$ to parallel plane containing $Q$ is $\frac{(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot (2\mathbf{i} + k\mathbf{j} - 7\mathbf{k})}{\sqrt{6^2 + 2^2 + 3^2}} = \frac{-9 - 2k}{7}$<br>$d = \left  \frac{5}{7} - \frac{-9 - 2k}{7} \right $<br>Correct method for the shortest distance                                                                                                                                                                                                                                                                                                                |                                                                                                                | M1       |
|              | $= \frac{1}{7} 2k + 14  = \frac{2}{7} k + 7 *$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | Correct completion                                                                                             | A1*      |
| (b)<br>Way 3 | $d = \frac{ \overrightarrow{PQ} \cdot \mathbf{n} }{ \mathbf{n} } = \frac{ (\mathbf{i} + (k - 2)\mathbf{j} - 8\mathbf{k}) \cdot (-42\mathbf{i} + 14\mathbf{j} - 21\mathbf{k}) }{\sqrt{42^2 + 14^2 + 21^2}}$<br>Correct method for the shortest distance                                                                                                                                                                                                                                                                                                                                                                                                                                |                                                                                                                | M1       |
|              | $= \frac{ -42 + 14k - 28 + 168 }{49} = \frac{ 14k + 98 }{49} = \frac{2}{7} k + 7 *$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | Correct completion                                                                                             | A1*      |
| (c)          | $\frac{2}{7} k + 7  = \frac{ 8(2) - 4k - 7 + 3 }{\sqrt{8^2 + 4^2 + 1^2}}$<br>Correctly attempts the distance between $(2, k, -7)$ and $\Pi_2$ and sets equal to the result from (a). May see alternative methods here for the distance between $(2, k, -7)$ and $\Pi_2$ e.g. finds the coordinates of a point on $\Pi_2$ e.g. $R(1, 1, -7)$ and then finds<br>$d = \frac{ \overrightarrow{RQ} \cdot (8\mathbf{i} - 4\mathbf{j} + \mathbf{k}) }{ 8\mathbf{i} - 4\mathbf{j} + \mathbf{k} } = \frac{ (\mathbf{i} + (k - 1)\mathbf{j}) \cdot (8\mathbf{i} - 4\mathbf{j} + \mathbf{k}) }{\sqrt{8^2 + 4^2 + 1^2}} = \left  \frac{8 - 4k + 4}{9} \right  = \left  \frac{12 - 4k}{9} \right $ |                                                                                                                | M1       |
|              | $\frac{2}{7}(k + 7) = \frac{1}{9}(12 - 4k) \Rightarrow k = \dots$ or $\frac{2}{7}(k + 7) = \frac{1}{9}(4k - 12) \Rightarrow k = \dots$<br>Attempts to solve one of these equations where their distance from $Q$ to $\Pi_2$ is of the form $ak + b$ where $a$ and $b$ are non-zero.<br>or<br>$\frac{2}{7}(k + 7) = \frac{1}{9}(12 - 4k) \Rightarrow \frac{4}{49}(k + 7)^2 = \frac{1}{81}(12 - 4k)^2$<br>$\Rightarrow 23k^2 - 462k - 441 = 0 \Rightarrow k = \dots$<br>Squares both sides and attempts to solve resulting quadratic.<br>Condone poor attempts at squaring the brackets and there is no requirement to follow the usual guidance for solving the quadratic              |                                                                                                                | dM1      |
|              | $k = -\frac{21}{23}$ or $k = 21$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      | One correct value. Must be 21 but allow equivalent exact fractions for $-\frac{21}{23}$                        | A1       |
|              | $k = -\frac{21}{23}$ and $k = 21$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | Both correct values. Must be 21 but allow equivalent exact fractions for $-\frac{21}{23}$ and no other values. | A1       |
|              | (4)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |                                                                                                                | (4)      |
|              | Total 11                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |                                                                                                                | Total 11 |

| Question Number | Scheme                                                                                                                                                                                                                                                                                                                                                               | Notes                                                                                                                                                                                                                                                      | Marks      |
|-----------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|
| <b>8(a)</b>     | $\frac{dy}{dx} = \frac{-2x}{1-x^2}$                                                                                                                                                                                                                                                                                                                                  | Correct derivative                                                                                                                                                                                                                                         | B1         |
|                 | $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4x^2}{(1-x^2)^2} = \frac{(1-x^2)^2 + 4x^2}{(1-x^2)^2}$ or $\frac{x^4 - 2x^2 + 1 + 4x^2}{(1-x^2)^2}$ or $\frac{x^4 + 2x^2 + 1}{(1-x^2)^2}$<br>Attempts $1 + \left(\frac{dy}{dx}\right)^2$ , finds common denominator and shows working in the numerator<br>condoning sign slips only. (The denominator may be expanded) |                                                                                                                                                                                                                                                            | M1         |
|                 | $= \frac{(1+x^2)^2}{(1-x^2)^2}$ or $\left(\frac{1+x^2}{1-x^2}\right)^2$                                                                                                                                                                                                                                                                                              | Fully correct expression with factorised numerator and denominator.                                                                                                                                                                                        | A1         |
|                 | $\int_{\frac{1}{2}}^{\frac{3}{4}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\frac{1+x^2}{1-x^2}\right) dx$ *                                                                                                                                                                                                               | Fully correct proof with no errors and integral as printed on the question paper but allow $x^2 + 1$ for $1 + x^2$ and allow<br>$\int_{\frac{1}{2}}^{\frac{3}{4}} \frac{(1+x^2)}{(1-x^2)} dx$ or $\int_{\frac{1}{2}}^{\frac{3}{4}} \frac{1+x^2}{1-x^2} dx$ | A1*        |
|                 |                                                                                                                                                                                                                                                                                                                                                                      |                                                                                                                                                                                                                                                            | <b>(4)</b> |

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |                |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------|
| (b) | $\frac{(x^2+1)}{(1-x^2)} = -1 + \frac{2}{1-x^2} \text{ or e.g. } -1 + \frac{1}{1-x} + \frac{1}{1+x}$ <p>Writes the improper fraction correctly</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | B1             |
|     | $\int \frac{k}{1-x^2} dx = \pm \alpha \ln \frac{1+x}{1-x}$ <p>Or e.g.</p> $\int \frac{k}{1-x^2} dx = \pm \alpha \ln(1+x) \pm \alpha \ln(1-x)$ <p>Or e.g.</p> $\int \frac{k}{1-x^2} dx = \pm \alpha \operatorname{artanh} x$ <p>Achieves an acceptable form for <math>\int \frac{k}{1-x^2} dx</math> (<math>k</math> constant)</p> <p>(may see partial fraction approach).</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | M1             |
|     | $\int -1 + \frac{2}{1-x^2} dx = -x + \ln \frac{1+x}{1-x}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | A1             |
|     | $\left[ -x + \ln \frac{1+x}{1-x} \right]_{\frac{1}{2}}^{\frac{3}{4}} = -\frac{3}{4} + \ln 7 - \left( -\frac{1}{2} + \ln 3 \right)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | dM1            |
|     | $= -\frac{1}{4} + \ln \frac{7}{3}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | A1             |
|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | (5)            |
|     | <p>Note that a common incorrect approach is:</p> $\int \frac{(1+x^2)}{(1-x^2)} dx = \int \left( \frac{1}{1-x^2} + \frac{x^2}{1-x^2} \right) dx = \frac{1}{2} \ln \frac{1+x}{1-x} + \dots$ $= \left[ \frac{1}{2} \ln \frac{1+x}{1-x} + \dots \right]_{\frac{1}{2}}^{\frac{3}{4}} = \dots$ <p>If there is no attempt at <math>\int \left( \frac{x^2}{1-x^2} \right) dx</math> this will generally score B0M1A0M0A0</p> <p>BUT</p> <p>If there is an attempt at <math>\int \left( \frac{x^2}{1-x^2} \right) dx</math> (however poor) and evidence that the limits have been applied this will generally score B0M1A0M1A0. Condone slips with the substitution of limits as long as the intention is clear.</p> <p><b>BUT</b> note that attempts that consider partial fractions such as <math>\frac{1+x^2}{1-x^2} \equiv \frac{A}{1-x} + \frac{B}{1+x}</math> will generally score no marks – <b>if you are unsure, send to review.</b></p> <p>Note also that <math>\frac{1+x^2}{1-x^2} \equiv \frac{A}{1-x} + \frac{B}{1+x} + C</math> is a correct form and could score full marks.</p> <p>Also, use of <math>\frac{(1+x^2)}{(1-x^2)} = \frac{1-x^2+2x^2}{1-x^2} = 1 + \frac{2x^2}{1-x^2}</math> with no attempt to deal with the <math>\frac{2x^2}{1-x^2}</math> as an improper fraction as in the main scheme is likely to score no marks.</p> |                |
|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | <b>Total 9</b> |

Example alternative approach to integration in part (b) by substitution:

|            |                                                                                                                                                                                                                                                                                                                                                                                                                                                               |     |
|------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| (b)        | $x = \tanh \theta \Rightarrow \int \frac{(1+x^2)}{(1-x^2)} dx = \int \frac{(1+\tanh^2 \theta)}{(1-\tanh^2 \theta)} \operatorname{sech}^2 \theta d\theta$ <p style="text-align: center;">Substitutes fully</p>                                                                                                                                                                                                                                                 | B1  |
|            | $\int \frac{(1+\tanh^2 \theta)}{(1-\tanh^2 \theta)} \operatorname{sech}^2 \theta d\theta = \int (1+\tanh^2 \theta) d\theta$ $= \int (2 - \operatorname{sech}^2 \theta) d\theta$ <p style="text-align: center;">Cancel and applies <math>\tanh^2 \theta = 1 - \operatorname{sech}^2 \theta</math></p>                                                                                                                                                          | M1  |
|            | $= \int (2 - \operatorname{sech}^2 \theta) d\theta = 2\theta - \tanh \theta$ <p style="text-align: center;">Correct integration</p>                                                                                                                                                                                                                                                                                                                           | A1  |
|            | $\left[ 2 \operatorname{artanh} x - x \right]_{\frac{1}{2}}^{\frac{3}{4}} = 2 \times \frac{1}{2} \ln \left( \frac{1+\frac{3}{4}}{1-\frac{3}{4}} \right) - \frac{3}{4} - \left( 2 \times \frac{1}{2} \ln \left( \frac{1+\frac{1}{2}}{1-\frac{1}{2}} \right) - \frac{1}{2} \right)$ <p style="text-align: center;">Evidence that the given limits have been applied. Condone slips as long as the intention is clear.<br/><b>Depends on the previous M.</b></p> | dM1 |
|            | $= -\frac{1}{4} + \ln \frac{7}{3}$ <p style="text-align: center;">cao</p>                                                                                                                                                                                                                                                                                                                                                                                     | A1  |
| <b>(5)</b> |                                                                                                                                                                                                                                                                                                                                                                                                                                                               |     |

There may be other attempts at  $\int \frac{1+x^2}{1-x^2}$  or  $\int \left( 1 + \frac{2x^2}{1-x^2} \right) dx$  by substitution.

