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EDEXCEL INTERNATIONAL A LEVEL MATHS: QUESTIONS BY TOPIC

Integration (FP3.4)

International A Level Mathematics · FP3: Further Pure Mathematics 3 (WFM03)

33 questions · 288 marks · Mark schemes included

Past papers October 2020 - June 2025 · 3 sub-topics

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 Piccadilly Business Centre, Unit C, Aldow Enterprise Park
Blackett Street, Manchester, M12 6AE

HOW TO USE THIS BOOKLET

Questions are grouped by sub-topic and run from the lowest to the highest mark tariff, then oldest paper first.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme; < QUESTION brings you back.

A gold bar in the left margin marks exactly where the question sits on the original page.

Diagrams, graphs, grids and tables are printed exactly as they appear in the original papers.

Each question is filed under its main topic; longer questions may also test skills from other topics.

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FP3.4a INTEGRATION USING HYPERBOLIC & TRIGONOMETRIC SUBSTITUTIONS		11 Qs · 84 marks	3
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QUESTION	SOURCE	MARKS	PAGE
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QUESTIONS

33 questions · 288 marks · 3 sub-topics

SUB-TOPIC FP3.4a

INTEGRATION USING HYPERBOLIC & TRIGONOMETRIC SUBSTITUTIONS

Sub-topic FP3.4a · 11 questions · 84 marks · lowest tariff first



4. (a) Determine

$$\int \frac{1}{\sqrt{9x^2 + 16}} \, dx \quad (2)$$

(b) Hence determine the exact value of

$$\int_{-2}^2 \frac{1}{\sqrt{9x^2 + 16}} \, dx$$

Give your answer in the form $a \ln(b + c\sqrt{13})$, where a , b and c are rational numbers.

(3)







Question 4 continued

(Total for Question 4 is 5 marks)



4. Using the substitution $x = 4 \cosh \theta$ show that

$$\int \frac{1}{(x^2 - 16)^{\frac{3}{2}}} dx = \frac{ax}{\sqrt{x^2 - 16}} + c \quad |x| > 4$$

where a is a constant to be determined and c is an arbitrary constant.

(6)

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Question 4 continued

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Question 4 continued

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Question 4 continued

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Q4

(Total 6 marks)



5. Determine

(i) $\int \frac{1}{\sqrt{x^2 - 3x + 5}} dx$

(3)

(ii) $\int \frac{1}{\sqrt{63 + 4x - 4x^2}} dx$

(4)

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Question 5 continued

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Question 5 continued

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(Total 7 marks)

Q5





Question 1 continued

(Total for Question 1 is 7 marks)



2. Determine

(i) $\int \frac{1}{\sqrt{4x^2 + 8x + 9}} \, dx$

(3)

(ii) $\int \operatorname{arcosh} 3x \, dx$

(4)



(Total for Question 2 is 7 marks)



2. Determine

(i) $\int \frac{1}{3x^2 + 12x + 24} dx$ (4)

(ii) $\int \frac{1}{\sqrt{27 - 6x - x^2}} dx$ (4)

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Question 2 continued

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Question 2 continued

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(Total 8 marks)

Q2



7. Using calculus, find the exact values of

(i) $\int_1^2 \frac{1}{x^2 - 4x + 5} \, dx$

(3)

$$(ii) \int_{\sqrt{3}}^3 \frac{\sqrt{x^2 - 3}}{x^2} dx$$

(5)

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Question 7 continued

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Question 7 continued

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Question 7 continued

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(Total 8 marks)

Q7



Question 5 continued

Lined area for writing the answer to Question 5 continued.





(Total for Question 5 is 8 marks)







Question 2 continued

(Total for Question 2 is 9 marks)





TOTAL FOR PAPER IS 75 MARKS



3.

In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.

(i) Determine

$$\int \frac{1}{4x^2 + 12x + 25} \, dx$$

(4)

(ii) Show that

$$\int_3^9 \frac{1}{\sqrt{x^2 + 4x - 17}} \, dx = \ln a$$

where a is an integer to be determined.

(6)



Question 3 continued



Question 3 continued

(Total for Question 3 is 10 marks)



SUB-TOPIC FP3.4b

REDUCTION FORMULAE

Sub-topic FP3.4b · 11 questions · 107 marks · lowest tariff first



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4. $I_n = \int x^n \cos x \, dx$

(a) Show that, for $n \geq 2$

$$I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2} \quad (4)$$

(b) Hence find the functions $f(x)$ and $g(x)$ such that

$$\int x^4 \cos x \, dx = f(x) \sin x + g(x) \cos x + c$$

where c is an arbitrary constant.

(5)

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Q4

6.

$$I_n = \int_0^{\sqrt{\frac{\pi}{2}}} x^n \cos(x^2) dx \quad n \geq 1$$

(a) Prove that, for $n \geq 5$

$$I_n = \frac{1}{2} \left(\frac{\pi}{2} \right)^{\frac{n-1}{2}} - \frac{1}{4} (n-1)(n-3) I_{n-4} \quad (6)$$

(b) Hence, determine the exact value of I_5 , giving your answer in its simplest form. (3)

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7.

$$I_n = \int \cosh^n 2x \, dx \quad n \geq 0$$

(a) Show that, for $n \geq 2$

$$I_n = \frac{\cosh^{n-1} 2x \sinh 2x}{2n} + \frac{n-1}{n} I_{n-2} \quad (5)$$

(b) Hence determine

$$\int (1 + \cosh 2x)^3 \, dx$$

collecting any like terms in your answer.

(4)

Question 7 continued

Lined area for writing the answer to Question 7 continued.



Question 7 continued

Lined area for writing the answer to Question 7.



Question 7 continued

Lined area for writing the answer to Question 7.

(Total for Question 7 is 9 marks)



Question 8 continued

Lined area for writing the answer to Question 8.



Question 8 continued

Lined area for writing the answer to Question 8.



Question 8 continued

Lined area for writing the answer to Question 8.

(Total for Question 8 is 9 marks)



Question 5 continued

Lined area for writing the answer to Question 5 continued.



Question 5 continued

Lined area for writing the answer to Question 5.



(Total for Question 3 is 5 marks)



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Q6

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5. $I_n = \int \sec^n x \, dx \quad n \geq 0$

(a) Prove that for $n \geq 2$

$$(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2} \quad (6)$$

(b) Hence, showing each step of your working, find the exact value of

$$\int_0^{\frac{\pi}{4}} \sec^6 x \, dx \tag{4}$$

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Question 5 continued

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(Total 10 marks)

Q5
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Q6

Question 7 continued

Lined area for writing the answer to Question 7 continued.





(Total for Question 7 is 10 marks)



Question 8 continued

Lined area for writing the answer to Question 8 continued.



Question 8 continued

Lined area for writing the answer to Question 8 continued.



Question 8 continued

Lined area for writing the answer to Question 8.

(Total for Question 8 is 11 marks)



Question 5 continued

Lined area for writing the answer to Question 5 continued.



Question 5 continued

Lined area for writing the answer to Question 5 continued.



Question 5 continued

Lined area for writing the answer to Question 5.

(Total for Question 5 is 11 marks)



SUB-TOPIC FP3.4c

ARC LENGTH & SURFACE AREA OF REVOLUTION

Sub-topic FP3.4c · 11 questions · 97 marks · lowest tariff first



1. The curve C has equation

$$y = \frac{1}{2} \operatorname{arcosh}(2x) \qquad \frac{7}{2} \leq x \leq 13$$

Using calculus, determine the exact length of the curve C .

Give your answer in the form $p\sqrt{q}$, where p and q are constants to be found.

(6)

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Question 1 continued

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Question 1 continued

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Lined area for writing the answer to Question 1 continued.



Question 1 continued

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Lined area for writing the answer to Question 1.

(Total 6 marks)

Q1

Box for marking the question.



2. A curve has equation

$$y = \sqrt{9 - x^2} \quad 0 \leq x \leq 3$$

(a) Using calculus, show that the length of the curve is $\frac{3\pi}{2}$

(4)

The curve is rotated through 2π radians about the x -axis.

(b) Using calculus, find the exact area of the surface generated.

(3)

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Question 2 continued

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Question 2 continued

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Lined area for writing the answer to Question 2 continued.



Question 2 continued

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Handwriting practice area with horizontal lines.

(Total 7 marks)

Q2

Grading box for Q2.



5. A curve has parametric equations

$$x = 4e^{\frac{1}{2}t} \quad y = e^t - t \quad 0 \leq t \leq 4$$

The curve is rotated through 2π radians about the x -axis.

Show that the area of the curved surface generated is

$$\pi(e^8 + Ae^4 + B)$$

where A and B are constants to be determined.

(7)



Question 5 continued

Lined area for writing the answer to Question 5 continued.



Question 5 continued

Lined area for writing the answer to Question 5 continued.



Question 5 continued

Lined area for writing the answer to Question 5.

(Total for Question 5 is 7 marks)



Question 2 continued

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Question 2 continued

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Lined area for writing the answer to Question 2 continued.



Question 2 continued

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(Total 8 marks)

Q2



Question 7 continued

Lined area for writing the answer to Question 7 continued.



Question 7 continued

Lined area for writing the answer to Question 7 continued.



Question 7 continued

(Total for Question 7 is 8 marks)



Question 8 continued

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Question 8 continued

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Lined area for writing the answer to Question 8 continued.



Question 8 continued

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(Total 9 marks)

Q8



6. A curve has parametric equations

$$x = a(\theta - \sin\theta)$$

$$y = a(1 - \cos \theta)$$

where a is a positive constant.

- (a) Show that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = ka^2 \sin^2 \frac{\theta}{2}$$

where k is a constant to be determined.

(4)

The part of the curve from $\theta = 0$ to $\theta = 2\pi$ is rotated through 2π radians about the x -axis.

- (b) Determine the area of the surface generated, giving your answer in terms of π and a .

[Solutions relying on calculator technology are not acceptable.]

(5)







Question 6 continued

Lined area for writing the answer to Question 6.

(Total for Question 6 is 9 marks)



7.

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

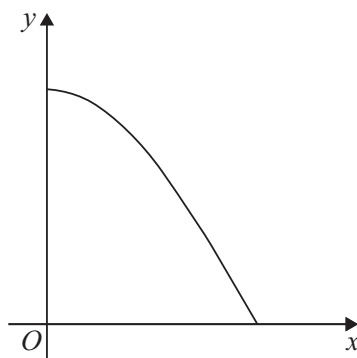


Figure 1

Figure 1 shows a sketch of the curve with equation

$$y = \cos 2x \quad 0 \leq x \leq \frac{\pi}{4}$$

The curve is rotated through 2π radians about the x -axis.

(a) Show that the area of the curved surface generated is given by

$$S = 2\pi \int_0^{\frac{\pi}{4}} \cos 2x \sqrt{1 + 4 \sin^2 2x} \, dx \quad (2)$$

(b) Hence, using the substitution $2 \sin 2x = \sinh \theta$, show that

$$S = \frac{\pi}{4} \left(\ln(a + \sqrt{b}) + a\sqrt{b} \right)$$

where a and b are integers to be determined.

(7)



Question 7 continued

Lined area for writing the answer to Question 7 continued.



Question 7 continued

Lined area for writing the answer to Question 7 continued.



Question 7 continued

Lined area for writing the answer to Question 7.

(Total for Question 7 is 9 marks)



Question 3 continued

Lined area for writing the answer to Question 3 continued.



Question 3 continued

Lined area for writing the answer to Question 3 continued.



(Total for Question 3 is 11 marks)



5.

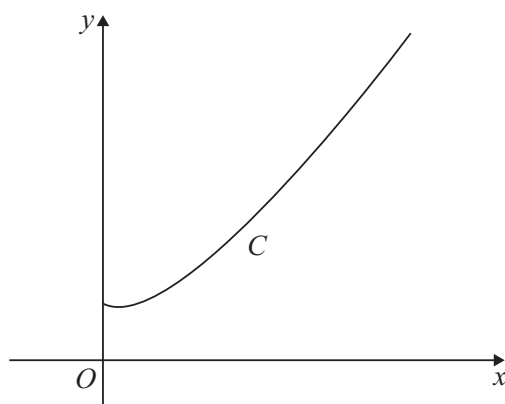


Figure 1

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

Figure 1 shows a sketch of the curve C defined by the parametric equations

$$x = (2t + 3)^{\frac{3}{2}} \quad y = \frac{3}{2}t^2 + 3t + 6 \quad -\frac{3}{2} \leq t \leq 3$$

(a) Show that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a(t + 2)^2$$

where a is an integer to be determined.

(4)

Hence, using algebraic integration, determine

(b) the exact length of C ,

(3)

(c) the exact area of the surface generated when C is rotated through 360° about the x -axis, giving your answer in the form $k\pi$ where k is a rational number.

(4)



Question 5 continued

(Total for Question 3 is 11 marks)



Question 7 continued

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Question 7 continued

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Question 7 continued

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(Total 12 marks)

Q7



MARK SCHEMES

official mark schemes, in the same order as the questions



Question Number	Scheme	Notes	Marks
1(a)	$\frac{dy}{dx} = 3\arcsin 2x + 3x \frac{1}{\sqrt{1-(2x)^2}} \times 2$ $\left(= 3\arcsin 2x + \frac{6x}{\sqrt{1-4x^2}} \right)$	M1: Obtains $p \arcsin qx + \frac{rx}{\sqrt{1-(sx)^2}} \text{ or }$ $p \arcsin qx + \frac{rx}{\sqrt{1-tx^2}}$ $p, q, r, s, t > 0$ A1: Correct derivative. Allow unsimplified and isw. Allow $\sin^{-1}$ and condone “arsin” but “arsinh” or “arcsinh” is M0	M1 A1
(b)	$x = \frac{1}{4} \Rightarrow \frac{dy}{dx} = \frac{\pi}{2} + \sqrt{3}$	$\frac{\pi}{2} + \sqrt{3}$ only but allow $\frac{1}{2}\pi$ or 0.5π. Terms as a sum in either order. Allow $a = \frac{1}{2}, b = \sqrt{3}$ Isw following a correct answer.	B1dep
	This is a “Hence” question so this mark can only be awarded following full marks in part (a)		
			Total 3

Question Number	Scheme	Notes	Marks
2(a)	$x = -\frac{4}{3}$	$x = -\frac{4}{3}$ or any equivalent equation . Allow $x = \pm \frac{4}{3}$	B1
			(1)
(b)(i) Way 1	$\frac{a}{e} = \frac{4}{3}$ $b^2 = a^2(e^2 - 1) \Rightarrow 5 = a^2\left(\frac{9a^2}{16} - 1\right)$	Uses $\frac{a}{e} = \pm \frac{4}{3}$ oe and a correct eccentricity formula and obtains an equation in a . Condone replacing b^2 with 25 if equation is otherwise correct	M1
	$9a^4 - 16a^2 - 80 = 0$ $\Rightarrow (9a^2 + 20)(a^2 - 4) = 0 \Rightarrow a^2 = \dots$	Solves a 3TQ in a^2 (or equation that would lead to a 3TQ) to find a positive real root (usual rules – but if no working seen they must obtain one positive real value of a^2 or a correct to 3 sf which is consistent with their equation). Do not award if confusion with variable e.g., " $(9a^2 + 20)(a^2 - 4) = 0 \Rightarrow a = 4$ " Requires previous M mark.	dM1
	$a = 2$	Not $a = \pm 2$ unless negative rejected	A1
			(3)
Way 2	$\frac{a}{e} = \frac{4}{3}$ $b^2 = a^2(e^2 - 1) \Rightarrow 5 = \left(\frac{4e}{3}\right)^2(e^2 - 1)$	Uses $\frac{a}{e} = \pm \frac{4}{3}$ oe and a correct eccentricity formula and obtains an equation in e . Condone replacing b^2 with 25 if equation is otherwise correct	M1
	$16e^4 - 16e^2 - 45 = 0$ $\Rightarrow (4e^2 - 9)(4e^2 + 5) = 0 \Rightarrow e^2 = \dots$	Solves a 3TQ in e^2 (or equation that would lead to a 3TQ) to find a positive real root (usual rules – but if no working seen they must obtain one positive real value of e^2 or e correct to 3 sf which is consistent with their equation). Do not award if confusion with variable e.g., " $(4e^2 - 9)(4e^2 + 5) = 0 \Rightarrow e = \frac{3}{2}$ " Requires previous M mark.	dM1
	$\left(e = \frac{3}{2} \Rightarrow\right) a = 2$	Not $a = \pm 2$ unless negative rejected but condone sight of " $e = \pm \frac{3}{2}$ " or " $e = -\frac{3}{2}$ "	A1
			(3)