Award the B mark for a correct full substitution into  $\int \frac{1+x^2}{1-x^2}$  or  $\int \left( 1 + \frac{2x^2}{1-x^2} \right)$

with  $x = f(\theta)$  where  $f$  is any trigonometric or hyperbolic function.

The first M mark can be scored if they reach something that is clearly directly “integrable”.

This will be hard to achieve for some choices like  $x = \cosh \theta$

Award the first A if the integration is correct - so that requires  $\int 1 dx = x$  as well if

$\int \left( 1 + \frac{2x^2}{1-x^2} \right)$  is being attempted.

The dependent M can be awarded if there is evidence that the given limits have been applied. So score M0 if their integration has led to something that is defined outside of the limits (such as  $\operatorname{arcosh} x$  or  $\operatorname{arcoth} x$ ). Then A1 for the correct answer.

| Question Number | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            | Notes                                                                                                                                                                                             | Marks |
|-----------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------|
| 9               | $\frac{x^2}{25} + \frac{y^2}{16} = 1, (5 \cos \theta, 4 \sin \theta)$                                                                                                                                                                                                                                                                                                                                                                                                                                             |                                                                                                                                                                                                   |       |
| (a)             | $\frac{dx}{d\theta} = -5 \sin \theta, \frac{dy}{d\theta} = 4 \cos \theta$<br>or<br>$\frac{2x}{25} + \frac{2y}{16} \frac{dy}{dx} = 0$ oe<br>or<br>$\frac{dy}{dx} = -\frac{4x}{25} \left(1 - \frac{x^2}{25}\right)^{-\frac{1}{2}}$ oe                                                                                                                                                                                                                                                                               | Correct derivatives or correct implicit differentiation or correct explicit differentiation.                                                                                                      | B1    |
|                 | $\frac{dy}{dx} = \frac{4 \cos \theta}{-5 \sin \theta}$                                                                                                                                                                                                                                                                                                                                                                                                                                                            | Divides their derivatives correctly or substitutes and rearranges                                                                                                                                 | M1    |
|                 | $M_N = \frac{5 \sin \theta}{4 \cos \theta}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | Correct perpendicular gradient rule – may be implied when they form the normal equation.                                                                                                          | M1    |
|                 | $y - 4 \sin \theta = \frac{5 \sin \theta}{4 \cos \theta} (x - 5 \cos \theta)$                                                                                                                                                                                                                                                                                                                                                                                                                                     | Correct straight line method (any complete method). <b>Must</b> use their gradient of the normal.                                                                                                 | M1    |
|                 | $5x \sin \theta - 4y \cos \theta = 9 \sin \theta \cos \theta^*$<br>or<br>$9 \sin \theta \cos \theta = 5x \sin \theta - 4y \cos \theta^*$                                                                                                                                                                                                                                                                                                                                                                          | Achieves the printed answer with no errors and allow this answer to be obtained from the previous line. Allow $5 \sin \theta x$ for $5x \sin \theta$ and $4 \cos \theta y$ for $4y \cos \theta$ . | A1*   |
|                 | Allow all marks if the gradient is seen as a function of $x$ and $y$ initially (even in the straight line equation) as long as this is recovered correctly.                                                                                                                                                                                                                                                                                                                                                       |                                                                                                                                                                                                   |       |
|                 | <b>Solutions that do not use calculus e.g. just quoting the equation of the normal as <math>y - 4 \sin \theta = \frac{5 \sin \theta}{4 \cos \theta} (x - 5 \cos \theta)</math> send to review however if they just quote e.g. <math>ax \sin \theta - by \sin \theta = (a^2 - b^2) \sin \theta \cos \theta</math> and then write down the given result this scores no marks.</b><br><b>But we would accept <math>\frac{dy}{dx} = \frac{4 \cos \theta}{-5 \sin \theta}</math> to be quoted for a full solution.</b> |                                                                                                                                                                                                   |       |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |                                                                                                                                                                                                   | (5)   |
| (b)             | $b^2 = a^2 (1 - e^2) \Rightarrow 16 = 25 (1 - e^2) \Rightarrow e = \frac{3}{5}$<br>$F \text{ is } (ae, 0) = \left(5 \times \frac{3}{5}, 0\right)$<br>Or e.g. " $c^2 = a^2 e^2 = a^2 - b^2 = 25 - 16 \Rightarrow a^2 e^2 = 9 \Rightarrow ae = \dots$ "<br>Fully correct strategy for $F$ (must be numerical so $(5e, 0)$ is M0                                                                                                                                                                                     |                                                                                                                                                                                                   | M1    |
|                 | $(3, 0)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          | Correct coordinates. $(\pm 3, 0)$ scores A0                                                                                                                                                       | A1    |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |                                                                                                                                                                                                   | (2)   |

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |                                                                                                                                                                                                                                                          |                 |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------|
| (c) | $x = \frac{9}{5} \cos \theta$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | Correct $x$ coordinate (of $Q$ )                                                                                                                                                                                                                         | B1              |
|     | $PF^2 = (5 \cos \theta - "3")^2 + (4 \sin \theta)^2$<br>or<br>$PF = \sqrt{(5 \cos \theta - "3")^2 + (4 \sin \theta)^2}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | Correct application of Pythagoras to find $PF$ or $PF^2$ . Their "3" should be positive but allow work in terms of $e$ e.g. "5e".                                                                                                                        | M1              |
|     | $= 25 \cos^2 \theta - 30 \cos \theta + 9 + 16 \sin^2 \theta$<br>$= 25 \cos^2 \theta - 30 \cos \theta + 9 + 16(1 - \cos^2 \theta)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | Applies $\sin^2 \theta = 1 - \cos^2 \theta$ to obtain a quadratic expression in $\cos \theta$ . If the correct identity is not seen explicitly then their working must imply that a correct identity has been used.<br><b>Depends on the previous M.</b> | dM1             |
|     | $PF = \pm(5 - 3 \cos \theta)$<br>$PF^2 = 9 \cos^2 \theta - 30 \cos \theta + 25$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               | Correct expression for $PF$ or $PF^2$ in terms of $\cos \theta$ with terms collected.                                                                                                                                                                    | A1              |
|     | Note that an alternative to using Pythagoras to find $PF$ is to use $PF = ePM$ where $M$ is the foot of the perpendicular from $P$ to the <b>positive</b> directrix.<br>$\text{Score M1 for } x = \frac{a}{e} = \frac{5}{3/5} \left( = \frac{25}{3} \right) \text{ (not } \pm \frac{25}{3} \text{)}$ $\text{and dM1A1 for } PF = ePM = \frac{3}{5} \left( \frac{25}{3} - 5 \cos \theta \right)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |                                                                                                                                                                                                                                                          |                 |
|     | $\frac{ QF }{ PF } = \frac{3 - \frac{9}{5} \cos \theta}{5 - 3 \cos \theta} = \frac{3 \left( 1 - \frac{3}{5} \cos \theta \right)}{5 \left( 1 - \frac{3}{5} \cos \theta \right)} \text{ or e.g. } \frac{3}{5} \times \frac{1 - \frac{3}{5} \cos \theta}{1 - \frac{3}{5} \cos \theta} = \frac{3}{5} = e^*$ <p style="text-align: center;"><b>or e.g.</b></p> $\frac{QF^2}{PF^2} = \frac{\left( 3 - \frac{9}{5} \cos \theta \right)^2}{9 \cos^2 \theta - 30 \cos \theta + 25} = \frac{9 - \frac{54}{5} \cos \theta + \frac{81}{25} \cos^2 \theta}{9 \cos^2 \theta - 30 \cos \theta + 25}$ $= \frac{9 \left( 1 - \frac{6}{5} \cos \theta + \frac{9}{25} \cos^2 \theta \right)}{25 \left( 1 - \frac{6}{5} \cos \theta + \frac{9}{25} \cos^2 \theta \right)} \text{ or e.g. } = \frac{9}{25} \times \frac{1 - \frac{6}{5} \cos \theta + \frac{9}{25} \cos^2 \theta}{1 - \frac{6}{5} \cos \theta + \frac{9}{25} \cos^2 \theta} = \frac{9}{25} \Rightarrow \frac{QF}{PF} = \frac{3}{5} = e^*$ <p>Fully correct working including factorisation or equivalent leading to showing that <math>\frac{ QF }{ PF } = e</math> with no errors and a conclusion "<math>= e</math>".</p> <p>Note that the value of <math>e</math> must have been seen earlier e.g. in part (b) or calculated independently somewhere in the question.</p> <p>Note that this mark depends on a ratio where the numerator and denominator are either both positive or both negative or modulus symbols are present throughout. This does not apply to the second case as both numerator and denominator must be positive as they are squared.</p> |                                                                                                                                                                                                                                                          | A1*             |
|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |                                                                                                                                                                                                                                                          | (5)             |
|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |                                                                                                                                                                                                                                                          | <b>Total 12</b> |

| Question   | Scheme                                                                                                                                                                        | Marks    |
|------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------|
| 7(a)       | $\frac{dy}{dx} = \frac{b \cos \theta}{-a \sin \theta}$ or $\frac{dy}{dx} = -\frac{2x}{a^2} \times \frac{b^2}{2y}$                                                             | M1       |
|            | $m_N = \frac{a \sin \theta}{b \cos \theta}$                                                                                                                                   | A1       |
|            | Equation is $y - b \sin \theta = \frac{a \sin \theta}{b \cos \theta} (x - a \cos \theta)$                                                                                     | M1       |
|            | $\Rightarrow by - b^2 \sin \theta = ax \tan \theta - a^2 \sin \theta \Rightarrow by = ax \tan \theta + (b^2 - a^2) \sin \theta^*$                                             | A1*      |
|            |                                                                                                                                                                               | (4)      |
| (b)        | $B$ is $(0, -b) \Rightarrow -b^2 = (b^2 - a^2) \sin \theta$                                                                                                                   | B1       |
|            | Area $OBP = \frac{3}{4} b^2 \Rightarrow \frac{1}{2} b \times a \cos \theta = \frac{3}{4} b^2$                                                                                 | M1       |
|            | $\Rightarrow 2a \cos \theta = 3b \Rightarrow b^2 = \frac{4}{9} \cancel{a^2} \cos^2 \theta = \left( \cancel{a^2} - \frac{4}{9} \cancel{a^2} \cos^2 \theta \right) \sin \theta$ | dM1      |
|            | $\Rightarrow 4(1 - \sin^2 \theta) = (9 - 4(1 - \sin^2 \theta)) \sin \theta$<br>$\Rightarrow 4 \sin^3 \theta + 4 \sin^2 \theta + 5 \sin \theta - 4 = 0$                        | ddM1     |
|            | $\Rightarrow (2 \sin \theta - 1)(2 \sin^2 \theta + 3 \sin \theta + 4) = 0 \Rightarrow \sin \theta = \frac{1}{2}^*$                                                            | A1*      |
|            |                                                                                                                                                                               | (5)      |
| (c)        | $b = \frac{4}{3} \times \frac{1}{2} a \times \sqrt{1 - \left(\frac{1}{2}\right)^2} = \dots \left( = \frac{\sqrt{3}}{3} a \right)$                                             | M1       |
|            | $P$ is $\left( \frac{\sqrt{3}}{2} a, \frac{\sqrt{3}}{6} a \right)$                                                                                                            | B1<br>A1 |
|            |                                                                                                                                                                               | (3)      |
| (12 marks) |                                                                                                                                                                               |          |

**Notes:**
**(a)**

**M1:** A correct method for  $\frac{dy}{dx}$  or  $-\frac{dx}{dy}$  in terms of  $\theta$  (from parametric differentiation) or in terms of  $x$  and  $y$  (from implicit or Cartesian approaches).

**A1:** Correct gradient for the normal in terms of  $\theta$

**M1:** Correct method for the normal through  $P$  applied. There must have been a correct attempt at the gradient of the normal (ie negative reciprocal of their  $\frac{dy}{dx}$ , or directly finding  $-\frac{dx}{dy}$ ). Allow recovery from gradients in terms of  $x$  and  $y$  only if correct substitution for them occurs. If using  $y = mx + c$  they must proceed as far as finding  $c$ .

**A1\*:** Correct completion to the equation given with no errors seen in working and an intermediate step (e.g. if using  $y = mx + c$  they must substitute  $c$  back in before proceeding to the given answer).

**(b)**


**B1:** States or identifies the point  $B$  is  $(0, -b)$  and substitutes into the normal equation to obtain the correct equation shown or equivalent.

**M1:** Correct attempt at the area and sets equal to  $\frac{3}{4}b^2$  to form a second equation. Note there are other possible methods for the area of the triangle e.g. “shoelace”/determinant method, or use of separate triangles. Some will essentially give the same result as in the scheme, but others will give more complicated formulae, making the next M much more challenging (and unlikely to be gained).

**dm1:** Solves the equations simultaneously to achieve an equation in  $\theta$  only. Alternatively, eliminates  $\sin \theta$  and forms a polynomial equation in  $a^2$  and  $b^2$ . This latter is unlikely to make further progress.

**ddM1:** Rearranges to an expanded (but not necessarily fully simplified) cubic in  $\sin \theta$ . For equivalent working in  $a^2$  or  $b^2$  they must proceed as far as finding a relationship between  $a^2$  and  $b^2$ .

**A1\*:** Fully correct work, showing the factorised cubic to justify the given value of  $\sin \theta$ . No need to show there is only one solution (we allow the factorised to imply the quadratic factor does not reduce). Note substitution of  $\sin \theta = \frac{1}{2}$  into the equation is A0 as it provides no evidence of no other solutions.

For working in  $a$  and  $b$  the factorisation to deduce the relationship between  $a^2$  and  $b^2$  must have been seen.

**(c)**

**M1:** Correct process to find  $b$  in terms of  $a$

**B1:**  $x$  coordinate correct

**A1:**  $y$  coordinate correct, accept  $\frac{a}{2\sqrt{3}}$ .

**(b) Working in  $a$  and  $b$ :**

$$\sin^2 \theta + \cos^2 \theta = 1 \Rightarrow \frac{b^4}{(b^2 - a^2)^2} + \frac{9b^2}{4a^2} = 1 \Rightarrow 4a^2b^4 + 9b^2(b^2 - a^2)^2 = 4a^2(b^2 - a^2)^2 \quad \text{dM1}$$

$$\Rightarrow 9b^6 - 18a^2b^4 + 17a^4b^2 - 4a^6 = 0 \Rightarrow (3b^2 - a^2)(3b^4 - 5a^2b^2 + 4a^4) = 0$$

$$\Rightarrow a^2 = 3b^2 \quad \text{ddM1}$$

$$\Rightarrow \sin \theta = -\frac{b^2}{b^2 - 3b^2} = -\frac{1}{-2} = \frac{1}{2} \quad \text{A1}$$

Note: A lot of solutions are not using that the coordinate of  $B$  is  $(0, -b)$  but instead use  $\sin \theta = \frac{1}{2}$  to

show  $\sin \theta = \frac{1}{2}$ . These will score at most B0M1M0M0A0. as they have no equations to solve simultaneously.

| Question Number | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            | Marks |
|-----------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------|
| 9(a)            | $b^2 = a^2(e^2 - 1) \Rightarrow 49 = 64(e^2 - 1) \Rightarrow e^2 = \dots$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         | M1    |
|                 | $49 = 64e^2 - 64 \Rightarrow e^2 = \frac{113}{64} \Rightarrow e = \frac{\sqrt{113}}{8}^*$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         | A1*   |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | (2)   |
| (b)             | $x = 8 \sec t, y = 7 \tan t \Rightarrow \frac{dy}{dx} = \frac{7 \sec^2 t}{8 \sec t \tan t} = \frac{7 \sec^2 \theta}{8 \sec \theta \tan \theta} \left( = \frac{7}{8} \operatorname{cosec} \theta \right)$ (Parametric diff.)<br>or<br>$\frac{x^2}{64} - \frac{y^2}{49} = 1 \Rightarrow \frac{x}{32} - \frac{2y}{49} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{49x}{64y} = \frac{7 \sec \theta}{8 \tan \theta}$ (Implicit diff.)<br>or<br>$\frac{x^2}{64} - \frac{y^2}{49} = 1 \Rightarrow y = 7 \sqrt{\frac{x^2}{64} - 1} \Rightarrow \frac{dy}{dx} = \frac{7}{2} \left( \frac{x^2}{64} - 1 \right)^{-\frac{1}{2}} \times \frac{x}{32} = \frac{7}{2} (\sec^2 \theta - 1)^{-\frac{1}{2}} \times \frac{8 \sec \theta}{32}$ | B1    |
|                 | $y - 7 \tan \theta = \frac{7 \sec \theta}{8 \tan \theta} (x - 8 \sec \theta)$<br>or<br>$'y = mx + c' \Rightarrow 7 \tan \theta = \frac{7 \sec \theta}{8 \tan \theta} \times 8 \sec \theta + c \Rightarrow c = \dots \left( -\frac{7}{\tan \theta} \right)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | M1    |
|                 | $y - 7 \tan \theta = \frac{7 \sec \theta}{8 \tan \theta} (x - 8 \sec \theta) \Rightarrow 8y \tan \theta - 56 \tan^2 \theta = 7x \sec \theta - 56 \sec^2 \theta$<br>$\Rightarrow \frac{x}{8} \sec \theta - \frac{y}{7} \tan \theta = \sec^2 \theta - \tan^2 \theta$<br>$\Rightarrow \frac{x}{8} \sec \theta - \frac{y}{7} \tan \theta = 1^*$                                                                                                                                                                                                                                                                                                                                                                                       | A1*   |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | (3)   |
| (c)             | $(0, -7 \cot \theta)$ o.e.                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | B1    |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | (1)   |
| (d)             | $\left( 0, \frac{113}{7} \tan \theta \right)$ o.e.                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                | B1    |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | (1)   |
| (e)             | Centre is $\left( 0, \frac{113}{14} \tan \theta - \frac{7}{2} \cot \theta \right)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                | B1    |
|                 | (Radius is) $\frac{113}{14} \tan \theta + \frac{7}{2} \cot \theta$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                | B1    |
|                 | $x^2 + \left( y - \left( \frac{113}{14} \tan \theta - \frac{7}{2} \cot \theta \right) \right)^2 = \left( \frac{113}{14} \tan \theta + \frac{7}{2} \cot \theta \right)^2$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          | M1    |
|                 | $y = 0 \Rightarrow x^2 + \left( \frac{113}{14} \tan \theta - \frac{7}{2} \cot \theta \right)^2 = \left( \frac{113}{14} \tan \theta + \frac{7}{2} \cot \theta \right)^2$<br>$\Rightarrow x^2 = 2 \times 2 \times \frac{113}{14} \tan \theta \times \frac{7}{2} \cot \theta = 113 \Rightarrow x = \pm \sqrt{113}$<br>Foci are at $(\pm ae, 0) = \left( \pm 8 \times \frac{\sqrt{113}}{8}, 0 \right) = (\pm \sqrt{113}, 0)$                                                                                                                                                                                                                                                                                                          | M1    |
|                 | Hence the circle passes through the foci of H*                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    | A1*   |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | (5)   |