Question Number	Scheme	Notes	Marks
1(a)	$\pm \overrightarrow{AB} = \pm \begin{pmatrix} 4 \\ -4 \\ -1 \end{pmatrix}, \pm \overrightarrow{BC} = \pm \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}, \pm \overrightarrow{AC} = \pm \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ Attempts any 2 of these vectors. Allow these to be written as coordinates.		M1
	E.g. $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 4 \\ -4 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -7 \\ 16 \end{pmatrix}$	Attempts the vector product of 2 appropriate vectors. If no working is shown, look for at least 2 correct elements.	dM1
	Area = $\frac{1}{2} \sqrt{3^2 + 7^2 + 16^2} = \frac{1}{2} \sqrt{314}$	Correct exact area. Allow recovery from sign errors in the vector product e.g. allow following a vector product of $\pm 3\mathbf{i} \pm 7\mathbf{j} \pm 16\mathbf{k}$	A1
	Note that a correct exact area of $\frac{1}{2} \sqrt{314}$ with no evidence of any incorrect work scores full marks		
			(3)
	Alternative 1 using cosine rule:		
	$\pm \overrightarrow{AB} = \pm \begin{pmatrix} 4 \\ -4 \\ -1 \end{pmatrix}, \pm \overrightarrow{BC} = \pm \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}, \pm \overrightarrow{AC} = \pm \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ Attempts any 2 of these vectors		M1
	$ \pm \overrightarrow{AB} = \sqrt{4^2 + 4^2 + 1^2}, \pm \overrightarrow{BC} = \sqrt{1^2 + 5^2 + 2^2}, \pm \overrightarrow{AC} = \sqrt{3^2 + 1^2 + 1^2}$ $\cos A = \frac{33 + 11 - 30}{2\sqrt{33}\sqrt{11}} = \frac{7\sqrt{3}}{33} \text{ or } \cos B = \frac{30 + 33 - 11}{2\sqrt{30}\sqrt{33}} = \frac{13\sqrt{2}}{3\sqrt{55}} \text{ or } \cos C = \frac{30 + 11 - 33}{2\sqrt{30}\sqrt{11}} = \frac{\sqrt{8}}{\sqrt{165}}$ (For reference $A = 68.44\dots^\circ$, $B = 34.27\dots^\circ$, $C = 77.27\dots^\circ$) Attempts the magnitude of all 3 sides and attempts the cosine of one of the angles using a correctly applied cosine rule or e.g. $\cos A = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\sqrt{33}\sqrt{11}} = \frac{12 - 4 - 1}{\sqrt{33}\sqrt{11}}$ Finds the magnitude of 2 sides and the cosine of the included angle using a correctly applied scalar product		dM1
	Area = $\frac{1}{2} \sqrt{11} \sqrt{33} \sin A = \frac{1}{2} \sqrt{314}$ or Area = $\frac{1}{2} \sqrt{30} \sqrt{33} \sin B = \frac{1}{2} \sqrt{314}$ or Area = $\frac{1}{2} \sqrt{30} \sqrt{11} \sin C = \frac{1}{2} \sqrt{314}$	Correct exact area. Allow recovery from sign errors in the vectors that do not affect the calculations e.g. allow $\pm \overrightarrow{AB} = \pm 4\mathbf{i} \pm 4\mathbf{j} \pm \mathbf{k}$, $\pm \overrightarrow{BC} = \pm \mathbf{i} \pm 5\mathbf{j} \pm 2\mathbf{k}$, $\pm \overrightarrow{AC} = \pm 3\mathbf{i} \pm \mathbf{j} \pm \mathbf{k}$ And allow work in decimals as long as a correct exact area is found.	A1
			(3)

Alternative 2 using scalar product:		
$\pm \overrightarrow{AB} = \pm \begin{pmatrix} 4 \\ -4 \\ -1 \end{pmatrix}, \pm \overrightarrow{BC} = \pm \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}, \pm \overrightarrow{AC} = \pm \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$		M1
Attempts any 2 of these vectors		
$A \text{ to } BC \text{ is } \sqrt{AB^2 - \left(\frac{\overrightarrow{AB} \cdot \overrightarrow{BC}}{BC} \right)^2} = \sqrt{\frac{157}{15}}$ or $B \text{ to } CA \text{ is } \sqrt{BC^2 - \left(\frac{\overrightarrow{BC} \cdot \overrightarrow{CA}}{CA} \right)^2} = \sqrt{\frac{314}{11}}$ or $C \text{ to } BA \text{ is } \sqrt{AC^2 - \left(\frac{\overrightarrow{AC} \cdot \overrightarrow{AB}}{AB} \right)^2} = \sqrt{\frac{314}{33}}$		dM1
Attempts one of the altitudes of triangle ABC using a correct method		
$\text{Area} = \frac{1}{2} \sqrt{30} \sqrt{\frac{157}{15}} = \frac{1}{2} \sqrt{314}$ or $\text{Area} = \frac{1}{2} \sqrt{11} \sqrt{\frac{314}{11}} = \frac{1}{2} \sqrt{314}$ or $\text{Area} = \frac{1}{2} \sqrt{33} \sqrt{\frac{314}{33}} = \frac{1}{2} \sqrt{314}$	Correct exact area. Allow work in decimals as long as a correct exact area is found.	A1
		(3)
Alternative 3 using vector products:		
$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 \\ 4 \\ -16 \end{pmatrix}, \mathbf{b} \times \mathbf{c} = \begin{pmatrix} 0 \\ -8 \\ 20 \end{pmatrix}, \mathbf{c} \times \mathbf{a} = \begin{pmatrix} -3 \\ -3 \\ 12 \end{pmatrix}$		M1
Attempts these vector products		
$\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} = \begin{pmatrix} -3 \\ -7 \\ 16 \end{pmatrix}$		dM1
Adds the appropriate vector products		
$\text{Area} = \frac{1}{2} \sqrt{3^2 + 7^2 + 16^2} = \frac{1}{2} \sqrt{314}$	Correct exact area. Allow work in decimals as long as a correct exact area is found.	A1
		(3)

(b)	$\pm \overrightarrow{AD} = \pm \begin{pmatrix} 2 \\ -2 \\ k-1 \end{pmatrix}, \pm \overrightarrow{BD} = \pm \begin{pmatrix} -2 \\ 2 \\ k \end{pmatrix}, \pm \overrightarrow{CD} = \pm \begin{pmatrix} -1 \\ -3 \\ k-2 \end{pmatrix}$		M1
	Attempts one of these vectors		
	E.g. $\overrightarrow{AB} \times \overrightarrow{AC} \cdot \overrightarrow{AD} = \begin{pmatrix} -3 \\ -7 \\ 16 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ k-1 \end{pmatrix} = -6 + 14 + 16k - 16$ E.g. $\overrightarrow{AB} \times \overrightarrow{AC} \cdot \overrightarrow{BD} = \begin{pmatrix} -3 \\ -7 \\ 16 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 2 \\ k \end{pmatrix} = 6 - 14 + 16k$ E.g. $\overrightarrow{AB} \times \overrightarrow{AC} \cdot \overrightarrow{CD} = \begin{pmatrix} -3 \\ -7 \\ 16 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -3 \\ k-2 \end{pmatrix} = 3 + 21 + 16k - 32$		dM1
	Attempts a suitable triple product to obtain a scalar quantity ($\frac{1}{6}$ not required here). They must be forming the triple product correctly e.g. not the magnitude of a vector. Do not be too concerned if they make slips as long as appropriate vectors are being used and a scalar quantity is obtained. Must be an attempt at the tetrahedron ABCD.		
	Volume = $\frac{1}{3} 8k - 4$	Correct volume. Must see modulus and must be 2 terms but allow equivalents e.g. $\frac{4}{3} 2k - 1 , \frac{1}{6} 16k - 8 , \frac{1}{6} 8 - 16k$ Award once a correct answer is seen and apply isw if necessary.	A1
			(3)
			Total 6

Question Number	Scheme	Notes	Marks
2(a)	$y = \ln(\tanh 2x) \Rightarrow \frac{dy}{dx} = \frac{1}{\tanh 2x} \times 2 \operatorname{sech}^2 2x$ or $y = \ln(\tanh 2x) \Rightarrow e^y = \tanh 2x \Rightarrow e^y \frac{dy}{dx} = 2 \operatorname{sech}^2 2x \Rightarrow \frac{dy}{dx} = \frac{2 \operatorname{sech}^2 2x}{\tanh 2x}$ M1: Applies the chain rule or eliminates the “ln” and differentiates implicitly to obtain to obtain $\frac{dy}{dx} = \frac{k \operatorname{sech}^2 2x}{\tanh 2x}$ A1: Correct derivative in any form Note that some candidates now convert to exponential form to complete this part – see below in the alternative for scoring the final M1A1		M1A1
	$= \frac{2 \cosh 2x}{\sinh 2x} \times \frac{1}{\cosh^2 2x} = \frac{2}{\sinh 2x \cosh 2x}$	Converts to sinh2x and cosh2x correctly to obtain $\frac{k}{\sinh 2x \cosh 2x}$	M1
	$= \frac{2}{\frac{1}{2} \sinh 4x} = 4 \operatorname{cosech} 4x$	Correct answer. Note that this is not a given answer so you can allow if e.g. a sinh becomes a sin but is then recovered but if there are any obvious errors this mark should be withheld.	A1
			(4)
	Alternative using exponentials:		
	$y = \ln(\tanh 2x) = \ln\left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right)$ $\frac{dy}{dx} = \frac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}} \left(\frac{(e^{2x} + e^{-2x})(2e^{2x} + 2e^{-2x}) - (e^{2x} - e^{-2x})(2e^{2x} - 2e^{-2x})}{(e^{2x} + e^{-2x})^2} \right)$ or $y = \ln(\tanh 2x) = \ln\left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right) = \ln(e^{2x} - e^{-2x}) - \ln(e^{2x} + e^{-2x})$ $\frac{dy}{dx} = \frac{2e^{2x} + 2e^{-2x}}{e^{2x} - e^{-2x}} - \frac{2e^{2x} - 2e^{-2x}}{e^{2x} + e^{-2x}}$ M1: Writes tanh2x correctly in terms of exponentials and applies the chain rule and quotient rule or uses the subtraction law of logs and applies the chain rule A1: Correct derivative in any form		M1A1
	$= \frac{2(e^{2x} + e^{-2x})^2 - 2(e^{2x} - e^{-2x})^2}{e^{4x} - e^{-4x}} = \frac{8}{e^{4x} - e^{-4x}} \quad \text{Obtains } \frac{k}{e^{4x} - e^{-4x}}$		M1
	$= \frac{4}{\sinh 4x} = 4 \operatorname{cosech} 4x$	Correct answer. Note that this is not a given answer so you can allow if e.g. a sinh becomes a sin but is then recovered but if there are any obvious errors this mark should be withheld.	A1

(b) Way 1	$4\operatorname{cosech}4x = 1 \Rightarrow \sinh 4x = 4 \Rightarrow 4x = \ln(4 + \sqrt{4^2 + 1})$		M1
	Changes to $\sinh 4x = \dots$ and uses the correct logarithmic form of arsinh to reach $4x = \dots$		
	$x = \frac{1}{4} \ln(4 + \sqrt{17})$	This value only. Allow e.g. $x = \ln(4 + \sqrt{17})^{\frac{1}{4}}$	A1
			(2)
(b) Way 2	$4\operatorname{cosech}4x = 1 \Rightarrow 4 \times \frac{2}{e^{4x} - e^{-4x}} = 1 \Rightarrow e^{8x} - 8e^{4x} - 1 = 0$		
	Changes to the correct exponential form to reach $\frac{k}{e^{4x} - e^{-4x}}$, obtains a 3TQ in e^{4x} , solves and takes $\ln$'s to reach $4x = \dots$ (usual rules for solving a 3TQ do not apply as long as the above conditions are met)		M1
	$x = \frac{1}{4} \ln(4 + \sqrt{17})$	This value only. Allow e.g. $x = \ln(4 + \sqrt{17})^{\frac{1}{4}}$	A1
			(2)
			Total 6

Question Number	Scheme	Notes	Marks
3(a)	$\mathbf{A} = \begin{pmatrix} 2 & k & 2 \\ 2 & 2 & k \\ 1 & 2 & 2 \end{pmatrix}$		
	$ \mathbf{A} = 2(4-2k) - k(4-k) + 2(4-2) = 0$ $\Rightarrow k^2 - 8k + 12 = 0 \Rightarrow k = \dots$ Attempts $\det \mathbf{A} = 0$ and solves 3TQ to obtain 2 values for k Note that the usual rules for solving a 3TQ do not need to be applied as long as 2 values for k are obtained. The attempt at the determinant should be a correct expression for their row or column so allow errors only when collecting terms Note that the rule of Sarrus gives $8 + k^2 + 8 - 4 - 4k - 4k = 0$		M1
	$k = 2, 6$	Correct values.	A1
	Marks for part (a) can only be scored in their attempt at (a) and not recovered from part (b)		
			(2)
(b)	$\begin{pmatrix} 2 & k & 2 \\ 2 & 2 & k \\ 1 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 4-2k & 4-k & 2 \\ 2k-4 & 2 & 4-k \\ k^2-4 & 2k-4 & 4-2k \end{pmatrix} \rightarrow \begin{pmatrix} 4-2k & k-4 & 2 \\ 4-2k & 2 & k-4 \\ k^2-4 & 4-2k & 4-2k \end{pmatrix}$ Applies the correct method to reach at least a matrix of cofactors Should be an attempt at the minors followed by $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ If there is any doubt then look for at least 6 correct cofactors		M1
	$\begin{pmatrix} 4-2k & k-4 & 2 \\ 4-2k & 2 & k-4 \\ k^2-4 & 4-2k & 4-2k \end{pmatrix} \rightarrow \begin{pmatrix} 4-2k & 4-2k & k^2-4 \\ k-4 & 2 & 4-2k \\ 2 & k-4 & 4-2k \end{pmatrix}$ dM1: Attempts adjoint matrix by transposing. Dependent on previous mark. A1: Correct adjoint		dM1 A1
	$\mathbf{A}^{-1} = \frac{1}{k^2-8k+12} \begin{pmatrix} 4-2k & 4-2k & k^2-4 \\ k-4 & 2 & 4-2k \\ 2 & k-4 & 4-2k \end{pmatrix}$ Fully correct inverse or follow through their incorrect determinant from part (a) where their determinant is a function of k		A1ft
	Ignore any labelling of the matrices and allow any type of brackets around the matrices		
			(4)
			Total 6

Question Number	Scheme	Notes	Marks
4	$x = 4 \cosh \theta \Rightarrow \frac{dx}{d\theta} = 4 \sinh \theta$ $\Rightarrow \int \frac{1}{(x^2 - 16)^{\frac{3}{2}}} dx = \int \frac{4 \sinh \theta}{(16 \cosh^2 \theta - 16)^{\frac{3}{2}}} d\theta$ Full attempt to use the given substitution. Award for $\int \frac{1}{(x^2 - 16)^{\frac{3}{2}}} dx = k \int \frac{\sinh \theta}{((4 \cosh \theta)^2 - 16)^{\frac{3}{2}}} d\theta$ Condone $4 \cosh^2 \theta$ for $(4 \cosh \theta)^2$		M1
	$= \int \frac{4 \sinh \theta}{(16 \sinh^2 \theta)^{\frac{3}{2}}} d\theta = \int \frac{4 \sinh \theta}{64 \sinh^3 \theta} d\theta$ Simplifies $(16 \cosh^2 \theta - 16)^{\frac{3}{2}}$ to the form $k \sinh^3 \theta$ which may be implied by: $\int \frac{1}{(x^2 - 16)^{\frac{3}{2}}} dx = k \int \frac{1}{\sinh^2 \theta} d\theta$ Note that this is not dependent on the first M		M1
	$= \int \frac{1}{16 \sinh^2 \theta} d\theta$ Fully correct simplified integral. Allow equivalents e.g. $\frac{1}{16} \int \operatorname{cosech}^2 \theta d\theta$, $\int \frac{1}{(4 \sinh \theta)^2} d\theta$, $\int (4 \sinh \theta)^{-2} d\theta$ etc. May be implied by subsequent work.		A1
	$= \int \frac{1}{16 \sinh^2 \theta} d\theta = \frac{1}{16} \int \operatorname{cosech}^2 \theta d\theta = -\frac{1}{16} \coth \theta (+c)$ Integrates to obtain $k \coth \theta$. Depends on both previous method marks.		dM1
	$= -\frac{1}{16} \frac{\cosh \theta}{\sinh \theta} + c = -\frac{1}{16} \frac{\frac{x}{4}}{\frac{\sqrt{x^2 - 16}}{\sqrt{16}}} + c \text{ or e.g. } -\frac{1}{4} \frac{\frac{x}{4}}{\sqrt{x^2 - 16}} + c$ Substitutes back correctly for x by replacing $\cosh \theta$ with $\frac{x}{4}$ or equivalent e.g. $4 \cosh \theta$ θ with x and $\sinh \theta$ with $\sqrt{\left(\frac{x}{4}\right)^2 - 1}$ or equivalent e.g. $4 \sinh \theta$ with $\sqrt{x^2 - 16}$ Depends on all previous method marks and must be fully correct work for their $-\frac{1}{16}$		dM1
	$\frac{-x}{16\sqrt{x^2 - 16}} (+c) \text{ oe e.g. } \frac{-\frac{1}{16}x}{\sqrt{x^2 - 16}} (+c)$ Correct answer. Award once the correct answer is seen and apply isw if necessary. Condone the omission of “+ c” Note that you can condone the omission of the “dθ” throughout		A1
			(6)
			Total 6