|                                               |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |     |
|-----------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| <b>ALT for final M1A1</b>                     | <p>Foci are at <math>(\pm ae, 0) = \left(\pm 8 \times \frac{\sqrt{113}}{8}, 0\right) = (\pm \sqrt{113}, 0)</math></p> <p>Substituting into the circle equation:</p> $LHS = (\pm \sqrt{113})^2 + \left(\frac{113}{14}\right)^2 \tan^2 \theta + \left(\frac{7}{2}\right)^2 \cot^2 \theta - 2\left(\frac{113}{14}\right)\left(\frac{7}{2}\right) = \left(\frac{113}{14}\right)^2 \tan^2 \theta + \left(\frac{7}{2}\right)^2 \cot^2 \theta + \frac{113}{2}$ $RHS = \left(\frac{113}{14}\right)^2 \tan^2 \theta + \left(\frac{7}{2}\right)^2 \cot^2 \theta + 2\left(\frac{113}{14}\right)\left(\frac{7}{2}\right) = \left(\frac{113}{14}\right)^2 \tan^2 \theta + \left(\frac{7}{2}\right)^2 \cot^2 \theta + \frac{113}{2}$ | M1  |
|                                               | $LHS = RHS$ so the circle passes through the foci of $H$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               | A1* |
| <b>ALT 2</b><br>Circle theorems and gradients | $m_{FR} = \frac{\frac{113}{7} \tan \theta - 0}{0 - \sqrt{113}} = -\frac{113}{7\sqrt{113}} \tan \theta$ <p>Or</p> $m_{FR} = \frac{\frac{113}{7} \tan \theta - 0}{0 + \sqrt{113}} = \frac{113}{7\sqrt{113}} \tan \theta$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | B1  |
|                                               | $m_{FQ} = \frac{-7 \cot \theta - 0}{0 - \sqrt{113}} = \frac{7}{\sqrt{113}} \cot \theta$ <p>Or</p> $m_{FQ} = \frac{-7 \cot \theta - 0}{0 + \sqrt{113}} = -\frac{7}{\sqrt{113}} \cot \theta$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | B1  |
|                                               | $m_{FR} \times m_{FQ} = -\frac{113}{7\sqrt{113}} \tan \theta \times \frac{7}{\sqrt{113}} \cot \theta = -1$ <p>Or</p> $m_{FR} \times m_{FQ} = \frac{113}{7\sqrt{113}} \tan \theta \times -\frac{7}{\sqrt{113}} \cot \theta = -1$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | M1  |
|                                               | <p>For both <math>m_{FR} \times m_{FQ} = -\frac{113}{7\sqrt{113}} \tan \theta \times \frac{7}{\sqrt{113}} \cot \theta = -1</math><br/>and</p> $m_{FR} \times m_{FQ} = \frac{113}{7\sqrt{113}} \tan \theta \times -\frac{7}{\sqrt{113}} \cot \theta = -1$                                                                                                                                                                                                                                                                                                                                                                                                                                                               | M1  |
|                                               | $m_{FR} \times m_{FQ} = -1$ for both foci, so the circle passing through $F$ has $QR$ as a diameter                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    | A1* |
|                                               |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | (5) |

|                                                         |                                                                                                                                       |                    |
|---------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------|--------------------|
| <b>ALT 3</b><br>Circle<br>theorems<br>and<br>Pythagoras | $FQ = \sqrt{113 + 49 \cot^2 \theta}$ or $FQ^2 = 113 + 49 \cot^2 \theta$                                                               | B1                 |
|                                                         | $FR = \sqrt{113 + \left(\frac{113}{7} \tan \theta\right)^2}$ or $FR^2 = 113 + \left(\frac{113}{7} \tan \theta\right)^2$ or better     | B1                 |
|                                                         | $FQ^2 + FR^2 = 49 \cot^2 \theta + \left(\frac{113}{7} \tan \theta\right)^2 + 226$                                                     | M1                 |
|                                                         | $QR^2 = \left(7 \cot \theta + \frac{113}{7} \tan \theta\right)^2 = 49 \cot^2 \theta + \left(\frac{113}{7} \tan \theta\right)^2 + 226$ | M1                 |
|                                                         | $QR^2 = FR^2 + FQ^2$ for both foci, so the circle passing through $F$ has $QR$ as a diameter                                          | A1*                |
|                                                         |                                                                                                                                       | (5)                |
|                                                         |                                                                                                                                       | <b>Total</b><br>12 |

**Notes**

(a)

**M1:** Uses a correct eccentricity formula with  $a^2 = 64$  and  $b^2 = 49$  and rearranges for  $e^2$ **A1\*:** Correct proof with sufficient working and no errors. Must have at least one line of correct working between their correct eccentricity formula with  $a^2 = 64$  and  $b^2 = 49$  substituted before the printed answer. It's not sufficient to go from a correct formula with values substituted directly to a simplified  $e^2$  or  $e$  without someintermediate working. Do not accept  $e = \pm \frac{\sqrt{113}}{8}$ . Some exemplars are below

$$b^2 = a^2(e^2 - 1) \Rightarrow 49 = 64(e^2 - 1) \Rightarrow e = \frac{\sqrt{113}}{8} \quad \text{M1A0* (no intermediate line of working shown)}$$

$$49 = 64(e^2 - 1) \Rightarrow e^2 = \frac{113}{64} \Rightarrow e = \frac{\sqrt{113}}{8} \quad \text{M1A0* (no intermediate line of working shown)}$$

$$49 = 64(e^2 - 1) \Rightarrow e^2 = \frac{49 + 64}{64} \Rightarrow e = \frac{\sqrt{113}}{8} \quad \text{M1A1* (Intermediate line of working before a simplified } e^2 \text{ or } e \text{ is seen).}$$

$$7^2 = 8^2(e^2 - 1) \Rightarrow e^2 - 1 = \frac{49}{64} \Rightarrow e = \frac{\sqrt{113}}{8} \quad \text{M1A1* (Intermediate line of working before a simplified } e^2 \text{ or } e \text{ is seen).}$$

**Note:** accept a verification method which shows that 'LHS = RHS' after a correct eccentricity formula with  $a^2 = 64$  and  $b^2 = 49$  is seen.

(b)

**B1:** Correct tangent gradient obtained using calculus. Need not be simplified for this mark, but it must be a fully correct derivative expression in terms of  $\theta$ .**M1:** Uses a correct straight-line method with their gradient and the point  $P$ . If they are using  $y = mx + c$  then they must make progress to finding a value for their ' $c$ '.**A1\*:** Correct proof with no errors and sufficient working shown. At least one line of intermediate working required between their correct line equation and the given answer.

(c)

**B1:** Correct coordinates or  $x = 0$  and  $y = -7 \cot \theta$  o.e. e.g.  $\left(0, -\frac{7}{\tan \theta}\right)$  or  $\left(0, -\frac{7 \cos \theta}{\sin \theta}\right)$ . Can award if seen under other parts of the question, but must be clearly labelled as point  $Q$ .

(d)

**B1:** Correct coordinates or  $x = 0$  and  $y = \frac{113}{7} \tan \theta$  o.e. e.g.  $\left(0, \frac{113 \sin \theta}{7 \cos \theta}\right)$  or  $\left(0, \frac{113}{7 \cot \theta}\right)$ . Can award if seen under other parts of the question, but must be clearly labelled as point  $R$ .

(e)

**B1:** Correct centre in any equivalent form e.g.

$$\left(0, \frac{113 \tan \theta - 49 \cot \theta}{14}\right) \text{ or } \left(0, \frac{113 \frac{\sin \theta}{\cos \theta} - 49 \frac{\cos \theta}{\sin \theta}}{14}\right) \text{ or } \left(0, \frac{113 \sin^2 \theta - 49 \cos^2 \theta}{14 \sin \theta \cos \theta}\right)$$

**B1:** Correct radius in any equivalent form e.g.  $\frac{113 \sin \theta}{14 \cos \theta} + \frac{7 \cos \theta}{2 \sin \theta}$  or  $\frac{113 \sin^2 \theta + 49 \cos^2 \theta}{14 \sin \theta \cos \theta}$  or  $\frac{64 \sin^2 \theta + 49}{14 \sin \theta \cos \theta}$

Ignore the labelling for this mark. Award this mark if the correct expression is seen.

**M1:** Uses their centre and radius to form the equation of the circle in the form  $x^2 + (y - \dots)^2 = r^2$

**M1:** Substitutes  $y = 0$ , solves for  $x$  and uses the result from part (a) to compare with the values of  $\pm ae$

**A1\*:** Fully correct work with conclusion

#### ALT for final M1A1

**M1:** Substitutes the correct foci into their circle equation, and makes some attempt to simplify both sides to show they are equal- may be seen with  $x$  squared which is fine.

**A1:** Shows that 'LHS = RHS' for their correct circle equation and gives a basic conclusion.

#### ALT 2 using circle theorems and gradients

**B1:** for finding the gradient of  $FR$  for either focus  $F$

**B1:** For finding the gradient of  $FQ$  for either focus  $F$

**M1:** For using  $m_{FR} \times m_{FQ} = -1$  to show that  $FQ$  is perpendicular to  $FR$  for either focus

**M1:** For showing that  $FQ$  is perpendicular to  $FR$  for both foci

**A1:** Concludes that the circle passing through the focus (point  $F$ ) has  $QR$  as a diameter

#### ALT 3 using circle theorems and converse of Pythagoras

**B1:** For finding  $FQ$  or  $FQ^2$

**B1:**  $FR$  or  $FR^2$

**M1:** For finding  $FQ^2 + FR^2$

**M1** For showing that  $QR^2 = FQ^2 + FR^2$

**A1:** Concludes that the circle passing through the focus (point  $F$ ) has  $QR$  as a diameter



| Question Number | Scheme                                                                                                                                                                                                                                                                                                                                                                           | Notes                                                                                                                                     | Marks |
|-----------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------|-------|
| 8(a)            | $b^2 = a^2(1 - e^2) \Rightarrow 4 = 9(1 - e^2) \Rightarrow e = \dots$<br>or e.g.<br>$e = \sqrt{1 - \frac{b^2}{a^2}} \Rightarrow e = \dots$                                                                                                                                                                                                                                       | Uses a correct formula with $a$ and $b$ correctly placed to find a value for $e$                                                          | M1    |
|                 | $e = \frac{\sqrt{5}}{3}$                                                                                                                                                                                                                                                                                                                                                         | Correct value (or equivalent)<br>$e = \pm \frac{\sqrt{5}}{3}$ scores A0                                                                   | A1    |
|                 |                                                                                                                                                                                                                                                                                                                                                                                  |                                                                                                                                           | (2)   |
| (b)(i)          | $(\pm ae, 0) = (\pm\sqrt{5}, 0)$ or $\left(\pm 3\frac{\sqrt{5}}{3}, 0\right)$<br>Correct foci. Must be coordinates but allow unsimplified and isw if necessary.<br>Follow through their $e$ so allow for $(\pm 3 \times \text{their } e, 0)$                                                                                                                                     |                                                                                                                                           | B1ft  |
| (ii)            | $x = \pm \frac{a}{e} = \pm \frac{9}{\sqrt{5}}$ or $x = \pm \frac{3}{\frac{\sqrt{5}}{3}}$<br>Correct directrices. Must be equations but allow unsimplified and isw if necessary.<br>Follow through their $e$ so allow for $x = \pm 3/\text{their } e$                                                                                                                             |                                                                                                                                           | B1ft  |
|                 |                                                                                                                                                                                                                                                                                                                                                                                  |                                                                                                                                           | (2)   |
|                 | <b>Special case:</b><br>Use of $a^2$ for $a$ and $b^2$ for $b$ <u>consistently</u> scores M0A0 in (a) and B1ft B1ft in (b)<br>This gives $e = \frac{\sqrt{65}}{9}$ , $(\pm\sqrt{65}, 0)$ , $x = \pm \frac{81}{\sqrt{65}}$                                                                                                                                                        |                                                                                                                                           |       |
| (c)             | $\frac{dx}{d\theta} = -3\sin\theta$ , $\frac{dy}{d\theta} = 2\cos\theta$<br>or<br>$\frac{2x}{9} + \frac{2y}{4} \frac{dy}{dx} = 0$<br>or<br>$y = \left(4 - \frac{4x^2}{9}\right)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = -\frac{4x}{9} \left(4 - \frac{4x^2}{9}\right)^{-\frac{1}{2}}$<br>$\Rightarrow \frac{dy}{dx} = \dots \left( = \frac{2\cos\theta}{-3\sin\theta} \right)$ | Correct strategy for the gradient of $l$ in terms of $\theta$ .<br>Allow $\frac{dy}{dx} = \frac{2\cos\theta}{-3\sin\theta}$ to be stated. | M1    |
|                 | $y - 2\sin\theta = \frac{2\cos\theta}{-3\sin\theta}(x - 3\cos\theta)$                                                                                                                                                                                                                                                                                                            | Correct straight line method (any complete method). Finding the equation of the normal is M0.                                             | M1    |
|                 | $-3y\sin\theta + 6\sin^2\theta = 2x\cos\theta - 6\cos^2\theta$<br>$2x\cos\theta + 3y\sin\theta = 6^*$                                                                                                                                                                                                                                                                            | Cso with at least one intermediate line of working                                                                                        | A1*   |
|                 |                                                                                                                                                                                                                                                                                                                                                                                  |                                                                                                                                           | (3)   |

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                            |     |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------|-----|
| (d) | $l_2: y = \frac{3 \sin \theta}{2 \cos \theta} x$                                                                                                                                                                                                                                                                                                                                                                                                                                                      | Correct equation for $l_2$ | B1  |
|     | $2x \cos \theta + 3y \sin \theta = 6, y = \frac{3 \sin \theta}{2 \cos \theta} x$<br>$\Rightarrow x = \dots, y = \dots$                                                                                                                                                                                                                                                                                                                                                                                | Complete method for $Q$    | M1  |
|     | $Q: \left( \frac{12 \cos \theta}{4 \cos^2 \theta + 9 \sin^2 \theta}, \frac{18 \sin \theta}{4 \cos^2 \theta + 9 \sin^2 \theta} \right)$<br>Correct coordinates. Allow as $x = \dots, y = \dots$ and allow equivalent correct expressions as long as they are single fractions<br>e.g. $x = \frac{12 \cos \theta}{4 + 5 \sin^2 \theta} \quad y = \frac{18 \sin \theta}{4 + 5 \sin^2 \theta}, \quad x = \frac{12 \cos \theta}{9 - 5 \cos^2 \theta} \quad y = \frac{18 \sin \theta}{9 - 5 \cos^2 \theta}$ |                            | A1  |
|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                            | (3) |

|              |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                                                                                    |                 |
|--------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------|-----------------|
| (e)          | At $Q$ , $\frac{y}{x} = \frac{3}{2} \tan \theta$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          | Uses their coordinates of $Q$ to attempt an equation connecting $x$ , $y$ and $\theta$ or states or uses the equation found in (d) | M1              |
|              | $x = \frac{12 \cos \theta}{4 \cos^2 \theta + 9 \sin^2 \theta} = \frac{12 \sec \theta}{4 + 9 \tan^2 \theta} \Rightarrow x^2 = \frac{144 \sec^2 \theta}{(4 + 9 \tan^2 \theta)^2} = \frac{144 \left(1 + \frac{4y^2}{9x^2}\right)}{\left(4 + 9 \times \frac{4y^2}{9x^2}\right)^2}$ <p style="text-align: center;">or</p> $y = \frac{18 \sin \theta}{4 \cos^2 \theta + 9 \sin^2 \theta} = \frac{12 \sec \theta \tan \theta}{4 + 9 \tan^2 \theta}$ $\Rightarrow y^2 = \frac{324 \sec^2 \theta \tan^2 \theta}{(4 + 9 \tan^2 \theta)^2} = \frac{324 \left(1 + \frac{4y^2}{9x^2}\right) \frac{4y^2}{9x^2}}{\left(4 + 9 \times \frac{4y^2}{9x^2}\right)^2}$ <p style="text-align: center;">Eliminates <math>\theta</math><br/><b>Depends on the first mark.</b></p> |                                                                                                                                    | dM1             |
|              | $\Rightarrow x^2 = \frac{x^2 (9x^2 + 4y^2)}{(x^2 + y^2)^2} \Rightarrow (x^2 + y^2)^2 = 9x^2 + 4y^2$ <p style="text-align: center;">or</p> $\Rightarrow 9 \times 16x^2 y^2 \left(1 + \frac{y^2}{x^2}\right)^2 = 4 \times 18^2 \left(1 + \frac{4y^2}{9x^2}\right) \Rightarrow (x^2 + y^2)^2 = 9x^2 + 4y^2$ <p style="text-align: center;">Correct equation or correct values for <math>\alpha</math> and <math>\beta</math>.</p>                                                                                                                                                                                                                                                                                                                            |                                                                                                                                    | A1              |
|              |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                                                                                    | (3)             |
| (e)<br>Way 2 | $x = \frac{12 \cos \theta}{4 + 5 \sin^2 \theta} \quad y = \frac{18 \sin \theta}{4 + 5 \sin^2 \theta} \Rightarrow (x^2 + y^2)^2 = \left( \frac{144 \cos^2 \theta + 324 \sin^2 \theta}{(4 + 5 \sin^2 \theta)^2} \right)^2$ <p style="text-align: center;">Uses their <math>Q</math> to obtain an expression for <math>(x^2 + y^2)^2</math> in terms of <math>\theta</math></p>                                                                                                                                                                                                                                                                                                                                                                              |                                                                                                                                    | M1              |
|              | $\left( \frac{144 \cos^2 \theta + 324 \sin^2 \theta}{(4 + 5 \sin^2 \theta)^2} \right)^2 = \left( \frac{144 + 180 \sin^2 \theta}{(4 + 5 \sin^2 \theta)^2} \right)^2 = \left( \frac{36(4 + 5 \sin^2 \theta)}{(4 + 5 \sin^2 \theta)^2} \right)^2 = \frac{1296}{(4 + 5 \sin^2 \theta)^2}$ $\frac{1296}{(4 + 5 \sin^2 \theta)^2} = \alpha x^2 + \beta y^2 = \alpha \frac{144 \cos^2 \theta}{(4 + 5 \sin^2 \theta)^2} + \beta \frac{324 \sin^2 \theta}{(4 + 5 \sin^2 \theta)^2} \Rightarrow \alpha = \dots, \beta = \dots$ <p style="text-align: center;">Substitutes into the given answer and solves for <math>\alpha</math> and <math>\beta</math><br/><b>Depends on the first mark.</b></p>                                                                 |                                                                                                                                    | dM1             |
|              | $(x^2 + y^2)^2 = 9x^2 + 4y^2$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | Correct expression or correct values for $\alpha$ and $\beta$ .                                                                    | A1              |
|              |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                                                                                    | <b>Total 13</b> |

| Question Number                                                  | Scheme                                                                                                                                                                                                                                                                                                     | Notes                                                                                                                                                                                                                                                      | Marks           |
|------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------|
| 7(c)                                                             |                                                                                                                                                                                                                                                                                                            |                                                                                                                                                                                                                                                            |                 |
| Way 2                                                            | $(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) = -5$                                                                                                                                                                                                            | M1: Finds a value for the scalar product of a position vector to a point the plane and their normal.<br>A1: -5 (or 5 if normal is in the opposite direction)                                                                                               | M1 A1           |
| Perp. distance formula                                           | $\frac{8x - 2y - 3z + 5 = 0}{\sqrt{8^2 + 2^2 + 3^2}}$ Shortest distance is<br>$\frac{ (8)(6) + (-2)(-3) + (-3)(-6) + 5 }{\sqrt{8^2 + 2^2 + 3^2}}$ $= \frac{77}{\sqrt{77}} \text{ or } \sqrt{77}$                                                                                                           | dM1: Uses distance formula with their normal and plane equation to reach a numerical expression for the distance. Condone sign slip on their -5 and their $d$ must not be zero.<br><b>Dependent on previous M mark.</b><br>A1: Correct exact distance. Isw | dM1 A1          |
|                                                                  |                                                                                                                                                                                                                                                                                                            |                                                                                                                                                                                                                                                            | (4)             |
| Way 3                                                            | Let $Q$ be the point on the plane $(1, 2, 3)$ then $\overrightarrow{PQ} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - (6\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}) = -5\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}$                                                                                                 | M1: Attempts vector from given point to a point on the plane<br>A1: Correct vector $(\pm)$                                                                                                                                                                 | M1 A1           |
| Projection/resolving formula                                     | Shortest distance is $ \overrightarrow{PQ} \cdot \mathbf{n}  =$<br>$\frac{ (-5\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}) \cdot (8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) }{\sqrt{8^2 + 2^2 + 3^2}} = \dots$ $= \frac{77}{\sqrt{77}} \text{ or } \sqrt{77}$                                                    | dM1: Uses formula with their vectors to reach a numerical expression for the distance<br><b>Dependent on previous M mark.</b><br>A1: Correct exact distance. Isw                                                                                           | dM1 A1          |
|                                                                  |                                                                                                                                                                                                                                                                                                            |                                                                                                                                                                                                                                                            | (4)             |
| Way 4                                                            | Line through given point in direction of normal is<br>$r = (6\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}) + \lambda(8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k})$<br>& meets plane " $8x - 2y - 3z + 5 = 0$ " when<br>$8(6 + 8\lambda) - 2(-3 - 2\lambda) - 3(-6 - 3\lambda) + 5 = 0$<br>$\Rightarrow \lambda = -1$ | M1: Uses line through given point in the direction of their normal and substitutes into their plane to find a value for $\lambda$ . The $d$ in their plane equation must not be zero<br>A1: Correct value                                                  | M1 A1           |
| Example of method involving the point where the line meets plane | $ -1(8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k})  = \sqrt{8^2 + 2^2 + 3^2}$<br>Or point of intersection is<br>$(6 - 8, -3 - (-2), -6 - (-3))$<br>$= (-2, -1, -3)$ and distance is<br>$\sqrt{(-2)^2 + (-1)^2 + (-3)^2}$ $\Rightarrow \sqrt{14}$                                                                | dM1: Attempts $ \lambda \mathbf{n} $ or finds point on the plane and obtains numerical expression for distance between this point and the given point<br><b>Dependent on previous M mark.</b><br>A1: Correct exact distance. Isw                           | dM1 A1          |
|                                                                  |                                                                                                                                                                                                                                                                                                            |                                                                                                                                                                                                                                                            | (4)             |
|                                                                  | Marks are scored through the way which is the best overall match for the attempt. Credit for work done in (b) is only available for part (c) if it is used in part (c).                                                                                                                                    |                                                                                                                                                                                                                                                            |                 |
|                                                                  |                                                                                                                                                                                                                                                                                                            |                                                                                                                                                                                                                                                            | <b>Total 10</b> |