Question Number	Scheme	Notes	Marks
5(i)	$x^2 - 3x + 5 = \left(x - \frac{3}{2}\right)^2 + \frac{11}{4}$	Correct completion of the square	B1
	$\int \frac{1}{\sqrt{x^2 - 3x + 5}} dx = \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 + \frac{11}{4}}} dx = \sinh^{-1} \frac{2x-3}{\sqrt{11}} (+c)$ M1: Use of $\sinh^{-1}$ A1: Fully correct expression (condone omission of + c) Allow equivalent correct expressions e.g. $\sinh^{-1} \frac{x-\frac{3}{2}}{\frac{\sqrt{11}}{2}} (+c)$, $\sinh^{-1} \frac{x-\frac{3}{2}}{\frac{\sqrt{11}}{2}} (+c)$ Allow equivalents for $\sinh^{-1}$ e.g. arsinh, arcsinh but not arsin or arcsin		M1A1
	You may see logarithmic forms for the answer: e.g. $\ln \left(\frac{2x-3}{\sqrt{11}} + \sqrt{\left(\frac{2x-3}{\sqrt{11}}\right)^2 + 1} \right)$, $\ln \left(x - \frac{3}{2} + \sqrt{\left(x - \frac{3}{2}\right)^2 + \frac{11}{4}} \right)$ but apply isw once a correct answer is seen.		
			(3)
(ii)	$63 + 4x - 4x^2 = -4 \left(x^2 - x - \frac{63}{4} \right)$ $= -4 \left(\left(x - \frac{1}{2} \right)^2 - \frac{64}{4} \right)$	Obtains $-4 \left(\left(x - \frac{1}{2} \right)^2 \pm \dots \right)$ or $-4 \left(x - \frac{1}{2} \right)^2 \pm \dots$ or $\dots - (2x-1)^2$	M1
	$-4 \left(\left(x - \frac{1}{2} \right)^2 - 16 \right)$ or $64 - 4 \left(x - \frac{1}{2} \right)^2$ or $64 - (2x-1)^2$	Correct completion of the square	A1
	$\int \frac{1}{\sqrt{63 + 4x - 4x^2}} dx = \frac{1}{2} \sin^{-1} \left(\frac{2x-1}{8} \right) (+c)$ M1: Use of $\sin^{-1}$ A1: Fully correct expression (condone omission of + c) Allow equivalent correct expressions e.g. $\frac{1}{2} \sin^{-1} \frac{x-\frac{1}{2}}{4} (+c)$, $-\frac{1}{2} \sin^{-1} \frac{\frac{1}{2}-x}{4} (+c)$ Allow equivalents for $\sin^{-1}$ e.g. arsin, arcsin but not arsinh or arcsinh		M1A1
			(4)
	In (ii) there are no marks for using $\int \frac{1}{\sqrt{63 + 4x - 4x^2}} dx = - \int \frac{1}{\sqrt{4x^2 - 63 - 4x}} dx$ But if completion of square attempted first allow M1A1 e.g. for $\int \frac{1}{\sqrt{63 + 4x - 4x^2}} dx = \int \frac{1}{\sqrt{64 - (2x-1)^2}} dx$ but then M0 for $= \int \frac{-1}{\sqrt{(2x-1)^2 - 64}} dx$		
			Total 7

Question Number	Scheme	Notes	Marks
1(i)	$(8) \int \frac{1}{16+x^2} dx = (8) \left(\frac{1}{4} \arctan \left(\frac{x}{4} \right) \right)$	Obtains ...arctan (kx) Allow $k = 1$	M1
	$2 \left[\arctan \left(\frac{x}{4} \right) \right]_4^{4\sqrt{3}} = 2(\arctan \sqrt{3} - \arctan 1) = \dots$	Substitutes the given limits, subtracts either way round and obtains a value (could be a decimal). The substitution does not need to be seen explicitly and may be implied by their value.	dM1
	$\frac{\pi}{6}$ or $p = \frac{1}{6}$ Correct exact value (or value for p) Accept equivalent exact expressions e.g. $\frac{2\pi}{12}$ or $p = \frac{2}{12}$ and isw if necessary.		A1
			(3)
(ii)	$2 \int \frac{1}{\sqrt{9-4x^2}} dx = 2 \left(\frac{1}{2} \arcsin \frac{2x}{3} \right) \left(\text{or e.g. } \arcsin \frac{x}{\frac{3}{2}} \right)$ M1: Obtains ...arcsin (kx). Allow $k = 1$ so allow just arcsin x. A1: Fully correct integration but allow unsimplified as above		M1 A1
	$\left[\arcsin \left(\frac{2x}{3} \right) \right]_3^k = \arcsin \left(\frac{2k}{3} \right) - \arcsin \left(\frac{1}{2} \right) = \frac{\pi}{12}$ $\Rightarrow \arcsin \left(\frac{2k}{3} \right) = \frac{\pi}{12} + \frac{\pi}{6} \Rightarrow \frac{2k}{3} = \sin \left(\frac{\pi}{4} \right) \Rightarrow \frac{2k}{3} = \frac{\sqrt{2}}{2} \Rightarrow k = \dots$ Substitutes the given limits, subtracts either way round, sets $= \frac{\pi}{12}$, uses $\arcsin \left(\frac{1}{2} \right) = \frac{\pi}{6}$ and the correct order of operations condoning sign errors only to reach a value for k e.g. $\pm \alpha \left(\arcsin \left(\frac{2k}{3} \right) - \frac{\pi}{6} \right) = \frac{\pi}{12} \Rightarrow \arcsin \left(\frac{2k}{3} \right) = \frac{\pi}{12\alpha} \pm \frac{\pi}{6} \Rightarrow k = \frac{3 \sin \left(\frac{\pi}{12\alpha} \pm \frac{\pi}{6} \right)}{2}$ Note that k may be inexact (decimal) or may be in terms of “sin” but must have a simplified argument e.g. $k = \frac{3 \sin \left(\frac{\pi}{4} \right)}{2}$		dM1
	$k = \frac{3\sqrt{2}}{4}$ or exact equivalent e.g., $\frac{3}{2\sqrt{2}}$ Note that a common incorrect answer is $k = \frac{3}{2} \sin \left(\frac{5\pi}{24} \right) (= 0.913\dots)$ which comes from an incorrect integral of $2 \arcsin \left(\frac{2x}{3} \right)$ (generally scoring 1010) Condone $x = \frac{3\sqrt{2}}{4}$		A1
			(4)
			Total 7

Question Number	Scheme	Marks
	There is no credit for attempts that do not use exponential definitions	
1(a) Working from LHS to RHS	$(2 \cosh 5x \cosh x) = 2 \frac{e^{5x} + e^{-5x}}{2} \times \frac{e^x + e^{-x}}{2} = \frac{e^{6x} + e^{4x} + e^{-4x} + e^{-6x}}{2}$	M1
	$= \frac{e^{6x} + e^{-6x}}{2} + \frac{e^{4x} + e^{-4x}}{2} = \cosh 6x + \cosh 4x^*$	A1*
		(2)
ALT Working from RHS to LHS	$\cosh 6x + \cosh 4x = \frac{e^{6x} + e^{-6x}}{2} + \frac{e^{4x} + e^{-4x}}{2} = \frac{e^{6x} + e^{4x} + e^{-4x} + e^{-6x}}{2} = 2 \frac{e^{5x} + e^{-5x}}{2} \times \frac{e^x + e^{-x}}{2}$	M1
	$= 2 \frac{e^{5x} + e^{-5x}}{2} \times \frac{e^x + e^{-x}}{2} = 2 \cosh 5x \cosh x$	A1*
(b)	$\cosh 6x + \cosh 4x = 8 \cosh x \Rightarrow 2 \cosh 5x \cosh x = 8 \cosh x$ $\Rightarrow 2 \cosh x (\cosh 5x - 4) = 0 \Rightarrow \cosh 5x = \dots \text{ or } 2 \cosh 5x = \dots$	M1
	$\cosh 5x = 4 \text{ or } 2 \cosh 5x = 8$	A1
	$\cosh 5x = 4 \Rightarrow 5x = \ln(4 + \sqrt{4^2 - 1})$ or $\cosh 5x = 4 \Rightarrow \frac{e^{5x} + e^{-5x}}{2} = 4 \Rightarrow e^{10x} - 8e^{5x} + 1 = 0 \Rightarrow e^{5x} = \frac{8 \pm \sqrt{60}}{2}$	M1
	$x = \pm \frac{1}{5} \ln(4 + \sqrt{15}) \text{ or } x = \frac{1}{5} \ln(4 \pm \sqrt{15})$	A1
		(4)
		Total 6

Notes:

(a)

M1: Expresses $\cosh 5x$ and $\cosh x$ correctly in terms of exponentials and attempts to multiply to obtain an expression involving four terms of the form $e^{\pm 6x}$ and $e^{\pm 4x}$ only.

A1*: Separates terms correctly from a correct expansion and achieves the required result with no errors. Must see one further line isolating the correct terms in exponential form before the stated result. Must clearly link the

two exponential expressions to $\cosh 6x$ and $\cosh 4x$ so e.g. $= \frac{e^{6x} + e^{-6x}}{2} + \frac{e^{4x} + e^{-4x}}{2}$ and then stop is insufficient

unless $\cosh 6x$ and $\cosh 4x$ have been correctly defined elsewhere in the work, or e.g. “=RHS” is seen

Note: One of the two's may be shown cancelled, but the correct exponential forms must be seen before any cancellation so e.g. $2 \cosh 5x \cosh x = e^{5x} + e^{-5x} \times \frac{e^x + e^{-x}}{2} = \frac{e^{6x} + e^{4x} + e^{-4x} + e^{-6x}}{2}$ is M0A0

Missing ‘h’ is A0

ALT

M1: Substitutes the correct exponential definitions of $\cosh 5x$ and $\cosh x$ into the RHS, combines over a common denominator and re-writes as “2 x a product involving $e^{\pm 5x}$ and $e^{\pm x}$ terms only

A1* Re-writes the expression in the correct form and achieves the required result with no errors. Must clearly

link the two exponential expressions to $\cosh 6x$ and $\cosh 4x$ so e.g. $= 2 \frac{e^{5x} + e^{-5x}}{2} \times \frac{e^x + e^{-x}}{2}$ and then stop is

insufficient unless $\cosh 5x$ and $\cosh x$ have been correctly defined elsewhere in the work, or e.g. “=LHS” is seen

Note: One of the two's may be shown cancelled, but the correct exponential forms must be seen before any cancellation so e.g. $\frac{e^{6x} + e^{4x} + e^{-4x} + e^{-6x}}{2} = e^{5x} + e^{-5x} \times \frac{e^x + e^{-x}}{2} = 2 \cosh 5x \cosh x$ is M0A0

Missing ‘h’ is A0

(b)

M1: Uses the result from part (a) to obtain a value for $\cosh 5x$ or $2 \cosh 5x$

A1: For $\cosh 5x = 4$ or $2 \cosh 5x = 8$ (No need to discount $\cosh x = 0$)

M1: Uses the correct logarithmic form for $\operatorname{arcosh} x$ to find a value for $5x$ (which could be implied by a correct value of x) or uses the correct exponential form for $\cosh 5x$ then forms and solves (usual rules) a 3TQ in e^{5x} (may be implied by a correct value for e^{5x})

A1: Both correct values in either correct form and no extras. Condone modulus signs instead of brackets around the $4 \pm \sqrt{15}$. Condone equivalent duplicate solutions.

Question Number	Scheme	Marks
In all parts, accept alternative notation for arsinh and arcosh such as $\sinh^{-1}$ and $\cosh^{-1}$ etc		
2(i)	$4x^2 + 8x + 9 = 4(x+1)^2 + 5$ or $4x^2 + 8x + 9 = (2x+2)^2 + 5$	B1
	$\int \frac{1}{\sqrt{4x^2 + 8x + 9}} dx = \int \frac{1}{\sqrt{4(x+1)^2 + 5}} dx = \frac{1}{2} \operatorname{ar sinh} \left(\frac{2(x+1)}{\sqrt{5}} \right) (+c) \text{ o.e.}$	M1A1
		(3)
(ii)	$\int \operatorname{arcosh} 3x dx = x \operatorname{arcosh} 3x - \int \frac{3x}{\sqrt{9x^2 - 1}} dx$	M1 A1
	$= \dots - \frac{1}{3} (9x^2 - 1)^{\frac{1}{2}}$ or $\dots - \frac{1}{3} \sinh(\operatorname{arcosh} 3x)$	M1
	$\int \operatorname{arcosh} 3x dx = x \operatorname{arcosh} 3x - \frac{1}{3} (9x^2 - 1)^{\frac{1}{2}} (+c)$ or $x \operatorname{arcosh} 3x - \left(x^2 - \frac{1}{9} \right)^{\frac{1}{2}} (+c)$ or $\int \operatorname{arcosh} 3x dx = x \operatorname{arcosh} 3x - \frac{1}{3} \sinh(\operatorname{arcosh} 3x) (+c)$	A1
		(4)
ALT	$u = \operatorname{ar cosh} 3x \Rightarrow x = \frac{1}{3} \cosh u \Rightarrow \frac{dx}{du} = \frac{1}{3} \sinh u$ $\int \operatorname{arcosh} 3x dx = \int \frac{1}{3} u \sinh u du = \frac{u}{3} \cosh u - \int \frac{1}{3} \cosh u du$	M1A1
	$\frac{u}{3} \cosh u - \frac{1}{3} \sinh u (+c)$	M1
	$x \operatorname{ar cosh} 3x - \frac{1}{3} \sinh(\operatorname{ar cosh} 3x) (+c)$ or $x \operatorname{arcosh} 3x - \frac{1}{3} (9x^2 - 1)^{\frac{1}{2}} (+c)$ or $x \operatorname{arcosh} 3x - \left(x^2 - \frac{1}{9} \right)^{\frac{1}{2}} (+c)$	A1
		(4)
		Total 7

General Principles for Further Pure Mathematics Marking
(But note that specific mark schemes may sometimes override these general principles).

Method mark for solving 3 term quadratic:

1. Factorisation

$(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$

$(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$

2. Formula

Attempt to use the correct formula (with values for a, b and c).

3. Completing the square

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$

Method marks for differentiation and integration:

1. Differentiation

Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)

2. Integration

Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)



Question Number	Scheme	Notes	Marks
7(i)	$x^2 - 4x + 5 = (x - 2)^2 + 1$	Attempts to complete the square. Allow for $(x - 2)^2 + c$, $c > 0$	M1
	$\int \frac{1}{(x-2)^2 + 1} dx = \arctan(x-2)$	Allow for $k \arctan f(x)$.	M1
	$[\arctan(x-2)]_1^2 = 0 - \left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$	$\frac{\pi}{4}$ cao	A1
			(3)
7(ii)	$\int \frac{\sqrt{x^2-3}}{x^2} dx = -\frac{\sqrt{x^2-3}}{x} + \int \frac{1}{\sqrt{x^2-3}} dx$ Uses integration by parts and obtains $A \frac{\sqrt{x^2-3}}{x} + B \int \frac{1}{\sqrt{x^2-3}} dx$		M1
	$= -\frac{\sqrt{x^2-3}}{x} + \operatorname{arcosh} \frac{x}{\sqrt{3}}$	$B \int \frac{1}{\sqrt{x^2-3}} dx = k \operatorname{arcosh} f(x)$	M1
		All correct	A1
	$\int_{\sqrt{3}}^3 \frac{\sqrt{x^2-3}}{x^2} dx = \left[-\frac{\sqrt{x^2-3}}{x} + \operatorname{arcosh} \frac{x}{\sqrt{3}} \right]_{\sqrt{3}}^3 = \left(-\frac{\sqrt{6}}{3} + \operatorname{arcosh} \sqrt{3} \right) - (0 + \operatorname{arcosh} 1)$ Applies the limits 3 and $\sqrt{3}$ Depends on both previous M marks		dM1
	$\operatorname{arcosh} \sqrt{3} - \frac{1}{3} \sqrt{6} = \ln(\sqrt{2} + \sqrt{3}) - \frac{1}{3} \sqrt{6}$	Accept either of these forms.	A1
			(5)
7(ii) ALT 1	$\int \frac{\sqrt{x^2-3}}{x^2} dx = \int \frac{\sqrt{3 \cosh^2 u - 3}}{3 \cosh^2 u} \sqrt{3} \sinh u du$	A complete substitution using $x = \sqrt{3} \cosh u$	M1
	$= \int \tanh^2 u du$	Obtains $k \int \tanh^2 u du$	M1
	$= \int (1 - \operatorname{sech}^2 u) du = u - \tanh u$	Correct integration	A1
	$\int_{\sqrt{3}}^3 \frac{\sqrt{x^2-3}}{x^2} dx = [u - \tanh u]_0^{\operatorname{arcosh} \sqrt{3}} = \operatorname{arcosh} \sqrt{3} - \tanh(\operatorname{arcosh} \sqrt{3}) - 0$ Applies the limits 0 and $\operatorname{arcosh} \sqrt{3}$ Depends on both previous M marks		dM1
	$\operatorname{arcosh} \sqrt{3} - \frac{1}{3} \sqrt{6} = \ln(\sqrt{2} + \sqrt{3}) - \frac{1}{3} \sqrt{6}$	Accept either of these forms.	A1

7(ii) ALT 2	$\int \frac{\sqrt{x^2-3}}{x^2} dx = \int \frac{\sqrt{3\sec^2 u - 3}}{3\sec^2 u} \sqrt{3} \sec u \tan u du$	A complete substitution using $x = \sqrt{3} \sec u$	M1
	$= \int \frac{\tan^2 u}{\sec u} du$	Obtains $k \int \frac{\tan^2 u}{\sec u} du$	M1
	$= \ln(\sec u + \tan u) - \sin u$	Correct integration	A1
	$\int_{\sqrt{3}}^3 \frac{\sqrt{x^2-3}}{x^2} dx = [\ln(\sec u + \tan u) - \sin u]_0^{\operatorname{arcsec} \sqrt{3}}$ $= \ln(\sec(\operatorname{arcsec} \sqrt{3}) + \tan(\operatorname{arcsec} \sqrt{3})) - \ln(\sec(0) + \tan(0)) - \sin(\operatorname{arcsec} \sqrt{3})$ Applies the limits 0 and $\operatorname{arcsec} \sqrt{3}$ Depends on both previous M marks		dM1
	$\int_{\sqrt{3}}^3 \frac{\sqrt{x^2-3}}{x^2} dx = \ln(\sqrt{2} + \sqrt{3}) - \frac{1}{3}\sqrt{6}$	Correct answer.	A1
			Total 8

Note that there may be other ways to perform the integration in part (ii) e.g. subsequent substitutions.