| Question Number                 | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                                | Notes                                                                                                                                                                                                                                                                                          | Marks      |
|---------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|
| <b>8(a)</b><br><br><b>Way 1</b> | $I_n = \int \cos^n x \, dx = \int \cos x \cos^{n-1} x \, (dx)$                                                                                                                                                                                                                                                                                                                                                                                                        | Correct split. Could be implied by their work                                                                                                                                                                                                                                                  | M1         |
|                                 | $= \sin x \cos^{n-1} x + \int (n-1) \cos^{n-2} x \sin^2 x \, (dx)$                                                                                                                                                                                                                                                                                                                                                                                                    | Obtains $p \sin x \cos^{n-1} x + \int q \cos^{n-2} x \sin^2 x \, (dx)$<br>oe<br><b>Requires previous M mark.</b>                                                                                                                                                                               | dM1        |
|                                 | $= \sin x \cos^{n-1} x + \int (n-1) \cos^{n-2} x (1 - \cos^2 x) \, (dx)$                                                                                                                                                                                                                                                                                                                                                                                              | Replaces $\sin^2 x$ with $1 - \cos^2 x$ to achieve a correct expression for $I_n$                                                                                                                                                                                                              | A1         |
|                                 | $= \sin x \cos^{n-1} x + (n-1)I_{n-2} - (n-1)I_n$<br>$\Rightarrow I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2} *$                                                                                                                                                                                                                                                                                                                                    | Proceeds to the given answer with at least one intermediate step and no errors. Condone missing “dx”s but there must be no missing arguments. Any clear bracketing error must be recovered <b>before</b> given answer.                                                                         | A1*        |
|                                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                                                                                                                                                                                                                                                                                                | <b>(4)</b> |
| <b>Way 2</b>                    | $I_n = \int \cos^n x \, dx = \int \cos^2 x \cos^{n-2} x \, (dx)$<br>$= \int (1 - \sin^2 x) \cos^{n-2} x \, (dx)$                                                                                                                                                                                                                                                                                                                                                      | Correct split <b>and</b> replaces $\cos^2 x$ with $1 - \sin^2 x$                                                                                                                                                                                                                               | M1         |
|                                 | $= \int (\cos^{n-2} x - \cos^{n-2} x \sin^2 x) \, (dx)$<br>$= \int \cos^{n-2} x \, (dx) - \int (\sin x \sin x \cos^{n-2} x) \, (dx) = \dots$<br>M1: Expands, splits and obtains $p \int \cos^{n-2} x \, (dx) + q \cos^{n-1} x \sin x + \int r \cos^n x \, (dx)$ oe<br><b>Requires previous M mark.</b><br>A1: Correct expression for $I_n$ : $\int \cos^{n-2} x \, (dx) - \left( -\frac{1}{n-1} \cos^{n-1} x \sin x + \int \frac{1}{n-1} \cos^n x \, (dx) \right)$ oe |                                                                                                                                                                                                                                                                                                | dM1<br>A1  |
|                                 | $= I_{n-2} + \frac{1}{n-1} \cos^{n-1} x \sin x - \frac{1}{n-1} I_n$<br>$\Rightarrow I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2} *$                                                                                                                                                                                                                                                                                                                  | Proceeds to the given answer with at least one intermediate step and no errors. Condone missing “dx”s but there must be no missing arguments. Any bracketing error must be recovered <b>before</b> given answer.                                                                               | A1*        |
|                                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                                                                                                                                                                                                                                                                                                | <b>(4)</b> |
| <b>(b)</b>                      | $I_n = \frac{1}{n} \left[ \cos^{n-1} x \sin x \right]_0^{\frac{\pi}{2}} + \frac{n-1}{n} I_{n-2}$ or $= \frac{1}{n} (n-1) I_{n-2}$<br>$I_2 = \frac{1}{2} \left[ \cos^{2-1} x \sin x \right]_0^{\frac{\pi}{2}} + \frac{2-1}{2} I_0$ or $= \frac{1}{2} I_0$                                                                                                                                                                                                              | Uses the RF to obtain an expression for $I_n$ in terms of $I_{n-2}$ or $I_2$ in terms of $I_0$<br>Condone if necessary if limits are absent.                                                                                                                                                   | M1         |
|                                 | $I_n = \frac{(n-1)(n-3)\dots 5 \times 3 \times 1}{n(n-2)(n-4)\dots 6 \times 4 \times 2} I_0$<br>with dots & at least 3 terms in each product (first 2 & last, or first & last 2)                                                                                                                                                                                                                                                                                      | Correct expression for $I_n$ in terms of $I_0$ oe following correct work including 2 applications of the reduction formula (which could be embedded) <b>prior</b> to this answer. $I_0$ may have been calculated previously but do not allow just the final printed answer to imply this mark. | A1         |
|                                 | e.g., $I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$ or $I_0 = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$ or $I_0 = \frac{\pi}{2} - 0$                                                                                                                                                                                                                                                                                                                                 | Correct value for $I_0$ - requires written evidence of integration (minimal)                                                                                                                                                                                                                   | B1         |
|                                 | $\therefore I_n = \frac{(n-1)(n-3)\dots 5 \times 3 \times 1}{n(n-2)(n-4)\dots 6 \times 4 \times 2} \times \frac{\pi}{2} *$<br>Allow extra terms in both products.                                                                                                                                                                                                                                                                                                     | Proceeds to given answer. Requires all previous marks. Withhold this mark if no $\frac{1}{k} [\cos^{k-1} x \sin x]_0^{\frac{\pi}{2}}$ is seen or expression just disappears – one such expression must be replaced by “0” or have substitution seen                                            | A1*        |
|                                 | Attempts via proof by induction will be reviewed.                                                                                                                                                                                                                                                                                                                                                                                                                     |                                                                                                                                                                                                                                                                                                | <b>(4)</b> |
|                                 | Attempts may be seen via $I_n = \frac{(n-1)(n-3)\dots 3}{n(n-2)\dots 4} I_2$ and $I_2 = \frac{1}{2} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{2}} = \frac{1}{2} \times \frac{\pi}{2}$                                                                                                                                                                                                                                                            |                                                                                                                                                                                                                                                                                                |            |

| Question Number | Scheme                                                                                                                                                                                                                                                                                                                                                        | Notes                                                                                                         | Marks           |
|-----------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------|-----------------|
| <b>8(c)</b>     | $\int_0^{\frac{\pi}{2}} \cos^6 x \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} \cos^6 x (1 - \cos^2 x) \, dx$                                                                                                                                                                                                                                                       | Replaces $\sin^2 x$ with $1 - \cos^2 x$<br>Can be implied by an attempt at $I_6 - I_8$                        | M1              |
|                 | $= I_6 - I_8 = \left( \frac{5 \times 3 \times 1}{6 \times 4 \times 2} - \frac{7 \times 5 \times 3 \times 1}{8 \times 6 \times 4 \times 2} \right) \frac{\pi}{2}$                                                                                                                                                                                              | Any correct numerical expression for the integral                                                             | A1              |
|                 | $\left( = \frac{5}{32} \pi - \frac{35}{256} \pi = \right) \frac{5}{256} \pi$ oe                                                                                                                                                                                                                                                                               | Correct exact value. Accept equivalent fractions and allow e.g., $\left( \frac{5}{128} \right) \frac{\pi}{2}$ | A1              |
|                 | <p><b>This is a “Hence” and requires clear use of <math>I_6 - I_8</math></b></p> <p>For the A marks there must be no evidence that the answer has been arrived at without using part (b). There is no credit in (b) for work in (c).</p> <p>Just "<math>I = \frac{5}{256} \pi</math>" is 0/3 but just "<math>I_6 - I_8 = \frac{5}{256} \pi</math>" is 3/3</p> |                                                                                                               |                 |
|                 |                                                                                                                                                                                                                                                                                                                                                               |                                                                                                               | <b>(3)</b>      |
|                 |                                                                                                                                                                                                                                                                                                                                                               |                                                                                                               | <b>Total 11</b> |

| Question Number | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         | Notes                                                                                                                                                                                                                                        | Marks      |
|-----------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|
| <b>9(a)(i)</b>  | $b^2 = a^2(1 - e^2) \Rightarrow 1 = 9(1 - e^2)$ $\Rightarrow e^2 = \dots \left(\frac{8}{9}\right), e = \frac{2\sqrt{2}}{3} \text{ or } \frac{\sqrt{8}}{3}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | M1: Uses a correct eccentricity formula with correct values for $a$ and $b$ and obtains a value for $e^2$ or $e$<br>A1: Correct value for $e$ (not $\pm$ )<br>Could be implied                                                               | M1 A1      |
|                 | Foci are $(\pm 2\sqrt{2}, 0)$ or $(\pm \sqrt{8}, 0)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | B1: Both correct foci as coordinates<br>Condone any use of a negative $e$<br><b>Note that this is not an ft mark.</b>                                                                                                                        | B1         |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |                                                                                                                                                                                                                                              | <b>(3)</b> |
| <b>(a)(ii)</b>  | $x = \pm \frac{9\sqrt{2}}{4} \text{ or } \pm \frac{9\sqrt{8}}{8} \text{ or } \pm \frac{9}{\sqrt{8}} \text{ oe}$ <p>Both correct <b>equations</b>. Requires single fraction.</p> <p><b>Allow ft:</b> <math>x = \pm \frac{3}{\text{their } e}</math> computed into a single fraction, condoning <math>e &lt; 0</math></p> <p>Allow "<math>x_1 = \dots, x_2 = \dots</math>"</p> <p>"<math>x = \pm \frac{a}{e}</math>"</p> <p>Condone, e.g., <math>= \frac{9\sqrt{2}}{4} \text{ or } -\frac{9\sqrt{2}}{4}</math> but just "<math>\frac{a}{e} = \pm \frac{9\sqrt{2}}{4}</math>" is B0</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                                                                                                                                                                                              | B1ft       |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |                                                                                                                                                                                                                                              | <b>(2)</b> |
| <b>(b)</b>      | $ PF_1  = e PM_1  \text{ or }  PF_2  = e PM_2  \text{ oe}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | States this definition of an ellipse.                                                                                                                                                                                                        | M1         |
| <b>Way 1</b>    | $ PF_1  +  PF_2  = e( PM_1  +  PM_2 ) \text{ or } e( M_1M_2 )$ $\frac{2\sqrt{2}}{3} \times 2 \times \frac{9\sqrt{2}}{4} \text{ oe}$ $\text{or }  PF_1  +  PF_2  =$ $= \frac{2\sqrt{2}}{3} \left( \frac{9\sqrt{2}}{4} - x \right) + \frac{2\sqrt{2}}{3} \left( \frac{9\sqrt{2}}{4} + x \right)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | Correct method for a numerical expression (or with cancelling "x"s) for $ PF_1  +  PF_2 $ with their $e$ and directrix.<br><b>One of the underlined expressions must be seen for the first approach.</b><br><b>Requires previous M mark.</b> | dM1        |
|                 | $= 6 *$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        | Fully correct proof. Modulus signs are not required.                                                                                                                                                                                         | A1*        |
| <b>Way 1</b>    | <p>If they work in <math>a</math> and <math>e</math>, <math>e \times 2 \times \frac{a}{e}</math> is only acceptable if <math>e( PM_1  +  PM_2 ) \text{ or } e( M_1M_2 )</math> is seen (as with using the values) and <math>e\left(\frac{a}{e} - x\right) + e\left(\frac{a}{e} + x\right) (\Rightarrow 2a)</math> is acceptable but note in both these general cases the second M mark becomes available when <math>a = 3</math> is substituted.</p> <p><b>The second M is not available for any work which relies on <math> PF_1  =  PF_2 </math></b></p> <p><b>Their proof needs to be shown to be valid for any position of <math>P</math></b></p> <p>So <math> PF_1  +  PF_2  = \frac{2\sqrt{2}}{3} \times \frac{9\sqrt{2}}{4} + \frac{2\sqrt{2}}{3} \times \frac{9\sqrt{2}}{4}</math> or using <math>e \times \frac{a}{e} + e \times \frac{a}{e}</math> cannot score the second M without <math>e( PM_1  +  PM_2 ) \text{ or } e( M_1M_2 )</math> being seen.</p> <p>If <math>e</math> appears as a value it must be correct for the final mark.</p> <p>Just <math> PF_1  +  PF_2  = 2a = 2 \times 3 = 6</math> is 0/3</p> <p>Having earned the first mark in Way 1, some candidates proceed to work with a specific point on the ellipse as in Way 2. Further credit is only available if they clearly state e.g., "<math> PF_1  +  PF_2 </math> is constant for any <math>P</math>"</p> |                                                                                                                                                                                                                                              | <b>(3)</b> |

| Question Number                                                  | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                   | Notes                                                                                                                                                                                                                                                  | Marks      |
|------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|
| <b>9(b)</b><br><br><b>Way 2</b><br><br>$PF_1 + PF_2 = k$         | $ PF_1  +  PF_2  =  QF_1  +  QF_2 $<br>where $P$ and $Q$ are any points on the ellipse<br>oe                                                                                                                                                                                                                                                                                                                             | States this oe definition of an ellipse, justified by explanation.<br>Accept e.g., " $ PF_1  +  PF_2 $ is constant for any $P$ "                                                                                                                       | M1         |
|                                                                  | e.g. $Q$ is where $E$ crosses positive $x$ -axis<br>$\Rightarrow  PF_1  +  PF_2  = 3 - "2\sqrt{2}" + 3 + "2\sqrt{2}"$<br>$Q$ is where $E$ crosses positive $y$ -axis<br>$\Rightarrow  PF_1  +  PF_2  = 2\sqrt{1^2 + "2\sqrt{2}"^2}$<br>$Q$ is on $E$ directly above $F_1$<br>$\Rightarrow  PF_1  +  PF_2  =$<br>$\sqrt{1 - \frac{("2\sqrt{2}"^2)}{9}} + \sqrt{(2 \times "2\sqrt{2}"^2 + 1 - \frac{("2\sqrt{2}"^2)}{9})}$ | Correct method for a numerical value for $ PF_1  +  PF_2 $ using another point on the ellipse and their foci.<br><b>Requires previous M mark.</b>                                                                                                      | dM1        |
|                                                                  | $= 6 *$                                                                                                                                                                                                                                                                                                                                                                                                                  | Fully correct proof. Modulus signs are not required.                                                                                                                                                                                                   | A1*        |
|                                                                  |                                                                                                                                                                                                                                                                                                                                                                                                                          |                                                                                                                                                                                                                                                        | <b>(3)</b> |
| <b>Way 3</b><br><br><b>Point in terms of <math>\theta</math></b> | $P(3\cos\theta, \sin\theta)$<br>$ PF_1 ^2 = (3\cos\theta - "2\sqrt{2}"^2 + \sin^2\theta$<br>or $ PF_2 ^2 = (3\cos\theta + "2\sqrt{2}"^2 + \sin^2\theta$                                                                                                                                                                                                                                                                  | Correct general point in parametric form and applies Pythagoras for the distance (or its square) to either of their foci. Allow in terms of $a$ , $b$ and $\theta$                                                                                     | M1         |
|                                                                  | $ PF_1  +  PF_2  =$<br>$\sqrt{8\cos^2\theta - 12\sqrt{2}\cos\theta + 9} + \sqrt{8\cos^2\theta + 12\sqrt{2}\cos\theta + 9}$                                                                                                                                                                                                                                                                                               | Correct method for $ PF_1  +  PF_2 $ with their foci. Two three term quadratic expressions required but allow the second to be implied if its correct square root is seen. Score when $a$ and $b$ are substituted.<br><b>Requires previous M mark.</b> | dM1        |
|                                                                  | $ PF_1  +  PF_2  =$<br>$3 - 2\sqrt{2}\cos\theta + 3 + 2\sqrt{2}\cos\theta = 6*$                                                                                                                                                                                                                                                                                                                                          | Fully correct proof. Modulus signs are not required. The intermediate step shown oe is required for this Way.                                                                                                                                          | A1*        |
|                                                                  |                                                                                                                                                                                                                                                                                                                                                                                                                          |                                                                                                                                                                                                                                                        | <b>(3)</b> |
| <b>Way 4</b><br><br><b>Point in terms of <math>x</math></b>      | $P\left(x, \sqrt{1 - \frac{x^2}{9}}\right)$ or $P\left(x, \sqrt{\frac{9 - x^2}{9}}\right)$<br>$ PF_1 ^2 = ("2\sqrt{2}" - x)^2 + 1 - \frac{x^2}{9}$<br>or $ PF_2 ^2 = (x + "2\sqrt{2}"^2 + 1 - \frac{x^2}{9}$                                                                                                                                                                                                             | Correct general point in terms of $x$ and applies Pythagoras for the distance (or its square) to either of their foci. Allow in terms of $a$ , $b$ and $x$ .                                                                                           | M1         |
|                                                                  | $ PF_1  +  PF_2  = \sqrt{\frac{8}{9}x^2 - 4\sqrt{2}x + 9} + \sqrt{\frac{8}{9}x^2 + 4\sqrt{2}x + 9}$                                                                                                                                                                                                                                                                                                                      | Correct method for $ PF_1  +  PF_2 $ with their foci. Two three term quadratic expressions required but allow the second to be implied if its correct square root is seen. Score when $a$ and $b$ are substituted.<br><b>Requires previous M mark.</b> | dM1        |
|                                                                  | $ PF_1  +  PF_2  = 3 - \frac{2\sqrt{2}}{3}x + 3 + \frac{2\sqrt{2}}{3}x = 6*$                                                                                                                                                                                                                                                                                                                                             | Fully correct proof. Modulus signs are not required. The intermediate step shown oe is required for this Way.                                                                                                                                          | A1*        |
|                                                                  |                                                                                                                                                                                                                                                                                                                                                                                                                          |                                                                                                                                                                                                                                                        | <b>(3)</b> |
|                                                                  | Creditworthy alternative approaches will be reviewed                                                                                                                                                                                                                                                                                                                                                                     |                                                                                                                                                                                                                                                        | <b>(3)</b> |