Marks can be awarded if the method leads to something that is integrable and should be awarded as in the main scheme e.g. M1 for a complete method, M2 for simplifying and reaching an expression that itself can be integrated or can be integrated after rearrangement, A1 for correct integration, dM3 for using appropriate limits and A2 as above.

Alternative approach:

$$\int \frac{\sqrt{x^2-3}}{x^2} dx = \int \frac{x^2-3}{x^2\sqrt{x^2-3}} dx = \int \frac{1}{\sqrt{x^2-3}} dx - \int \frac{3}{x^2\sqrt{x^2-3}} dx = \operatorname{arcosh} \frac{x}{\sqrt{3}} - \dots$$

Can score **M0M1A0dM0A0** if there is no creditable attempt at the second integral.

If the second integral is attempted, it must be using a suitable method

e.g. with either $x = \sqrt{3} \cosh u$ or $x = \sqrt{3} \sec u$:

$$\int \frac{3}{x^2\sqrt{x^2-3}} dx = \int \frac{3}{3\cosh^2 u \sqrt{3}\cosh^2 u - 3} \sqrt{3} \sinh u du = \int \operatorname{sech}^2 u du = \tanh u + c$$

or

$$\int \frac{3}{x^2\sqrt{x^2-3}} dx = \int \frac{3}{3\sec^2 u \sqrt{3}\sec^2 u - 3} \sqrt{3} \sec u \tan u du = \int \cos u du = \sin u + c$$

In these cases the first M can then be awarded and the other marks as defined with the appropriate limits used.

Question Number	Scheme	Notes	Marks
3(b)	$f(x) = \frac{1}{3} \operatorname{arsinh}(\sqrt{x^2-1}) - \arctan x$		
	$f'(x) = \frac{1}{3\sqrt{x^2-1}} - \frac{1}{1+x^2}$	$f'(x) = \frac{A}{\sqrt{x^2-1}} \pm \frac{1}{1 \pm x^2} \quad A = \frac{1}{3}, 3 \text{ or } 1$	M1 (B1 on ePen)
	$1+x^2 = 3\sqrt{x^2-1}$ $1+2x^2+x^4 = 9x^2-9$	Sets $\frac{A}{\sqrt{x^2-1}} \pm \frac{1}{1+x^2} = 0 \quad A = \frac{1}{3}, 3 \text{ or } 1$ Denominator of derivative of arctan x must now be $1+x^2$ Cross multiplies and squares to obtain the correct form for both sides so do not condone e.g., $(1+x^2)^2 = 1+x^4$ May see the quartic obtained through equivalent work.	M1
	$x^4 - 7x^2 + 10 = 0 \Rightarrow (x^2-2)(x^2-5) = 0 \Rightarrow x^2 = 2, 5$ Solves a 3TQ in x^2 (usual rules and one correct root if no working). No requirement to see the terms collected. Ignore labelling of solutions so allow e.g., " $x = 2, 5$ ". One correct value for their equation if no working, which may be for x or x^2 , so just look for the values. May change the variable. Allow for a correct solution with no working from solving a three term quartic of the correct form on a calculator. Allow if value for x^2 is negative or if roots are complex. Requires previous M marks.		ddM1
	$x = \sqrt{2}, \sqrt{5}$	Both exact and no other solutions e.g., $\pm$ is A0 unless negatives rejected. Must not reject either correct solution.	A1
			(4)
			Total 7

Question Number	Scheme/Notes	Marks
4(a)	$\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$	
	There is no credit for proofs that do not use exponential definitions	
	$\left\{ \sinh A \cosh B + \cosh A \sinh B = \right\}$ $\frac{e^A - e^{-A}}{2} \times \frac{e^B + e^{-B}}{2} + \frac{e^A + e^{-A}}{2} \times \frac{e^B - e^{-B}}{2} \text{ or }$ $\text{e.g., } \frac{(e^A - e^{-A})(e^B + e^{-B}) + (e^A + e^{-A})(e^B - e^{-B})}{4}$ Replaces two of the four hyperbolic functions with correct exponential expressions. Condone poor bracketing. If they immediately start expanding this mark is only implied by completely correct work (i.e., with exponential definitions correct) and <u>not just the fractions shown in the A1* note</u>	M1
	$= \frac{e^{A+B} - e^{B-A} + e^{A-B} - e^{-A-B} + e^{A+B} + e^{B-A} - e^{A-B} - e^{-A-B}}{4}$ Expands numerator (or numerators if 2 separate fractions). Allow for sign errors only with coefficients and indices and must see at least four terms (but count terms which have been crossed out by cancelling) Allow this mark for: $= \frac{e^A e^B - e^{-A} e^B + e^A e^{-B} - e^{-A} e^{-B} + e^A e^B + e^{-A} e^B - e^A e^{-B} - e^{-A} e^{-B}}{4}$ Must see at least four terms as before but the last mark will not be available unless the requirements shown below are satisfied.	M1
	$= \frac{2e^{A+B} - 2e^{-(A+B)}}{4} \text{ or } \frac{2(e^{A+B} - e^{-(A+B)})}{4} \text{ or } \frac{e^{A+B} - e^{-(A+B)}}{2} \text{ or } \frac{1}{2}(e^{A+B} - e^{-(A+B)}) \text{ or } \frac{e^{A+B}}{2} - \frac{e^{-(A+B)}}{2}$ $= \sinh(A+B) *$ Reaches $\sinh(A+B)$ with no errors. Condone if the "sinh A cosh B + cosh A sinh B = " is missing at the start but the "$= \sinh(A+B)$" or "$= \text{LHS}$" must be seen. All bracketing correct where required but condone an unclosed bracket. One of the expressions shown or similar must be seen and allow $-A-B$ used for $-(A+B)$. Allow a "meet in the middle" proof and condone a "1=1" style approach provided it is complete. In both these cases a minimal conclusion is required e.g., "shown" but allow if both "LHS = ..." and "...=RHS" are seen. Do not condone sinh and/or cosh written as sin/cos for this mark	A1*
	Attempts that start with the LHS and do not revert to a "meet in the middle" approach: Score the second M provided an eight term expanded numerator is achieved. The first M is for two explicitly clear correct replacements of hyperbolic expressions with two of $\sinh A$, $\cosh B$, $\cosh A$ and $\sinh B$. Condone if the $\sinh(A+B) =$ is missing at the start in these cases but the RHS or "...=RHS" must be seen.	
(3)		

Question Number	Scheme	Notes	Marks
4(b)	Condone the use of e.g., B for α or k for R for the first three marks but allow the A mark if recovered which may be via a correct expression which might be in (c)		
	$10 \sinh x + 8 \cosh x = R \sinh x \cosh \alpha + R \cosh x \sinh \alpha$ $\Rightarrow R \sinh \alpha = 8, \quad R \cosh \alpha = 10$ Equates coefficients to obtain the two correct equations. This mark could be implied by <u>either</u> correct elimination, i.e., $R^2 = 10^2 - 8^2 \text{ or } \tanh \alpha = \frac{8}{10} \text{ provided incorrect equations are not seen.}$		B1 (M1 on ePen)
	A complete attempt at finding a <u>positive</u> value for R: By elimination: $R^2 (\cosh^2 \alpha - \sinh^2 \alpha) = 10^2 - 8^2 \Rightarrow R^2 = 36 \Rightarrow R = 6$ Allow this mark for $R = \sqrt{10^2 + 8^2} = 2\sqrt{41}$ or $\sqrt{164}$. May just see e.g., $R = 2\sqrt{41}$ Following a positive value obtained for α where $\alpha = k \ln p$, $k > 0$, $p > 1$: $\alpha = \frac{1}{2} \ln 9 = \ln 3 \Rightarrow R \cosh(\ln 3) = 10 \Rightarrow R \left(\frac{e^{\ln 3} + e^{-\ln 3}}{2} \right) = 10 \Rightarrow R = \dots \left\{ \frac{5}{3} R = 10 \Rightarrow R = 6 \right\}$ $\text{or } R \sinh(\ln 3) = 8 \Rightarrow R \left(\frac{e^{\ln 3} - e^{-\ln 3}}{2} \right) = 8 \Rightarrow R = \dots \left\{ \frac{4}{3} R = 8 \Rightarrow R = 6 \right\}$ Correct exponential definitions must be used but can be implied by correct work. Allow if the 10 and 8 are mixed up and allow slips in solving		1st M1
	A complete attempt at finding a <u>positive</u> value for α where $\alpha = k \ln p$, $k > 0$, $p > 1$: By elimination: $\tanh \alpha = \frac{8}{10} \Rightarrow \alpha = \operatorname{artanh} \left(\frac{4}{5} \right) = \frac{1}{2} \ln \left(\frac{1 + \frac{4}{5}}{1 - \frac{4}{5}} \right) = \dots \left\{ = \frac{1}{2} \ln 9 = \ln 3 \right\}$ A correct logarithmic form must be used with a valid value for $\operatorname{artanh} (<1)$ Following a positive value obtained for R: $\sinh \alpha = \frac{8}{\text{"6"}} \Rightarrow \alpha = \operatorname{arsinh} \left(\frac{8}{\text{"6"}} \right) = \ln \left(\frac{8}{\text{"6"}} + \sqrt{\left(\frac{8}{\text{"6"}} \right)^2 + 1} \right) \quad \{ = \ln 3 \}$ $\cosh \alpha = \frac{10}{\text{"6"}} \Rightarrow \alpha = \operatorname{arcosh} \left(\frac{10}{\text{"6"}} \right) = \ln \left(\frac{10}{\text{"6"}} + \sqrt{\left(\frac{10}{\text{"6"}} \right)^2 - 1} \right) \quad \{ = \ln 3 \}$ A correct logarithmic form must be used with a valid value if using $\operatorname{arcosh} (>1)$ The appropriate logarithmic forms could be implied by correct values. Allow this mark if e.g., $\frac{8}{10}$ is erroneously simplified but the value must be valid for the inverse hyperbolic function. If an exponential form is used to evaluate an inverse hyperbolic the form must be correct and the solving of any resulting 3TQ (most likely in e^α or e^x) must satisfy usual rules with one root correct if no working. Note that using $\tanh$ leads to a 2TQ which they must get one correct root for. They must also proceed to $\alpha = k \ln p$, $k > 0$, $p > 1$		2nd M1
	$6 \sinh(x + \ln 3) \text{ or } R = 6 \text{ and } \alpha = \ln 3 \text{ (or } p = 3)$ Correct expression but allow values for R and α (or p). If all the values are not seen in (b) then allow if they are seen in (c) and they could be seen embedded in a correct expression. A0 for additional solutions e.g., $6 \sinh(x \pm \ln 3)$		A1

General Principles for Further Pure Mathematics Marking*(But note that specific mark schemes may sometimes override these general principles)***Method mark for solving 3 term quadratic:****1. Factorisation** $(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$ $(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$ **2. Formula**Attempt to use the correct formula (with values for a , b and c).**3. Completing the square**Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$ **Method marks for differentiation and integration:****1. Differentiation**Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)**2. Integration**Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)**Use of a formula**

Where a method involves using a formula that has been learnt, the advice given in recent examiners' reports is that the formula should be quoted first.

Normal marking procedure is as follows:

Method mark for quoting a correct formula and attempting to use it, even if there are small errors in the substitution of values.Where the formula is not quoted, the method mark can be gained by implication from correct working with values but may be lost if there is any mistake in the working.**Exact answers**

Examiners' reports have emphasised that where, for example, an exact answer is asked for, or working with surds is clearly required, marks will normally be lost if the candidate resorts to using rounded decimals.

Question Number	Scheme	Notes	Marks
8(a)	$\frac{dx}{dy} = \frac{y}{4}$ or $2y \frac{dy}{dx} = 8$ or $\frac{dy}{dx} = \left(\frac{1}{2}\right)(2\sqrt{2})x^{-\frac{1}{2}}$ or $\left(\frac{1}{2}\right)(2\sqrt{2})\left(\frac{2\sqrt{2}}{y}\right)$ oe Any correct equation in $\frac{dx}{dy}$ or $\frac{dy}{dx}$ in terms of y or x		B1
	$\left(\int \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int \sqrt{1 + \left(\frac{y}{4}\right)^2} (dy) \text{ or } \left(\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot \frac{dx}{dy} dy = \int \sqrt{1 + \left(\frac{4}{y}\right)^2} \cdot \frac{y}{4} (dy)\right)$ Forms $\int \sqrt{1 + \left(\frac{dx}{dy}\right)^2} (dy)$ or $\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot \frac{dx}{dy} (dy)$ correctly with their derivative		M1
	$x = 18 \Rightarrow y^2 = 144 \Rightarrow \beta = 12, \alpha = 24$ $\Rightarrow (\text{perimeter of } R =) 24 + 2 \int_0^{12} \sqrt{1 + \frac{y^2}{16}} dy$	Correct expression	A1
			(3)



(b)	$y = 4 \sinh u \Rightarrow \frac{dy}{du} = 4 \cosh u$	Correct derivative. Condone $\frac{dy}{dx} = 4 \cosh u$	B1
	$\int \sqrt{1 + \frac{y^2}{16}} dy = \int \sqrt{1 + \frac{(4 \sinh u)^2}{16}} (4 \cosh u) (du)$ $(= 4 \int \cosh^2 u du)$	Full substitution, correct for their $\frac{dy}{du}$	M1
	$\int \cosh^2 u du = \int \left(\frac{1}{2} \cosh 2u + \frac{1}{2} \right) du = \frac{1}{4} \sinh 2u + \frac{1}{2} u$ or $\int \left(\frac{e^u + e^{-u}}{2} \right)^2 du = \int \left(\frac{e^{2u}}{4} + \frac{1}{2} + \frac{e^{-2u}}{4} \right) du = \frac{e^{2u}}{8} + \frac{1}{2} u - \frac{e^{-2u}}{8}$ dM1: Uses $\cosh^2 u = \pm \frac{1}{2} \cosh 2u \pm \frac{1}{2}$ and integrates to obtain $a \sinh 2u + bu$ or uses $k(e^u + e^{-u})$ for $\cosh u$, expands and integrates to obtain $ae^{2u} + bu + ce^{-2u}$ A1: Correct integration		dM1 A1
	Perimeter of R:		
	$= 24 + (2)(4) \left[\frac{1}{4} \sinh 2u + \frac{1}{2} u \right]_0^{\text{arsinh } 3 = \ln(3 + \sqrt{10})}$ $= 24 + 2 \left[2 \sinh u \sqrt{1 + \sinh^2 u} + 2u \right]_0^{\text{arsinh } 3 = \ln(3 + \sqrt{10})}$ $= 24 + 2 \left[(2)(3)\sqrt{1 + 3^2} + 2 \ln(3 + \sqrt{10}) \right]$	$= 24 + (2)(4) \left[\frac{e^{2u}}{8} + \frac{1}{2} u - \frac{e^{-2u}}{8} \right]_0^{\ln(3 + \sqrt{10})}$ $= 24 + e^{2 \ln(3 + \sqrt{10})} - e^{-2 \ln(3 + \sqrt{10})} + 4 \ln(3 + \sqrt{10})$ $24 + (3 + \sqrt{10})^2 - \frac{1}{(3 + \sqrt{10})^2} + 4 \ln(3 + \sqrt{10})$	ddM1
	Substitutes $\text{arsinh } 3$ and/or $\ln(3 + \sqrt{3^2 + 1})$ into their expression using correct identities or correctly removes exponentials to obtain a numerical expression in constants and lns only Accept use of calculator here e.g. $\sinh(2 \text{arsinh } 3) = 6\sqrt{10}$		
	$24 + 12\sqrt{10} + 4 \ln(3 + \sqrt{10})$ or, e.g., $4(6 + 3\sqrt{10} + \ln(3 + \sqrt{10}))$	Correct answer – any exact simplified equivalent	A1
Note: Integration by calculator is likely to access the first two marks only			(6)
			Total 9
TOTAL FOR PAPER: 75 MARKS			

matching their quartic, but allow answers only from a calculator – but must be correct and exact for their equation.

M1: Proceeds to a solution in terms of natural logarithms from a positive solution to their equation using the correct formula. Not dependent, so this may be scored from decimal solutions to the quartic, for proceeding from $e^x = \dots$ to $x = \ln \dots$

A1: One value correct.

A1: Both values correct, accepting equivalents, and no others.

There may be other variations via substitutions etc.

M1: Attempts to use the correct identities to form an equation in one “variable”.

A1: Correct simplified equation.

M1: Full attempt to solve the equation to at least $\cosh x = \dots$ or $e^x = \dots$ (exact)

M1: Forms a solution in terms of natural logarithms from a valid solution to their equation using the correct formula.

A1: One value correct.