| Question Number | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    | Notes                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | Marks    |
|-----------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------|
| 9(c)            | $x^2 + 9(2x + c)^2 = 9$ or $\frac{x^2}{9} + (2x + c)^2 = 1$                                                                                                                                                                                                                                                                                                                                                                                                                                                               | Substitutes line into the ellipse equation.<br>Condone slips provided intention clear.                                                                                                                                                                                                                                                                                                                                                                                                                                            | M1       |
|                 | $37x^2 + 36cx + 9c^2 - 9 = 0$<br>or e.g., $\frac{37}{9}x^2 + 4cx + c^2 - 1 = 0$                                                                                                                                                                                                                                                                                                                                                                                                                                           | Correct quadratic with $x^2$ terms collected<br>(could be implied)                                                                                                                                                                                                                                                                                                                                                                                                                                                                | A1       |
|                 | $\frac{1}{2}(\text{sum of roots}) \Rightarrow (x =) \frac{-18c}{37}$<br>or<br>$(x =) \frac{1}{2} \left( \frac{-36c + \sqrt{(36c)^2 - 4(37)(9c^2 - 9)}}{2(37)} + \frac{-36c - \sqrt{(36c)^2 - 4(37)(9c^2 - 9)}}{2(37)} \right)$<br>M1: Correct attempt at $\frac{1}{2}(\text{sum of roots})$ , i.e., $-\frac{b}{2a}$ for their quadratic.<br>Ignore how the expression is labelled.<br><b>Requires previous M mark.</b><br>A1: Any correct <b>equation</b> in $x$ and $c$<br>Allow this mark if e.g., $x$ is seen as $M_x$ |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | dM1 A1   |
|                 | $\Rightarrow c = -\frac{37}{18}x \Rightarrow y = 2x + \left(-\frac{37}{18}\right)x$<br>or $x = -\frac{18}{37}c \Rightarrow y = 2 \times -\frac{18}{37}c + c \Rightarrow \dots \left(y = \frac{c}{37} \Rightarrow \frac{y}{x} = -\frac{1}{18}\right)$                                                                                                                                                                                                                                                                      | Substitutes their $c = px$ into the line to obtain an equation in <b><math>x</math> and <math>y</math></b> only.<br>Allow e.g., $x_M$ and $y_M$ and condone e.g., suffixes of $P$ & $Q$<br>This may also be achieved by e.g., finding $y$ in terms of $c$ and then eliminating $c$ with their equation in $x$ and $c$<br>Must not be using " $M_x$ " or " $M_y$ " etc. but imply this mark from a locus equation in $x$ and $y$ or $x_{\dots}$ and $y_{\dots}$ with appropriate suffixes<br><b>Requires both previous M marks</b> | ddM1     |
|                 | $\Rightarrow y_{\dots} = -\frac{1}{18}x_{\dots}$ oe<br>$\therefore l$ passes through the origin oe *                                                                                                                                                                                                                                                                                                                                                                                                                      | Obtains correct equation for locus (accept equivalents) <b>and</b> makes conclusion e.g., "passes/goes through origin/O/(0,0)" but allow "shown"/"as required"/"QED" etc.<br><b>Requires all previous marks.</b>                                                                                                                                                                                                                                                                                                                  | A1*      |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | (6)      |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | Total 13 |
| PAPER TOTAL: 75 |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |          |



| Question Number | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | Notes                                                                                                                                                                        | Marks          |
|-----------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------|
| 1               | $7 \cosh x + 3 \sinh x = 2e^x + 7 \Rightarrow$ $7 \left( \frac{e^x + e^{-x}}{2} \right) + 3 \left( \frac{e^x - e^{-x}}{2} \right) = 2e^x + 7$ $\left\{ \frac{7}{2}e^x + \frac{7}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} = 2e^x + 7 \right\}$                                                                                                                                                                                                                                                                                                                                                    | Substitutes at least one correct exponential form for either of the hyperbolic terms and achieves an <b>equation</b> in exponentials and constants alone                     | M1             |
|                 | $\Rightarrow 7(e^{2x} + 1) + 3(e^{2x} - 1) = 4e^{2x} + 14e^x$ $\left\{ \Rightarrow 5e^{2x} + 2 = 2e^{2x} + 7e^x \right\}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | Multiplies through by $e^x$ to obtain any equation that would form a 3TQ in $e^x$ if like terms were collected                                                               | M1             |
|                 | $\Rightarrow 6e^{2x} - 14e^x + 4 = 0 \quad \{3e^{2x} - 7e^x + 2 = 0\}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | A correct three term quadratic in $e^x$ . Could be implied by a correct root even if terms have not been collected.                                                          | A1             |
|                 | $\Rightarrow (3e^x - 1)(e^x - 2) = 0 \Rightarrow e^x = \dots$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          | Solves their 3TQ - usual rules. One correct root for their quadratic if no working. Ignore labelling of the roots even if e.g., " $x$ " is used.                             | M1             |
|                 | $x = \ln 2, \ln \frac{1}{3}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | Both correct and simplified but do not isw if there are <b>other answers</b> .<br>Allow $-\ln \frac{1}{2}$ for $\ln 2$<br>and $-\ln 3$ or $\ln 3^{-1}$ for $\ln \frac{1}{3}$ | A1             |
|                 | Answer only is 0/5                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |                                                                                                                                                                              | <b>Total 5</b> |
|                 | Note that it is possible to multiply through by $e^{-x}$ to form an equation in $e^{-2x}$ , $e^{-x}$ and constants. Score as main scheme, e.g.,<br>$\frac{7}{2}e^x + \frac{7}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} = 2e^x + 7$ $\Rightarrow \frac{7}{2} + \frac{7}{2}e^{-2x} + \frac{3}{2} - \frac{3}{2}e^{-2x} = 2 + 7e^{-x} \quad (\text{M1})$ $\Rightarrow 2e^{-2x} - 7e^{-x} + 3 = 0 \quad (\text{A1})$ $(2e^{-x} - 1)(e^{-x} - 3) = 0 \Rightarrow e^{-x} = \frac{1}{2}, 3 \quad (\text{M1})$ $\Rightarrow e^x = 2, \frac{1}{3} \Rightarrow x = \ln 2, \ln \frac{1}{3} \quad (\text{A1})$ |                                                                                                                                                                              |                |

| Question Number | Scheme                                                                                                                                                                                                                                                                                                                                                                                                                             | Notes                                                                                                                 | Marks           |
|-----------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------|-----------------|
| 8(a)            | Asymptotes are $y = \pm 2x$                                                                                                                                                                                                                                                                                                                                                                                                        | $y = \pm 2x$ oe e.g. $x = \pm \frac{y}{2}$                                                                            | B1              |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                                                                                       | (1)             |
| 8(b)            | $4 = e^2 - 1 \Rightarrow e = \sqrt{5}$                                                                                                                                                                                                                                                                                                                                                                                             | Uses the correct eccentricity formula with $a = 1$ and $b = 2$ to find a value for $e$ .                              | M1              |
|                 | Foci are $(\pm\sqrt{5}, 0)$                                                                                                                                                                                                                                                                                                                                                                                                        | Both required.                                                                                                        | A1              |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                                                                                       | (2)             |
| 8(c)            | $8x - 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{4x}{y} = \frac{4 \sec \theta}{2 \tan \theta}$ or $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{2 \sec^2 \theta}{\sec \theta \tan \theta}$<br>M1: $Ax + By \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = f(\theta)$ or $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = f(\theta)$<br>A1: Correct gradient in terms of $\theta$ |                                                                                                                       | M1A1            |
|                 | <p>Explicit differentiation may be seen:</p> $y^2 = 4x^2 - 4 \Rightarrow y = (4x^2 - 4)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(4x^2 - 4)^{-\frac{1}{2}} \times 8x = \frac{4 \sec \theta}{\sqrt{4 \sec^2 \theta - 4}}$ <p>Score M1 for <math>\frac{dy}{dx} = kx(4x^2 - 4)^{\frac{1}{2}} = f(\theta)</math> and A1 for correct gradient in terms of <math>\theta</math></p>                                           |                                                                                                                       |                 |
|                 | E.g. $y - 2 \tan \theta = \frac{4 \sec \theta}{2 \tan \theta}(x - \sec \theta)$                                                                                                                                                                                                                                                                                                                                                    | Correct straight line method using their gradient in terms of $\theta$ and $x = \sec \theta$ ,<br>$y = 2 \tan \theta$ | M1              |
|                 | $y \tan \theta - 2 \tan^2 \theta = 2x \sec \theta - 2 \sec^2 \theta$<br>$\Rightarrow y \tan \theta - 2 \tan^2 \theta = 2x \sec \theta - 2(1 + \tan^2 \theta)$                                                                                                                                                                                                                                                                      |                                                                                                                       |                 |
|                 | $y \tan \theta = 2x \sec \theta - 2^*$                                                                                                                                                                                                                                                                                                                                                                                             | Obtains the given answer with sufficient working shown as above.                                                      | A1cso           |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                                                                                       | (4)             |
| 8(d)            | $VP : V(-1, 0); P(\sec \theta, 2 \tan \theta) \Rightarrow y = \frac{2 \tan \theta}{\sec \theta + 1}(x + 1)$<br>or<br>$WQ : W(1, 0); Q(\sec \theta, -2 \tan \theta) \Rightarrow y = \frac{-2 \tan \theta}{\sec \theta - 1}(x - 1)$<br>M1: Correct straight line method for either VP or WQ<br>A1: One correct equation in any form                                                                                                  |                                                                                                                       | M1A1            |
|                 | $y = \frac{-2 \tan \theta}{\sec \theta - 1}(x - 1), y = \frac{2 \tan \theta}{\sec \theta + 1}(x + 1)$                                                                                                                                                                                                                                                                                                                              | Both equations correct in any form.                                                                                   | A1              |
|                 | $\frac{2 \tan \theta}{\sec \theta + 1}(x + 1) = \frac{-2 \tan \theta}{\sec \theta - 1}(x - 1) \Rightarrow x / y = \dots$                                                                                                                                                                                                                                                                                                           | Attempt to solve and makes progress to achieve either $x = \dots$ or $y = \dots$ in terms of $\theta$ only.           | M1              |
|                 | $x = \cos \theta$ or $y = 2 \sin \theta$                                                                                                                                                                                                                                                                                                                                                                                           | One correct coordinate                                                                                                | A1              |
|                 | $x = \cos \theta$ and $y = 2 \sin \theta$                                                                                                                                                                                                                                                                                                                                                                                          | Both correct                                                                                                          | A1              |
|                 | $x^2 + \frac{y^2}{4} = 1$ or $a = 1, b = 2$                                                                                                                                                                                                                                                                                                                                                                                        | Correct equation or correct values for $a$ and $b$                                                                    | A1              |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                                                                                       | (7)             |
|                 |                                                                                                                                                                                                                                                                                                                                                                                                                                    |                                                                                                                       | <b>Total 14</b> |