A1: Both values correct, accepting equivalents, and no others. Equivalents are e.g.

$$x = \pm \ln \left(\frac{3}{2} + \frac{\sqrt{5}}{2} \right), \quad \pm \ln \left(\frac{3}{2} - \frac{\sqrt{5}}{2} \right) \quad \text{Condone missing brackets.}$$

Question	Scheme	Marks
2.(a)	$kx^2 - y^2 = 9$	
	$\frac{x^2}{9/k} - \frac{y^2}{9} = 1 \Rightarrow a^2 = \frac{9}{k}, b^2 = 9$ or Focus $(6, 0) \Rightarrow 6 = ae$	B1
	$\frac{x^2}{9/k} - \frac{y^2}{9} = 1 \Rightarrow a^2 = \frac{9}{k}, b^2 = 9$ and Focus $(6, 0) \Rightarrow 6 = ae$	M1
	E.g. $b^2 = a^2(e^2 - 1) \Rightarrow 9 = a^2\left(\frac{36}{a^2} - 1\right) = 36 - a^2 \Rightarrow a^2 = \dots(27) \Rightarrow k = \dots$ or $b^2 = a^2(e^2 - 1) \Rightarrow 9 = \frac{36}{e^2}(e^2 - 1) \Rightarrow 9e^2 = 36e^2 - 36 \Rightarrow e^2 = \dots\left(\frac{36}{27}\right) \Rightarrow k = \dots$ $b^2 = a^2(e^2 - 1) \Rightarrow 9 = \frac{9}{k}\left(\frac{36k}{9} - 1\right) = 36 - \frac{9}{k} \Rightarrow k = \dots$	M1
	$\left(\Rightarrow \frac{9}{k} = 27\right) \Rightarrow k = \frac{1}{3}$	A1
		(4)
(b)	E.g. $a = \sqrt{27}$, directrix is $x = \pm \frac{a}{e} \Rightarrow (x =) \pm \frac{\sqrt{27}}{6/\sqrt{27}} = \dots$	M1
	$x = \frac{9}{2}$	A1
		(2)
(6 marks)		

Notes
(a)

B1: Either Correctly identifies b^2 and a^2 (or b and a) in terms of k for the hyperbola (may be implied). (isw if they put $\pm \dots$ square rooting.) May be implied by working. May be seen in (b)

Or Uses the focus to set up a correct equation relating a and e or k and e . Note that

$a^2 + b^2 = (ae)^2 = 36$ is an alternative correct equation. May be implied by working.

M1: Both of the above correctly attempted or implied in the working but accept $a^2 = 9k$ or $\frac{k}{9}$ as an attempt (and similar for b^2).

M1: Full method to find k using $b^2 = a^2(e^2 - 1)$ with their $ae = 6$ and their a and b . E.g. uses the eccentricity equation with their b and with their a or e in terms of a , e , or k appropriately, to set up and solve an equation in one variable, reaching a value for k . Condone confusion between a and a^2 or b and b^2 when substituting into the formulae for this mark.

A1: Correct k

(b)

M1: Full method to find $(\pm)\frac{a}{e}$ using their a and e (finding these if not already seen) or k . They must be using their a correctly, M0 if they have correctly identified a^2 but use it incorrectly here. Allow if $\pm$ is used and/or the x is missing for this mark.

A1: Correct equation only from fully correct work, and must be an equation not just value. A0 if $\pm$ given, they must be choosing the positive to correspond to the focus.



Question	Scheme	Marks
3(i)	$\int \frac{1}{4x^2 + 12x + 25} dx$	
	$\{4x^2 + 12x + 25 = 4(x^2 + 3x) + 25\}$ $= 4\left(x + \frac{3}{2}\right)^2 + \dots \quad \text{or} \quad 4x^2 + 12x + 25 = (2x + 3)^2 + \dots$	M1
	$4\left(x + \frac{3}{2}\right)^2 + 16 \quad \text{or} \quad (2x + 3)^2 + 16$	A1
	$\int \frac{1}{4x^2 + 12x + 25} dx = \frac{1}{4} \int \frac{1}{\left(x + \frac{3}{2}\right)^2 + 4} dx = k \arctan\left(\frac{"x + \frac{3}{2}" }{"2"}\right) (+c)$ $\text{or } \int \frac{1}{4x^2 + 12x + 25} dx = \int \frac{1}{(2x + 3)^2 + 4^2} dx = k \arctan\left(\frac{"2x + 3"}{"4"}\right) (+c)$	M1
	$= \frac{1}{8} \arctan\left(\frac{2x + 3}{4}\right) (+c) \quad (\text{oe})$	A1
		(4)
(ii)	$\int_3^9 \frac{1}{\sqrt{x^2 + 4x - 17}} dx = \ln a$	
	$x^2 + 4x - 17 = (x + 2)^2 - 4 - 17$	B1
	$\int \frac{1}{\sqrt{x^2 + 4x - 17}} dx = \int \frac{1}{\sqrt{(x + 2)^2 - 21}} dx = \operatorname{arcosh}\left(\frac{x + 2}{\sqrt{21}}\right) \text{ or }$ $\ln\left(x + 2 + \sqrt{(x + 2)^2 - 21}\right) \text{ or } \ln\left(\frac{x + 2}{\sqrt{21}} + \sqrt{\left(\frac{x + 2}{\sqrt{21}}\right)^2 - 1}\right)$	M1 A1
	Alt: $u = x + 2 \Rightarrow \int \frac{1}{\sqrt{u^2 - 21}} du = \operatorname{arcosh}\left(\frac{u}{\sqrt{21}}\right) \text{ or } \ln(u + \sqrt{u^2 - 21}) \text{ etc}$	
	$\int_3^9 \frac{1}{\sqrt{x^2 + 4x - 17}} dx = \operatorname{arcosh}\left(\frac{11}{\sqrt{21}}\right) - \operatorname{arcosh}\left(\frac{5}{\sqrt{21}}\right) \text{ or }$ $\ln(11 + \sqrt{11^2 - 21}) - \ln(5 + \sqrt{5^2 - 21})$	M1
	$\left(= \ln\left(\frac{11}{\sqrt{21}} + \sqrt{\frac{121}{21} - 1}\right) - \ln\left(\frac{5}{\sqrt{21}} + \sqrt{\frac{25}{21} - 1}\right) \right) = \ln\left(\frac{11 + \sqrt{100}}{5 + \sqrt{4}}\right)$	dM1
	$= \ln 3$	A1
		(6)
(10 marks)		
Notes:		

Question Number	Scheme	Notes	Marks
1(a)	$4\sinh^3 x + 3\sinh x = 4\left(\frac{e^x - e^{-x}}{2}\right)^3 + 3\left(\frac{e^x - e^{-x}}{2}\right)$ $= 4\left(\frac{e^{3x} - 3e^x + 3e^{-x} - e^{-3x}}{8}\right) + 3\left(\frac{e^x - e^{-x}}{2}\right)$ Uses $\sinh x = \frac{e^x - e^{-x}}{2}$ on both $\sinh$ terms and attempts to cube the bracket (min accepted is a linear x a quadratic bracket)		M1
	$= \frac{1}{2}e^{3x} - \frac{3}{2}e^x + \frac{3}{2}e^{-x} - \frac{1}{2}e^{-3x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x}$ $= \frac{e^{3x} - e^{-3x}}{2} = \sinh 3x^*$		A1*
			(2)
(b)	$\sinh 3x = 19\sinh x \Rightarrow 4\sinh^3 x + 3\sinh x = 19\sinh x$ $\Rightarrow 4\sinh^3 x - 16\sinh x = 0$ Uses the result from (a) and combines terms		M1
	$(\sinh x = 0 \text{ or } \sinh^2 x = 4)$	$\sinh^2 x = 4 \text{ or } \sinh x = (\pm)2$	A1
	$(0, 0)$	States the origin as one intersection	B1
	$\ln(2 + \sqrt{5})$ and $-\ln(2 + \sqrt{5})$	Two correct non-zero x values (allow e.g. $\ln(-2 + \sqrt{5})$ for $-\ln(2 + \sqrt{5})$)	A1
	$(\ln(2 + \sqrt{5}), 38)$ and $(-\ln(2 + \sqrt{5}), -38)$	Two correct points (allow e.g. $\ln(-2 + \sqrt{5})$ for $-\ln(2 + \sqrt{5})$)	A1
			(5)
Alternative for (b) using exponentials			
	$\sinh 3x = 19\sinh x \Rightarrow \frac{e^{3x} - e^{-3x}}{2} = \frac{19(e^x - e^{-x})}{2} \Rightarrow \dots$ Substitutes the correct exponential forms and collects terms to one side		M1
	$\Rightarrow e^{6x} - 19e^{4x} + 19e^{2x} - 1 = 0$	Correct equation (or equivalent)	A1
	$(0, 0)$	States the origin as one intersection	B1
	$\frac{1}{2}\ln(9 + 4\sqrt{5})$ or $\frac{1}{2}\ln(9 - 4\sqrt{5})$	Two correct non-zero x values (oe)	A1
	$\left(\frac{1}{2}\ln(9 + 4\sqrt{5}), 38\right)$ and $\left(\frac{1}{2}\ln(9 - 4\sqrt{5}), -38\right)$	Two correct points (oe)	A1
			Total 7

Question	Scheme	Marks
4(a)	$\begin{vmatrix} 2 & 0 & -1 \\ k & 3 & 2 \\ -2 & 1 & k \end{vmatrix} = 2 \begin{vmatrix} 3 & 2 \\ 1 & k \end{vmatrix} - 0 \begin{vmatrix} k & 2 \\ -2 & k \end{vmatrix} + (-1) \begin{vmatrix} k & 3 \\ -2 & 1 \end{vmatrix} = 2(3k-2) - (k+6) = \dots$	M1
	$= 6k - 4 - k - 6 = 5k - 10^*$	A1*
		(2)
(b)	$\mathbf{M}^T = \begin{pmatrix} 2 & k & -2 \\ 0 & 3 & 1 \\ -1 & 2 & k \end{pmatrix} \text{ or minors } \begin{pmatrix} 3k-2 & k^2+4 & k+6 \\ 1 & 2k-2 & 2 \\ 3 & 4+k & 6 \end{pmatrix} \text{ or}$ $\text{cofactors } \begin{pmatrix} 3k-2 & -k^2-4 & k+6 \\ -1 & 2k-2 & -2 \\ 3 & -4-k & 6 \end{pmatrix}$	M1
	$\text{Adjugate matrix is } \begin{pmatrix} 3k-2 & -1 & 3 \\ -k^2-4 & 2k-2 & -4-k \\ k+6 & -2 & 6 \end{pmatrix} (\geq 6 \text{ entries correct})$	M1
	$\text{Hence } \mathbf{M}^{-1} = \frac{1}{5k-10} \begin{pmatrix} 3k-2 & -1 & 3 \\ -k^2-4 & 2k-2 & -4-k \\ k+6 & -2 & 6 \end{pmatrix}$	dM1A1
		(4)
(c)	Images of A, B and C are $(5, 4k-18, 3k-16)$, $(0, 7-2k, 9-4k)$ and $(0, 4k-2, 8k-14)$	M1 A1
	$(\pm) 50 = \frac{1}{6} \begin{vmatrix} 5 & 4k-18 & 3k-16 \\ 0 & 7-2k & 9-4k \\ 0 & 4k-2 & 8k-14 \end{vmatrix} \Rightarrow (\pm) 300 = 5(\dots) (= 200k - 400) \Rightarrow k = \dots$	M1
	$(300 = 200k - 400 \Rightarrow) k = \frac{7}{2} \quad \text{or} \quad (-300 = 200k - 400 \Rightarrow) k = \frac{1}{2}$	A1
	$k = \frac{1}{2} \text{ and } k = \frac{7}{2}$	A1
		(5)
	Using volume scale factor. Attempts $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 4 & -8 & 3 \\ -2 & 5 & -4 \\ 4 & -6 & 8 \end{vmatrix} = 4(40-24) + 8(-16+16) + 3(12-20) = \dots$	M1
	$\text{Volume of } T \text{ is } \frac{1}{6} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \left \frac{1}{6} \begin{vmatrix} 4 & -8 & 3 \\ -2 & 5 & -4 \\ 4 & 6 & -8 \end{vmatrix} \right = \dots \frac{20}{3}$	A1
	$\text{Volume image of } T = \det \mathbf{M} \times \frac{20}{3} \Rightarrow \frac{20}{3} 5k-10 = 50 \Rightarrow k = \dots$	M1

	$\left(\frac{20}{3}(5k-10)=50 \Rightarrow\right) k=\frac{7}{2}$ or $\left(\frac{20}{3}(10-5k)=50 \Rightarrow\right) k=\frac{1}{2}$	A1
	$k=\frac{1}{2}$ and $k=\frac{7}{2}$	A1
		(5)

(11 marks)**Notes:****(a)**

M1: Correct method for expanding the determinant to reach a linear expression in k . Expect expansion along the top row, but may expand along any row or column. Sarrus gives $6 + k - (6 + 4)$.

A1*: Correct expression from correct work.

(b)

M1: Begins the process of finding the inverse by attempting either the transpose, or the matrix of minors or cofactors. Look for at least 6 correct entries.

M1: Proceeds to find the adjugate matrix (may include the reciprocal determinant). Again look for 6 correct entries.

dM1: Full method to find the inverse matrix, so divides their adjugate by the determinant.

Depends on both previous marks.

A1: Fully correct inverse.

(c)

M1: Attempts to find the image vectors of A , B and C under the transformation. (O mapping to O may be assumed). May be implied by at least two correct entries in one of the three vectors – but must be finding all three.

A1: Correct image vectors. Allow unsimplified and isw if necessary.

M1: Use their image vectors in a suitable scalar triple product to find the volume, and set volume equal to 50 and attempts to solve for k . Must include the $1/6$ but may appear later.

$$\text{Usually } \frac{1}{6}(200k - 400) = 50 \text{ leading to } k = \frac{7}{2}$$

A1: One correct value for k obtained, either $k = \frac{7}{2}$ or $k = \frac{1}{2}$

A1: Both values of k correctly found. $k = \frac{7}{2}$ and $k = \frac{1}{2}$

Alt method using determinant as volume scale factor.

M1: Attempts an appropriate scalar triple product. May have rows in different order.

A1: Correct volume for tetrahedron T . Need not be simplified, so $\frac{40}{6}$ is fine here.

M1: Uses the determinant as the volume scale factor to set up at least one equation in k using their volume and the given volume and attempts to solve for k . The $1/6$ may have been missing.

$$\text{Usually } \frac{20}{3}(5k-10)=50 \text{ leading to } k=\frac{7}{2}$$

A1: One correct value for k obtained, either $k = \frac{7}{2}$ or $k = \frac{1}{2}$

A1: Both values of k correctly found. $k = \frac{7}{2}$ and $k = \frac{1}{2}$



Question	Scheme	Marks
5(a)	$(5\mathbf{i} + \mathbf{j}) \times (8\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & 0 \\ 8 & -2 & 3 \end{vmatrix} = \dots$	M1
	Or $\left. \begin{aligned} (u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \cdot (5\mathbf{i} + \mathbf{j}) &= 0 \\ (u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \cdot (8\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) &= 0 \end{aligned} \right\} \Rightarrow \begin{aligned} 5u + v &= 0 \\ 8u - 2v + 3w &= 0 \end{aligned} \Rightarrow u, v, w = \dots$	
	$\mathbf{n} = 3\mathbf{i} - 15\mathbf{j} - 18\mathbf{k}$ or $\alpha(3\mathbf{i} - 15\mathbf{j} - 18\mathbf{k})$ for any $\alpha \neq 0$	A1
(b)	(i) $\mathbf{r} = (2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}) + s(8\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) + t(5\mathbf{i} + \mathbf{j})$	B1
		(1)
	(ii) $(2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}) \cdot (3\mathbf{i} - 15\mathbf{j} - 18\mathbf{k}) = \dots (= -6)$	M1
	So $\mathbf{r} \cdot (3\mathbf{i} - 15\mathbf{j} - 18\mathbf{k}) = -6$ oe such as $\mathbf{r} \cdot (-\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}) = 2$	A1
		(2)
(c) Way 1	Distance from plane in (b) to origin is $\frac{\pm 6}{\sqrt{3^2 + 15^2 + 18^2}}$ oe e.g. $\frac{2}{\sqrt{1^2 + 5^2 + 6^2}}$ Or attempts similar for parallel plane containing l_1 , e.g. $\frac{(\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) \cdot (3\mathbf{i} - 15\mathbf{j} - 18\mathbf{k})}{\sqrt{3^2 + 15^2 + 18^2}} = \dots$	M1
	$= \pm \frac{2}{\sqrt{62}}$ (oe evaluated) or $\mp \frac{21}{\sqrt{62}}$ if considering other plane.	A1
	Both $\frac{\pm 6}{\sqrt{3^2 + 15^2 + 18^2}}$ oe and $\frac{(\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) \cdot (3\mathbf{i} - 15\mathbf{j} - 18\mathbf{k})}{\sqrt{3^2 + 15^2 + 18^2}} = \dots$ attempted	M1
	Hence shortest distance between lines is $\frac{2}{\sqrt{62}} + \frac{21}{\sqrt{62}} = \dots$	M1
	$= \frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$	A1
		(5)
Way 2	$\overline{AB} = \pm((\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) - (2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k})) = \pm(-\mathbf{i} + 6\mathbf{j} - 9\mathbf{k})$	M1 A1
	$d = AB \cos \theta = \frac{\overline{AB} \cdot \mathbf{n}}{ \mathbf{n} } = \frac{\pm(-\mathbf{i} + 6\mathbf{j} - 9\mathbf{k}) \cdot (3\mathbf{i} - 15\mathbf{j} - 18\mathbf{k})}{\sqrt{3^2 + 15^2 + 18^2}}$ oe	M1
	$= \frac{\pm(-3 - 90 + 162)}{\sqrt{558}} = \frac{\pm 69}{\sqrt{558}} = \dots$	M1
	$= \frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$	A1
		(5)



Way 3	$(2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}) + \mu(8\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) - ((\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) + \lambda(5\mathbf{i} + \mathbf{j}))$ $= (1 + 8\mu - 5\lambda)\mathbf{i} + (-6 - 2\mu - \lambda)\mathbf{j} + (9 + 3\mu)\mathbf{k}$	M1 A1
	$((1 + 8\mu - 5\lambda)\mathbf{i} + (-6 - 2\mu - \lambda)\mathbf{j} + (9 + 3\mu)\mathbf{k}) \cdot (5\mathbf{i} + \mathbf{j}) = 0$ $\Rightarrow 38\mu - 26\lambda = 1$ $((1 + 8\mu - 5\lambda)\mathbf{i} + (-6 - 2\mu - \lambda)\mathbf{j} + (9 + 3\mu)\mathbf{k}) \cdot (8\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 0$ $\Rightarrow 77\mu - 38\lambda = -47$ $\Rightarrow \lambda = -\frac{207}{62}, \mu = -\frac{70}{31}$	M1
	$(2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}) + \mu(8\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) - ((\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) + \lambda(5\mathbf{i} + \mathbf{j}))$ $= -\frac{23}{62}\mathbf{i} + \frac{115}{62}\mathbf{j} + \frac{69}{31}\mathbf{k}$ $d = \sqrt{\left(\frac{23}{62}\right)^2 + \left(\frac{115}{62}\right)^2 + \left(\frac{69}{31}\right)^2}$	M1
	$= \frac{23}{\sqrt{62}} \text{ or } \frac{23\sqrt{62}}{62}$	A1
		(5)
(10 marks)		

Notes:

Accept equivalent vector notation, e.g. column vectors, throughout.

(a)

M1: Any correct method to find a vector perpendicular to the two direction vectors of the lines.

Look for the cross product between the two direction vectors, but may use dot products and solving equations. In the latter case the method should lead to values for u , v and w .

For the vector product, if no method is shown look for at least 2 correct components.

A1: Any correct vector, a scalar multiple of $-\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$

(b)

B1: Any correct equation. Must have $\mathbf{r} = \dots$ or e.g. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dots$

M1: Uses their normal vector from (a) with any point on the plane (probably $(2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k})$) to find p

Condone slips with the calculation so $(2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}) \cdot (3\mathbf{i} - 15\mathbf{j} - 18\mathbf{k})$ evaluated as a scalar is sufficient for M1. May also be implied by $p = -6$

A1: Any correct equation of the correct form.

(c)
Way 1

M1: Uses the plane equation from (b) (or otherwise) OR the parallel plane containing l_1 to find the distance of one of these planes to the origin.

A1: Correct distance between one of the planes and the origin, accept $\pm$ here.

M1: Attempts distance of both the parallel planes containing l_1 and l_2 from the origin.



M1: Correct method for finding the distance between lines – i.e. subtracts their distances either way round.

A1: Correct answer. Accept $\frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$

Way 2

M1: Subtracts position vectors of points on the lines (either way around). Implied by two correct coordinates if method not shown. (Forms suitable hypotenuse.)

A1: Correct vector or as coordinates, either direction.

M1: Correct formula for the distance using their vectors, $d = AB \cos \theta = \frac{\overline{AB} \cdot \mathbf{n}}{|\mathbf{n}|}$ with their $\overline{AB}$ and $\mathbf{n}$.

M1: Complete evaluation of the formula.

A1: Correct answer. Accept $\frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$ but must be positive.

Way 3

M1: Subtracts position vectors of general points on each line (either way around). Implied by two correct coordinates if method not shown.

A1: Correct vector or as coordinates, either direction.

M1: Forms scalar product of the general vector with both direction vectors, sets = 0 and solves simultaneously

M1: Substitutes the values of their parameters back into the general vector and attempts its magnitude

A1: Correct answer. Accept $\frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$ but must be positive.

Question	Scheme	Marks
6(a) Way 1	$I_n = \int_0^{\sqrt{\frac{\pi}{2}}} x^{n-1} \cdot x \cos(x^2) \, dx = \left[x^{n-1} \cdot \frac{1}{2} \sin(x^2) \right]_0^{\sqrt{\frac{\pi}{2}}} - \int_0^{\sqrt{\frac{\pi}{2}}} (n-1)x^{n-2} \cdot \frac{1}{2} \sin(x^2) \, dx$	M1A1
	$= \left[x^{n-1} \cdot \frac{1}{2} \sin(x^2) \right]_0^{\sqrt{\frac{\pi}{2}}} - \frac{1}{2}(n-1) \int_0^{\sqrt{\frac{\pi}{2}}} x^{n-3} \cdot x \sin(x^2) \, dx$	dM1A1
	$= \left[x^{n-1} \cdot \frac{1}{2} \sin(x^2) \right]_0^{\sqrt{\frac{\pi}{2}}} - \frac{1}{2}(n-1) \left(\left[x^{n-3} \cdot -\frac{1}{2} \cos(x^2) \right]_0^{\sqrt{\frac{\pi}{2}}} - \int_0^{\sqrt{\frac{\pi}{2}}} (n-3)x^{n-4} \cdot -\frac{1}{2} \cos(x^2) \, dx \right)$	
	$= \left(\frac{1}{2} \left(\sqrt{\frac{\pi}{2}} \right)^{n-1} \sin \frac{\pi}{2} - 0 \right) - \frac{1}{2}(n-1) \left[(0-0) + \frac{1}{2}(n-3)I_{n-4} \right]$	dM1
	$= \frac{1}{2} \left(\frac{\pi}{2} \right)^{\frac{n-1}{2}} - \frac{1}{4}(n-1)(n-3)I_{n-4}^*$	A1*
		(6)
Way 2	$I_n = \left[\frac{x^{n+1}}{n+1} \cdot \cos(x^2) \right]_0^{\sqrt{\frac{\pi}{2}}} - \int_0^{\sqrt{\frac{\pi}{2}}} \frac{x^{n+1}}{n+1} \cdot -2x \sin(x^2) \, dx$	M1A1
	$= \left[\frac{x^{n+1}}{n+1} \cdot \cos(x^2) \right]_0^{\sqrt{\frac{\pi}{2}}} + \frac{2}{n+1} \int_0^{\sqrt{\frac{\pi}{2}}} x^{n+2} \sin(x^2) \, dx$	dM1A1
	$= \left[\frac{x^{n+1}}{n+1} \cdot \cos(x^2) \right]_0^{\sqrt{\frac{\pi}{2}}} + \frac{2}{n+1} \left(\left[\frac{x^{n+3}}{n+3} \cdot \sin(x^2) \right]_0^{\sqrt{\frac{\pi}{2}}} - \int_0^{\sqrt{\frac{\pi}{2}}} \frac{x^{n+3}}{n+3} \cdot 2x \cos(x^2) \, dx \right)$	
	$= (0-0) + \frac{2}{n+1} \left(\frac{1}{n+3} \left(\sqrt{\frac{\pi}{2}} \right)^{n+3} \sin \frac{\pi}{2} - 0 - \frac{2}{n+3} I_{n+4} \right)$	dM1
	$\Rightarrow I_{n+4} = \frac{1}{2} \left(\frac{\pi}{2} \right)^{\frac{n+3}{2}} - \frac{1}{4}(n+1)(n+3)I_n \text{ so replacing } n \text{ by } n-4 \text{ gives}$	A1*
	$I_n = \frac{1}{2} \left(\frac{\pi}{2} \right)^{\frac{n-1}{2}} - \frac{1}{4}(n-1)(n-3)I_{n-4}^*$	
		(6)
(b)	$I_1 = \int_0^{\sqrt{\frac{\pi}{2}}} x \cos(x^2) \, dx = \left[\frac{1}{2} \sin(x^2) \right]_0^{\sqrt{\frac{\pi}{2}}} = \frac{1}{2}$	B1
	$I_5 = \frac{1}{2} \left(\frac{\pi}{2} \right)^{\frac{5-1}{2}} - \frac{1}{4}(5-1)(5-3) \times \frac{1}{2}$	M1
	$= \frac{\pi^2}{8} - 1 \text{ oe e.g. } \frac{\pi^2 - 8}{8}, \frac{1}{2} \left(\frac{\pi}{2} \right)^2 - 1$	A1
		(3)



(9 marks)

Notes:

(a) Way 1

M1: Applies integration by parts in the correct direction having made the ‘split’ and obtains:

$$\left[\pm \alpha x^{n-1} \sin(x^2) \right] \pm \beta \int x^{n-2} \cdot \sin(x^2) \, dx$$

A1: Fully correct expression

 dM1: Applies integration by parts in the correct direction to $\beta \int x^{n-2} \cdot \sin(x^2) \, dx$ and obtains:

$$\left[\pm \alpha x^{n-3} \cos(x^2) \right] \pm \beta \int x^{n-4} \cos(x^2) \, dx$$

Depends on the previous M mark.

A1: Correct second application of parts e.g.

$$\int x^{n-2} \cdot \sin(x^2) \, dx = \left[x^{n-3} \cdot -\frac{1}{2} \cos(x^2) \right] - \int (n-3)x^{n-4} \cdot -\frac{1}{2} \cos(x^2) \, dx$$

 dM1: Applies the limits completely to their result and replaces final integral by I_{n-4} . The substitution of limits may have been carried out in stages throughout the work, or may be applied after integration by parts twice has been carried out. **Depends on both previous M marks.**

There must some explicit evidence that the limits have been applied but this may be taken

 from either the $\left[x^{n-1} \cdot \frac{1}{2} \sin(x^2) \right]_0^{\sqrt{\frac{\pi}{2}}} = \text{e.g. } \sqrt{\frac{\pi}{2}}^{n-1} \cdot \frac{1}{2} \sin\left(\sqrt{\frac{\pi}{2}}^2\right), \sqrt{\frac{\pi}{2}}^{n-1} \cdot \frac{1}{2}, \frac{1}{2} \left(\frac{\pi}{2}\right)^{\frac{n-1}{2}} - 0$

 or $\left[x^{n-3} \cdot -\frac{1}{2} \cos(x^2) \right]_0^{\sqrt{\frac{\pi}{2}}} = \text{e.g. } 0 - 0, 0$

A1*: Achieves the printed answer from completely correct work with no errors seen and evidence of the given limits being applied.

Way 2

M1: Applies integration by parts in the correct direction and obtains:

$$\left[\pm \alpha x^{n+1} \cos(x^2) \right] \pm \beta \int x^{n+1} \cdot x \sin(x^2) \, dx$$

A1: Fully correct expression

 dM1: Applies integration by parts in the correct direction to $\beta \int x^{n+1} \cdot x \sin(x^2) \, dx$ and obtains:

$$\left[\pm \alpha x^{n+3} \sin(x^2) \right] \pm \beta \int x^{n+3} \cdot x \cos(x^2) \, dx$$

Depends on the previous M mark.

A1: Correct second application of parts e.g.

$$\int x^{n+2} \cdot \sin(x^2) \, dx = \left[\frac{x^{n+3}}{n+3} \cdot \sin(x^2) \right] - \int \frac{x^{n+3}}{n+3} \cdot 2x \cos(x^2) \, dx$$



dM1: Applies the limits completely to their result and replaces final integral by I_{n+4} . The substitution of limits may have been carried out in stages throughout the work, or may be applied after integration by parts twice has been carried out. **Depends on both previous M marks.**

There must some explicit evidence that the limits have been applied but this may be taken

from either the $\left[\frac{x^{n+1}}{n+1} \cdot \cos(x^2) \right]_0^{\sqrt{\frac{\pi}{2}}} = \text{e.g. } 0 - 0, 0$ or

$$\left[\frac{x^{n+3}}{n+3} \cdot \sin(x^2) \right]_0^{\sqrt{\frac{\pi}{2}}} = \text{e.g. } \frac{\sqrt{\frac{\pi}{2}}^{n+3}}{n+3} \cdot \sin\left(\sqrt{\frac{\pi}{2}}^2\right), \frac{\sqrt{\frac{\pi}{2}}^{n+3}}{n+3} \cdot \sin\left(\frac{\pi}{2}\right), \frac{\sqrt{\frac{\pi}{2}}^{n+3}}{n+3} \cdot (1), \frac{\left(\frac{\pi}{2}\right)^{\frac{n+3}{2}}}{n+3} - 0$$

A1*: Achieves the printed answer from completely correct work with no errors and evidence of the given limits being applied with a clear statement that n is replaced by $n - 4$

(b)

B1: Correct I_1 . May be seen after attempting the reduction.

M1: Applies the reduction formula with their I_1 and $n = 5$ to reach a value. Condone slips with evaluating $\frac{1}{4}(n-1)(n-3)$ as long as the intention is clear.

A1: Correct answer.

Note: Beware incorrect work in (a) leading to what appears to be a correct form e.g.

$$I_n = \int_0^{\sqrt{\frac{\pi}{2}}} x^n \cos(x^2) \, dx = \left[x^n \cdot \frac{\sin(x^2)}{2x} \right]_0^{\sqrt{\frac{\pi}{2}}} - \int_0^{\sqrt{\frac{\pi}{2}}} nx^{n-1} \cdot \frac{\sin(x^2)}{2x} \, dx$$

This scores M0 at the start and hence will usually score no marks in part (a)

Question Number	Scheme	Notes	Marks
1	$7 \cosh x + 3 \sinh x = 2e^x + 7 \Rightarrow$ $7 \left(\frac{e^x + e^{-x}}{2} \right) + 3 \left(\frac{e^x - e^{-x}}{2} \right) = 2e^x + 7$ $\left\{ \frac{7}{2}e^x + \frac{7}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} = 2e^x + 7 \right\}$	Substitutes at least one correct exponential form for either of the hyperbolic terms and achieves an equation in exponentials and constants alone	M1
	$\Rightarrow 7(e^{2x} + 1) + 3(e^{2x} - 1) = 4e^{2x} + 14e^x$ $\left\{ \Rightarrow 5e^{2x} + 2 = 2e^{2x} + 7e^x \right\}$	Multiplies through by e^x to obtain any equation that would form a 3TQ in e^x if like terms were collected	M1
	$\Rightarrow 6e^{2x} - 14e^x + 4 = 0 \quad \{3e^{2x} - 7e^x + 2 = 0\}$	A correct three term quadratic in e^x . Could be implied by a correct root even if terms have not been collected.	A1
	$\Rightarrow (3e^x - 1)(e^x - 2) = 0 \Rightarrow e^x = \dots$	Solves their 3TQ - usual rules. One correct root for their quadratic if no working. Ignore labelling of the roots even if e.g., “ x ” is used.	M1
	$x = \ln 2, \ln \frac{1}{3}$	Both correct and simplified but do not isw if there are other answers . Allow $-\ln \frac{1}{2}$ for $\ln 2$ and $-\ln 3$ or $\ln 3^{-1}$ for $\ln \frac{1}{3}$	A1
	Answer only is 0/5		Total 5
	Note that it is possible to multiply through by e^{-x} to form an equation in e^{-2x} , e^{-x} and constants. Score as main scheme, e.g., $\frac{7}{2}e^x + \frac{7}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} = 2e^x + 7$ $\Rightarrow \frac{7}{2} + \frac{7}{2}e^{-2x} + \frac{3}{2} - \frac{3}{2}e^{-2x} = 2 + 7e^{-x} \quad (M1)$ $\Rightarrow 2e^{-2x} - 7e^{-x} + 3 = 0 \quad (A1)$ $(2e^{-x} - 1)(e^{-x} - 3) = 0 \Rightarrow e^{-x} = \frac{1}{2}, 3 \quad (M1)$ $\Rightarrow e^x = 2, \frac{1}{3} \Rightarrow x = \ln 2, \ln \frac{1}{3} \quad (A1)$		

Question Number	Scheme	Notes	Marks
5(a)	$4x^2 + 4x + 17 = 4\left(x^2 + x + \frac{17}{4}\right) = 4\left[\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{17}{4}\right] = (2x+1)^2 + 16$ or $4x^2 + 4x + 17 = 4x^2 + 4px + q \Rightarrow 4px = 4x \Rightarrow p = 1, q + p^2 = 17 \Rightarrow q = 16$ B1: Either p or q correct B1: Both correct values in part (a). Allow from any/no work. Values may be embedded within expression $(2x + p)^2 + q$.		B1 B1
(2)			
(b)	$A = 8, B = 4$	Both correct values (accept if embedded)	B1
(1)			
(c)	Note that this is a Hence question and there is no credit for work on the original fraction		
	$\int \frac{8x+5}{\sqrt{4x^2+4x+17}} dx = \int \frac{1}{\sqrt{(2x+1)^2+16}} dx + \int \frac{8x+4}{\sqrt{4x^2+4x+17}} dx$ $= \frac{1}{2} \operatorname{arsinh}\left(\frac{2x+1}{4}\right) + 2(4x^2+4x+17)^{\frac{1}{2}}$ or $\frac{1}{2} \ln\left(\frac{2x+1}{4} + \sqrt{\left(\frac{2x+1}{4}\right)^2 + 1}\right) + 2(4x^2+4x+17)^{\frac{1}{2}}$ or $\frac{1}{2} \ln\left(2x+1 + \sqrt{(2x+1)^2+16}\right) + 2((2x+1)^2+16)^{\frac{1}{2}}$	M1: For ...$\operatorname{arsinh}(f(x))$ $f(x) \neq k$ or logarithmic equivalent i.e., ...$\ln\left(f(x) + \sqrt{(f(x))^2 + c}\right) \quad c \neq 0$ M1: For ...$(4x^2+4x+17)^{\frac{1}{2}}$ or ...$((2x+1)^2+16)^{\frac{1}{2}}$ A1: Fully correct integration	M1 M1 A1
	Allow for equivalents in e.g., u if substitutions are used e.g., $u = 2x+1 \Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u^2+16}} du \Rightarrow \frac{1}{2} \operatorname{arsinh}\left(\frac{u}{4}\right) \quad u = 4x^2+4x+17 \Rightarrow \int \frac{1}{\sqrt{u}} du \Rightarrow 2\sqrt{u}$ $4 \sinh u = 2x+1 \Rightarrow \int \frac{2 \cosh u}{\sqrt{16 \cosh^2 u}} du \Rightarrow \frac{1}{2} \int du = \frac{1}{2} u$ Score the M marks for appropriate forms (sign/coefficient errors only). If they continue working in terms of u the limits applied for the ddM1 must be correct for their substitution which for the above examples would be 3 & $\frac{5}{3}$, 25 & $\frac{169}{9}$ and $\operatorname{arsinh}\left(\frac{3}{4}\right)$ & $\operatorname{arsinh}\left(\frac{5}{12}\right)$		
	$\int_{\frac{1}{3}}^1 \frac{8x+5}{\sqrt{4x^2+4x+17}} dx = \frac{1}{2} \operatorname{arsinh}\left(\frac{3}{4}\right) - \frac{1}{2} \operatorname{arsinh}\left(\frac{5}{12}\right) + 2\sqrt{25} - 2\sqrt{\frac{169}{9}}$ $\Rightarrow \frac{1}{2} \ln\left(\frac{3}{4} + \sqrt{\left(\frac{3}{4}\right)^2 + 1}\right) - \frac{1}{2} \ln\left(\frac{5}{12} + \sqrt{\left(\frac{5}{12}\right)^2 + 1}\right) + 2\sqrt{25} - \frac{26}{3}$ Condone replacement of $\operatorname{arsinh}\left(\frac{x}{a}\right)$ with $\ln\left(x + \sqrt{x^2 + a^2}\right)$ with $a \neq 1$ instead of using $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$	Substitutes and subtracts with the given limits and uses the appropriate form for arsinh twice (if required). Results from separate integrals must be combined. Allow slips but the $f\left(\frac{1}{3}\right)$ terms (and no others) must be subtracted. Not implied by just the final answer. Requires both previous M marks.	ddM1
	$\operatorname{arsinh}(\dots)$ may be evaluated using correct exp definition & solving a exponential 3TQ		
	$\left\{ = \frac{1}{2} \ln 2 - \frac{1}{2} \ln \frac{3}{2} + 10 - \frac{26}{3} \right\} = \frac{4}{3} + \frac{1}{2} \ln \frac{4}{3}$ Correct answer in correct form. May be no further work following substitution but there must be nothing incorrect. Allow $k = \frac{4}{3}$ if $k + \frac{1}{2} \ln k$ is seen. Allow $\frac{1}{2} \ln \frac{4}{3} + \frac{4}{3}$		A1
	Algebraic integration must be used. Answer or 1.47717... only scores no marks		
(5)			
Total 8			

Question Number	Scheme	Marks
Accept vectors in i, j, k form or column vector form throughout		
3(a)	$\lambda = (1 \times 1) + (-1 \times 2) + (4 \times 1) = \dots$	M1
	$\lambda = 3$	A1
		(2)
(b)	$\begin{pmatrix} 1 & -1 & 4 \\ 3 & a & b \\ a & 1 & b \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \text{"}\lambda\text{"} \\ \text{"}2\lambda\text{"} \\ \text{"}\lambda\text{"} \end{pmatrix} \Rightarrow \begin{matrix} 3+2a+b = \text{"}2\lambda\text{"} \\ a+2+b = \text{"}\lambda\text{"} \end{matrix} \Rightarrow a = \dots, b = \dots$	M1
	$a = 2, b = -1$	A1
		(2)
(c)(i)	$ \mathbf{M} - \lambda \mathbf{I} = \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & b \\ a & 1 & -1-\lambda \end{vmatrix}$ $= (1-\lambda)[(\lambda-2)(\lambda+1)+1] + 3(-1-\lambda) + 2 + 4[3-2(2-\lambda)]$ (via first row) $= (1-\lambda)[(\lambda-2)(\lambda+1)+1] - 3[(1+\lambda)-4] + 2[1-4(2-\lambda)]$ (via first column) $= (1-\lambda)(2-\lambda)(-1-\lambda) + 2 + 12 - 8(2-\lambda) + (1-\lambda) + 3(-1-\lambda)$ (Sarrus)	M1
	$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda-3)(\lambda^2 + \lambda - 2) = 0 \Rightarrow \lambda = \dots$	dM1
	$\lambda = 1, -2, (3)$	A1
		(3)
(ii)	$\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} x - y + 4z = x \\ 3x + 2y - z = y \\ 2x + y - z = z \end{matrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dots \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} x - y + 4z = -2x \\ 3x + 2y - z = -2y \\ 2x + y - z = -2z \end{matrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dots \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$	M1
	$\lambda = 1 \rightarrow \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \text{ or } \lambda = -2 \rightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$	A1
	$\lambda = 1 \rightarrow \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \text{ and } \lambda = -2 \rightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$	A1
		(3)
		Total 10

Notes

(a)

M1: Correct strategy to obtain the eigenvalue, multiplying the first row of matrix M with the given eigenvector, equating to λ and solving. Condone one slip for this mark but need to see some attempt at multiplying elements together.

A1: Correct eigenvalue obtained. Answer of 3 with no working scores both marks.

(b)

M1: Uses their eigenvalue from part (a) to form and solve 2 equations in a and b (must obtain a value for both)

A1: Correct values

(c)(i)

M1: Attempts to expand $\det(M - \lambda I)$. This may be along any row or down any column, or via a “shoelace” approach (rule of Sarrus). Condone sign slips, but the overall structure should be correct. Working must be shown- do not accept solutions that just state the factorised form. The expansion must be seen in some form. Accept alternative approaches. Allow this mark to be scored if the attempt is seen under other parts of the question, or if a and b are not yet found.

Note: Attempts at using $Mx = \lambda x$ to obtain simultaneous equations are unlikely to make much progress

dM1: Sets the characteristic polynomial (which must be a four-term cubic) equal to zero (which may be implied) and solves via any valid method, including calculator, to find two other eigenvalues (not 3). This is a dependent mark, so as a minimum an attempt at the expansion must have been seen, but the cubic may not be seen fully simplified. If no working is shown, then their values must be correct for their unfactorised cubic (or apply general rules for a 3TQ if they factorise first)

A1: Correct remaining eigenvalues obtained

(ii)

M1: Applies $Mx = \lambda x$ with one of their calculated eigenvalues, however found (not using 3), forms simultaneous equations and proceeds to solve to find one eigenvector, or finds the cross product of any two rows of $M - \lambda I$ with their eigenvalues to obtain one eigenvector. Condone $x = \dots$, $y = \dots$, $z = \dots$ for this mark.

A1: One correct eigenvector. Allow any multiple of the correct vector. Must be a vector.

A1: Both correct eigenvectors. Allow any multiples of the correct vectors. Must be a vector.

Question Number	Scheme	Marks
4.	$y = \operatorname{arsinh} x + \operatorname{arsinh} \left(\frac{1}{x} \right) \quad x > 0$	
(a)	$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+\frac{1}{x^2}}} \times -x^{-2}$	M1
	$= \frac{1}{\sqrt{1+x^2}} - \frac{1}{x^2 \sqrt{1+\frac{1}{x^2}}} = \frac{1}{\sqrt{1+x^2}} - \frac{1}{x \sqrt{1+x^2}} = \frac{x-1}{x \sqrt{1+x^2}} *$ or $= \frac{1}{\sqrt{1+x^2}} - \frac{x^{-2}}{\sqrt{1+\frac{1}{x^2}}} = \frac{\sqrt{1+\frac{1}{x^2}} - x^{-2} \sqrt{1+x^2}}{\sqrt{1+x^2} \sqrt{1+\frac{1}{x^2}}} = \frac{\sqrt{1+x^2} - \frac{1}{x} \sqrt{1+x^2}}{1+x^2} = \frac{1-\frac{1}{x}}{\sqrt{x^2+1}} = \frac{x-1}{x \sqrt{1+x^2}} *$	A1*
		(2)
(b)	$\frac{x-1}{x \sqrt{1+x^2}} = 0 \Rightarrow x=1$	B1
	$y = \operatorname{arsinh} 1 + \operatorname{arsinh} \left(\frac{1}{1} \right) = 2 \operatorname{arsinh} 1 = 2 \ln(1 + \sqrt{1^2 + 1}) \text{ o.e.}$	M1
	$y = \ln(3 + 2\sqrt{2}) \text{ or } \ln(1 + \sqrt{2})^2 \text{ or } 2 \ln(1 + \sqrt{2})$	A1
		(3)
Alt for M1A1	$y = 2 \operatorname{arsinh} 1 \Rightarrow \sinh \frac{y}{2} = 1 \Rightarrow \frac{e^{\frac{y}{2}} - e^{-\frac{y}{2}}}{2} = 1 \Rightarrow e^y - 2e^{\frac{y}{2}} - 1 = 0$ $\Rightarrow e^{\frac{y}{2}} = 1 \pm \sqrt{2} \Rightarrow \frac{y}{2} = \dots$	M1
	$y = \ln(3 + 2\sqrt{2}) \text{ or } \ln(1 + \sqrt{2})^2 \text{ or } 2 \ln(1 + \sqrt{2})$	A1
		Total 5

Notes

(a)

M1: Differentiates to the form $\frac{1}{\sqrt{1+x^2}} + \frac{kx^{-2}}{\sqrt{1+\frac{1}{x^2}}}$.

Note: There may be approaches involving e.g. $y = \operatorname{arsinh} x \Rightarrow x = \sinh y \Rightarrow \frac{dx}{dy} = \dots \Rightarrow \frac{dy}{dx} = \dots$ for one or both parts of the expression for y , but the attempt must reach an expression for $\frac{dy}{dx}$ of the required form before the

M mark can be awarded.

A1*: Correct proof with no errors and sufficient working shown. Must have at least the underlined steps shown above for the approach taken to score this mark.

(b)

B1: Deduces $x = 1$ when $\frac{dy}{dx} = 0$

M1: Substitutes their value of x into the given equation and applies the logarithmic form of arsinh

A1: Correct value or equivalent e.g. $\ln(1+\sqrt{2})^2$, $2\ln(1+\sqrt{2})$ but do not condone poor bracketing here.

ALT for final two marks

M1: Substitutes their value of x into the given equation then uses the exponential form for $\sinh$ to form and solve a quadratic in $e^{\frac{y}{2}}$ (usual rules) reaching as far as a value for $\frac{y}{2}$

A1: Correct value or equivalent e.g. $\ln(1+\sqrt{2})^2$, $2\ln(1+\sqrt{2})$ but do not condone poor bracketing here. ISW once a correct answer of the correct form is seen.

Question Number	Scheme	Marks
5(a)	$\int_1^6 x^n (3x-2)^{-\frac{1}{2}} dx = \left[\frac{2}{3} x^n (3x-2)^{\frac{1}{2}} \right]_{(1)}^{(6)} - \frac{2}{3} \int_{(1)}^{(6)} nx^{n-1} (3x-2)^{\frac{1}{2}} dx$	M1A1
	$= \dots - \frac{2}{3} \int_{(1)}^{(6)} nx^{n-1} (3x-2)(3x-2)^{-\frac{1}{2}} dx$	dM1
	$= \frac{2}{3} \times 6^n \times 4 - \frac{2}{3} \times 1 - \frac{2}{3} n \times 3I_n + \frac{4}{3} nI_{n-1}$ $\left(I_n = \frac{8}{3} \times 6^n - \frac{2}{3} - 2nI_n + \frac{4}{3} nI_{n-1} \right)$	ddM1
	$3I_n = 8 \times 6^n - 2 - 6nI_n + 4nI_{n-1}$ $\Rightarrow (3+6n)I_n = 4nI_{n-1} + 8 \times 6^n - 2^*$	A1*
		(5)
(b)	$I_0 = \int_1^6 (3x-2)^{-\frac{1}{2}} dx = \left[\frac{2}{3} (3x-2)^{\frac{1}{2}} \right]_1^6 = \frac{8}{3} - \frac{2}{3} = 2$	B1
	$21I_3 = 12I_2 + 1728 - 2 \left(\text{Implied by } 21I_3 = \frac{9966}{5} \text{ or } I_3 = \frac{3322}{35} \right)$ Or $15I_2 = 8I_1 + 288 - 2 \left(\text{Implied by } 15I_2 = 334 \text{ or } I_2 = \frac{334}{15} \right)$ Or $9I_1 = 4I_0 + 48 - 2 \left(\text{Implied by } 9I_1 = 54 \right)$	M1
	$21I_3 = 12 \left(\frac{8}{15} I_1 + \frac{286}{15} \right) + 1728 - 2 = 12 \left(\frac{8}{15} \left(\frac{4}{9} I_0 + \frac{46}{9} \right) + \frac{286}{15} \right) + 1726 = \dots \left(\frac{9966}{5} \right)$ or $= \frac{12}{15} (8I_1 + 288 - 2) + 1726 = \frac{32}{45} (4I_0 + 48 - 2) + \frac{9774}{5} = \dots \left(\frac{9966}{5} \right)$	M1
	$I_3 = \frac{3322}{35}$	A1
		(4)
		Total 9

Question Number	Scheme	Notes	Marks
	Mark (a) and (b) together but do not credit work for (a) that is seen in (c)		
5(a)	$\begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8x \\ 8y \\ 8z \end{pmatrix} \text{ or } \begin{pmatrix} -2 & -2 & -1 \\ -2 & -2 & -1 \\ -1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dots$ Correct method for obtaining the eigenvector		M1
	i – j	Any multiple of this vector	A1
			(2)
(b)	$ \mathbf{M} - \lambda \mathbf{I} = \begin{vmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix}$ $\Rightarrow \underline{(6-\lambda)}((6-\lambda)(5-\lambda)-1) + \underline{2(2(\lambda-5)-1)} - \underline{1(2+6-\lambda)}$ Correct attempt at the determinant of $\mathbf{M} - \lambda \mathbf{I}$. The terms with single underlining should be correct with correct signs but allow minor slips in the brackets with double underlining. Note that the rule of Sarrus gives $(6-\lambda)(6-\lambda)(5-\lambda) - 2 - 2 - (6-\lambda) - (6-\lambda) - 4(5-\lambda)$		M1
	$\Rightarrow \lambda^3 - 17\lambda^2 + 90\lambda - 144 = 0 \Rightarrow \lambda = \dots$	Solves $\mathbf{M} - \lambda \mathbf{I} = 0$ to obtain 2 different distinct real eigenvalues excluding 8	M1
	$\Rightarrow \lambda = 3, 6, (8)$	For 3 and 6	A1
			(3)



(c)	$(\mathbf{D} =) \begin{pmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$	Correct D with distinct non-zero eigenvalues in any order. Follow through their non-zero 3 and 6. Ignore labelling and score for sight of the correct or correct ft matrix.	B1ft
	$\begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dots \quad \text{NB } \mathbf{v}_2 = k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dots \quad \text{NB } \mathbf{v}_3 = k \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ Attempts eigenvectors for their other 2 distinct eigenvalues not including 8 May use e.g. $(\mathbf{M} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$		M1
	$(\mathbf{P} =) \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{pmatrix}$ Forms a complete P from normalised eigenvectors using their eigenvector from part (a) and their other 2 eigenvectors formed from their other 2 different distinct eigenvalues in any order. Ignore labelling and score for forming this matrix which may be seen as part of a calculation.		M1
	$\mathbf{D} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \text{ and } \mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{pmatrix}$ All fully correct and consistent and correctly labelled but the labelling may be implied by their working.		A1
			(4)
			Total 9

Question Number	Scheme	Notes	Marks
6(a) Way 1	$\int \frac{x^n}{\sqrt{x^2+3}} dx = \int x^{n-1} x (x^2+3)^{-\frac{1}{2}} dx \text{ or } \int \frac{x^n}{\sqrt{x^2+3}} dx = \int x^{n-1} d(x^2+3)^{\frac{1}{2}}$ Applies $x^n = x^{n-1} \times x$ to $\int \frac{x^n}{\sqrt{x^2+3}} dx$ but may be implied by subsequent work		M1
	$\int x^{n-1} x (x^2+3)^{-\frac{1}{2}} dx = x^{n-1} (x^2+3)^{\frac{1}{2}} - \int (n-1) x^{n-2} (x^2+3)^{\frac{1}{2}} dx$ dM1: Applies integration by parts to obtain $\alpha x^{n-1} (x^2+3)^{\frac{1}{2}} - \beta \int x^{n-2} (x^2+3)^{\frac{1}{2}} dx$ (NB α, β may be functions of n) Note that if a correct formula for parts is quoted first and parts is applied in the correct direction then we can condone slips in signs as long as the expression is of the above form. If you are unsure – send to review. A1: Correct expression		dM1A1
	$= x^{n-1} (x^2+3)^{\frac{1}{2}} - \int (n-1) x^{n-2} (x^2+3) (x^2+3)^{-\frac{1}{2}} dx$ Applies $(x^2+3)^{\frac{1}{2}} = (x^2+3)(x^2+3)^{-\frac{1}{2}}$ having made an attempt at integration by parts in the correct direction		M1
	$= x^{n-1} (x^2+3)^{\frac{1}{2}} - (n-1) \int x^n (x^2+3)^{-\frac{1}{2}} dx - 3(n-1) \int x^{n-2} (x^2+3)^{-\frac{1}{2}} dx$ $= x^{n-1} (x^2+3)^{\frac{1}{2}} - (n-1) I_n - 3(n-1) I_{n-2}$ Splits into 2 integrals involving I_n and I_{n-2} Depends on all the previous method marks		dM1
	$\Rightarrow I_n = \frac{x^{n-1}}{n} (x^2+3)^{\frac{1}{2}} - \frac{3(n-1)}{n} I_{n-2} *$ Obtains the printed answer. You can condone the odd missing “dx” but if there are any clear errors e.g. invisible brackets that are not recovered, sign errors etc. then this mark should be withheld.		A1*
			(6)

6(a) Way 2	$\int \frac{x^n}{\sqrt{x^2+3}} dx = \int x^{n-2} x^2 (x^2+3)^{-\frac{1}{2}} dx$ Applies $x^n = x^{n-2} \times x^2$	M1
	$\int x^{n-2} x^2 (x^2+3)^{-\frac{1}{2}} dx = \int x^{n-2} (x^2+3-3) (x^2+3)^{-\frac{1}{2}} dx$ $= \int x^{n-2} (x^2+3)^{\frac{1}{2}} dx - \int 3x^{n-2} (x^2+3)^{-\frac{1}{2}} dx$ dM1: Writes x^2 as (x^2+3-3) to obtain $\alpha \int x^{n-2} (x^2+3)^{\frac{1}{2}} dx - \beta \int x^{n-2} (x^2+3)^{-\frac{1}{2}} dx$ A1: Correct expression	dM1A1
	$\int x^{n-2} (x^2+3)^{\frac{1}{2}} dx = \frac{x^{n-1}}{n-1} (x^2+3)^{\frac{1}{2}} - \frac{1}{n-1} \int x^n (x^2+3)^{-\frac{1}{2}} dx$ Applies integration by parts on $\int x^{n-2} (x^2+3)^{\frac{1}{2}} dx$ to obtain $\alpha x^{n-1} (x^2+3)^{\frac{1}{2}} - \beta \int x^n (x^2+3)^{-\frac{1}{2}} dx$ Note that if a correct formula for parts is quoted first and parts is applied in the correct direction then we can condone slips in signs as long as the expression is of the above form. If you are unsure – send to review.	M1
	$I_n = \frac{x^{n-1}}{n-1} (x^2+3)^{\frac{1}{2}} - \frac{1}{n-1} I_n - 3I_{n-2}$ Brings all together and introduces I_n and I_{n-2} Depends on all the previous method marks	dM1
	$\Rightarrow I_n = \frac{x^{n-1}}{n} (x^2+3)^{\frac{1}{2}} - \frac{3(n-1)}{n} I_{n-2} *$ Obtains the printed answer. You can condone the odd missing “dx” but if there are any clear errors e.g. invisible brackets that are not recovered, sign errors etc. then this mark should be withheld.	A1*

(b) Way 1	$I_5 = \frac{x^4}{5}(x^2 + 3)^{\frac{1}{2}} - \frac{12}{5}I_3$ Applies the reduction formula once to obtain I_5 in terms of I_3 Allow slips on coefficients only	M1
	$I_5 = \frac{x^4}{5}(x^2 + 3)^{\frac{1}{2}} - \frac{12}{5}\left(\frac{x^2}{3}(x^2 + 3)^{\frac{1}{2}} - \frac{6}{3}I_1\right)$ Applies the reduction formula again to obtain an expression for I_5 in terms of I_1 and allow "I_1" or what they think is I_1 Allow slips on coefficients only	M1
	E.g. $I_5 = \frac{x^4}{5}(x^2 + 3)^{\frac{1}{2}} - \frac{12}{5}\left(\frac{x^2}{3}(x^2 + 3)^{\frac{1}{2}} - \frac{6}{3}(x^2 + 3)^{\frac{1}{2}}\right)$ Or e.g. $I_5 = \frac{x^4}{5}(x^2 + 3)^{\frac{1}{2}} - \frac{4}{5}x^2(x^2 + 3)^{\frac{1}{2}} + \frac{24}{5}(x^2 + 3)^{\frac{1}{2}}$ Any correct expression in terms of x only	A1
	$I_5 = \frac{1}{5}(x^2 + 3)^{\frac{1}{2}}(x^4 - 4x^2 + 24) + k$ Must include the "+ k" but allow other letter e.g. + c	A1
		(4)
		Total 10
(b) Way 2	NB $I_1 = (x^2 + 3)^{\frac{1}{2}}$	
	$I_3 = \frac{x^2}{3}(x^2 + 3)^{\frac{1}{2}} - \frac{6}{3}I_1$ Applies the reduction formula once to obtain I_3 in terms of I_1 and allow "I_1" or what they think is I_1 Allow slips on coefficients only	M1
	$I_5 = \frac{x^4}{5}(x^2 + 3)^{\frac{1}{2}} - \frac{12}{5}\left(\frac{x^2}{3}(x^2 + 3)^{\frac{1}{2}} - 2I_1\right)$ Applies the reduction formula again to obtain an expression for I_5 in terms of I_1 and allow "I_1" or what they think is I_1 Allow slips on coefficients only	M1
	E.g. $I_5 = \frac{x^4}{5}(x^2 + 3)^{\frac{1}{2}} - \frac{12}{5}\left(\frac{x^2}{3}(x^2 + 3)^{\frac{1}{2}} - \frac{6}{3}(x^2 + 3)^{\frac{1}{2}}\right)$ Or e.g. $I_5 = \frac{x^4}{5}(x^2 + 3)^{\frac{1}{2}} - \frac{4}{5}x^2(x^2 + 3)^{\frac{1}{2}} + \frac{24}{5}(x^2 + 3)^{\frac{1}{2}}$ Any correct expression in terms of x only	A1
	$I_5 = \frac{1}{5}(x^2 + 3)^{\frac{1}{2}}(x^4 - 4x^2 + 24) + k$ Must include the "+ k" but allow other letter e.g. + c	A1

Note that (b) is hence so must involve use of the reduction formula so a direct attempt at I_5 scores no marks.
 However some candidates may apply the reduction formula once as in Way 1 and then attempt I_3 directly, in which case all marks are available as the reduction formula has been used but there must be a credible attempt at I_3 to reach an expression of the required form.

See below for an example:

(b) Way 3	$I_5 = \frac{x^4}{5}(x^2+3)^{\frac{1}{2}} - \frac{12}{5}I_3$ Applies the reduction formula once to obtain I_5 in terms of I_3 Allow slips on coefficients only	M1
	$I_3 = \int \frac{x^3}{(x^2+3)^{\frac{1}{2}}} dx$ $u = x^2 + 3 \Rightarrow I_3 = \int \frac{(u-3)^{\frac{3}{2}}}{u^{\frac{1}{2}}} \frac{du}{2(u-3)^{\frac{1}{2}}} = \frac{1}{2} \int \frac{(u-3)}{u^{\frac{1}{2}}} du = \frac{1}{3}u^{\frac{3}{2}} - 6u^{\frac{1}{2}}$ $= \frac{1}{3}(x^2+3)^{\frac{3}{2}} - 6(x^2+3)^{\frac{1}{2}}$ $I_5 = \frac{x^4}{5}(x^2+3)^{\frac{1}{2}} - \frac{12}{5} \left(\frac{1}{3}(x^2+3)^{\frac{3}{2}} - 6(x^2+3)^{\frac{1}{2}} \right)$ M1: A credible attempt to find I_3 and then expresses I_5 in terms of x A1: Any correct expression in terms of x only	M1A1
	$I_5 = \frac{1}{5}(x^2+3)^{\frac{1}{2}}(x^4 - 4x^2 + 24) + k$ Must include the “+ k” but allow other letter e.g. + c	A1



Question Number	Scheme	Notes	Marks
3.	$\mathbf{M} = \begin{pmatrix} 3 & 1 & p \\ 1 & 1 & 2 \\ -1 & p & 2 \end{pmatrix}$		
(a)	$\det \mathbf{M} = \begin{vmatrix} 3 & 1 & p \\ 1 & 1 & 2 \\ -1 & p & 2 \end{vmatrix}$ $= 3(2 - 2p) - 1(2 + 2) + p(p + 1)$	Attempts determinant. Requires at least 2 correct “terms”. May use other rows/columns or rule of Sarrus.	M1
	$= p^2 - 5p + 2$	Correct simplified determinant.	A1
	$p^2 - 5p + 2 = 0 \Rightarrow p = \dots$	Solves 3TQ	M1
	$\frac{5 \pm \sqrt{17}}{2}$	Correct values.	A1
			(4)
(b)	Minors $\begin{pmatrix} 2-2p & 4 & p+1 \\ (2-p^2) & 6+p & (3p+1) \\ 2-p & (6-p) & 2 \end{pmatrix}$	Attempts the matrix of minors. If there is any doubt look for at least 6 correct elements. May be implied by their matrix of cofactors.	M1 (B1 on EPEN)
	Cofactors $\begin{pmatrix} 2-2p & -4 & p+1 \\ -(2-p^2) & 6+p & -(3p+1) \\ 2-p & -(6-p) & 2 \end{pmatrix}$	Attempts cofactors.	M1
		Correct matrix	A1
	$\mathbf{M}^{-1} = \frac{1}{p^2 - 5p + 2} \begin{pmatrix} 2-2p & p^2 - 2 & 2-p \\ -4 & 6+p & p-6 \\ p+1 & -3p-1 & 2 \end{pmatrix}$	Transposes matrix of cofactors and divides by determinant.	M1
		Follow though their det $\mathbf{M}$ from part (a) but the adjoint matrix must be correct.	A1ft
			(5)
			Total 9

Question Number	Scheme	Notes	Marks
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Question Number	Scheme	Notes	Marks
6(a)	$\int e^x \sin^n x \, dx = e^x \sin^n x - n \int e^x \sin^{n-1} x \cos x \, dx$ Applies integration by parts to obtain $\pm e^x \sin^n x \pm \alpha \int e^x \sin^{n-1} x \cos x \, dx$		M1
	$= e^x \sin^n x - n \left\{ e^x \sin^{n-1} x \cos x - \int e^x ((n-1) \sin^{n-2} x \cos^2 x - \sin^n x) \, dx \right\}$ M1: Applies integration by parts to $\pm \alpha \int e^x \sin^{n-1} x \cos x \, dx$ to obtain $\pm e^x \sin^{n-1} x \cos x \pm \int e^x (\alpha \sin^{n-2} x \cos^2 x - \beta \sin^n x) \, dx$ Or equivalent e.g. $\pm e^x \sin^{n-1} x \cos x \pm \int e^x (\alpha \sin^{n-2} x - \beta \sin^n x) \, dx$ (if Pythagoras applied first) A1: Fully correct expression for I_n from parts applied twice.		dM1A1
	$= e^x \sin^n x - n \left\{ e^x \sin^{n-1} x \cos x - \int e^x ((n-1) \sin^{n-2} x (1 - \sin^2 x) - \sin^n x) \, dx \right\}$ Applies $\cos^2 x = 1 - \sin^2 x$		dM1
	$= e^x \sin^n x - n \left\{ e^x \sin^{n-1} x \cos x - \int e^x ((n-1) \sin^{n-2} x - (n-1) \sin^n x - \sin^n x) \, dx \right\}$ $= e^x \sin^n x - n \left\{ e^x \sin^{n-1} x \cos x - \int e^x ((n-1) \sin^{n-2} x - n \sin^n x) \, dx \right\}$ $= e^x \sin^n x - n e^x \sin^{n-1} x \cos x + n(n-1) I_{n-2} - n^2 I_n \Rightarrow I_n = \dots$ Completes by introducing I_{n-2} and I_n and makes I_n the subject		dM1
	$I_n = \frac{e^x \sin^{n-1} x}{n^2 + 1} (\sin x - n \cos x) + \frac{n(n-1)}{n^2 + 1} I_{n-2} *$ Fully correct proof with no errors but allow e.g. the occasional missing “dx” but any clear errors must be recovered before final answer e.g. missing brackets.		A1*
			(6)

(b)	$I_4 = \frac{e^x \sin^3 x}{17} (\sin x - 4 \cos x) + \frac{12}{17} I_2$ or $I_2 = \frac{e^x \sin x}{5} (\sin x - 2 \cos x) + \frac{2}{5} I_0$ Applies the reduction formula once	M1
	$= \frac{e^x \sin^3 x}{17} (\sin x - 4 \cos x) + \frac{12}{17} \left(\frac{e^x \sin x}{5} (\sin x - 2 \cos x) + \frac{2}{5} I_0 \right)$ $= \frac{e^x \sin^3 x}{17} (\sin x - 4 \cos x) + \frac{12e^x \sin x}{85} (\sin x - 2 \cos x) + \frac{24}{85} e^x$ Applies the reduction formula again and uses $I_0 = \int e^x dx = e^x$ to obtain an expression in terms of x	M1
	$\int_0^{\frac{\pi}{2}} e^x \sin^4 x dx = \left[\frac{e^x \sin^3 x}{17} (\sin x - 4 \cos x) + \frac{12e^x \sin x}{85} (\sin x - 2 \cos x) + \frac{24}{85} e^x \right]_0^{\frac{\pi}{2}}$ $= \frac{e^{\frac{\pi}{2}}}{17} + \frac{12e^{\frac{\pi}{2}}}{85} + \frac{24e^{\frac{\pi}{2}}}{85} - \frac{24}{85}$ Uses the limits 0 and $\frac{\pi}{2}$ and subtracts. Depends on both previous marks.	dM1
	$= \frac{41e^{\frac{\pi}{2}}}{85} - \frac{24}{85}$ Correct expression or correct values e.g. $A = \dots, B = \dots$	A1
		(4)
	Note that the limits may be applied as they go e.g.: M1: $I_4 = \frac{e^{\frac{\pi}{2}}}{17} (1-0) + \frac{12}{17} I_2$ $I_2 = \frac{e^{\frac{\pi}{2}}}{5} (1-0) + \frac{2}{5} I_0$ $I_0 = e^{\frac{\pi}{2}} - 1$ M1M1: $I_4 = \frac{e^{\frac{\pi}{2}}}{17} + \frac{12}{17} \left(\frac{e^{\frac{\pi}{2}}}{5} + \frac{2}{5} (e^{\frac{\pi}{2}} - 1) \right)$ A1: $= \frac{41e^{\frac{\pi}{2}}}{85} - \frac{24}{85}$	
		Total 10

Question Number	Scheme	Notes	Marks
7.	$I_n = \int \frac{x^n}{\sqrt{10-x^2}} dx \quad n \in \mathbb{N}, x < \sqrt{10}$		
(a)	$I_n = \int \frac{x^n}{\sqrt{10-x^2}} dx = \int \frac{x^{n-1} \times x}{\sqrt{10-x^2}} dx$	Writes x^n as $x \times x^{n-1}$	M1
	$\int \frac{x^{n-1} \times x}{\sqrt{10-x^2}} dx = -x^{n-1} (10-x^2)^{\frac{1}{2}} + (n-1) \int x^{n-2} (10-x^2)^{\frac{1}{2}} dx$ dM1: Uses integration by parts to obtain $\int \frac{x^{n-1} \times x}{\sqrt{10-x^2}} dx = \alpha x^{n-1} (10-x^2)^{\frac{1}{2}} + \beta \int x^{n-2} (10-x^2)^{\frac{1}{2}} dx$ A1: Correct expression		dM1A1
	$= \dots + (n-1) \int x^{n-2} (10-x^2) (10-x^2)^{-\frac{1}{2}} dx$ $= \dots + 10(n-1) \int x^{n-2} (10-x^2)^{-\frac{1}{2}} dx - (n-1) \int x^n (10-x^2)^{-\frac{1}{2}} dx$ Applies $(10-x^2)^{\frac{1}{2}} = (10-x^2)(10-x^2)^{-\frac{1}{2}}$ and splits into 2 integrals		dM1
	$= \dots + 10(n-1)I_{n-2} - (n-1)I_n \Rightarrow nI_n$	Introduces I_{n-2} and I_n and makes progress to the given result	dM1
	$nI_n = 10(n-1)I_{n-2} - x^{n-1} (10-x^2)^{\frac{1}{2}} *$ Fully correct proof with no errors (recovery of missing brackets counts as an error) as does missing dx		A1*
			(6)
(b)	$I_1 = \int_0^1 \frac{x}{\sqrt{10-x^2}} dx = \left[-(10-x^2)^{\frac{1}{2}} \right]_0^1 = (-3 + \sqrt{10})$ Correct method for I_1 Limits can be substituted later		M1
	$5I_5 = 10 \times 4I_3 + \dots$	Applies the reduction formula at least once Allow with 3 or $\left[-x^4 (10-x^2)^{\frac{1}{2}} \right]_0^1$	M1
	$I_5 = 8I_3 - \frac{3}{5} = 8 \left(\frac{20}{3} I_1 - 1 \right) - \frac{3}{5} = \frac{160}{3} I_1 - \frac{43}{5}$ $I_5 = \frac{160}{3} (\sqrt{10} - 3) - \frac{43}{5}$ Completes the process using their I_1 to obtain a numerical value for I_5 Limits must now be substituted		M1
	$= \frac{1}{15} (800\sqrt{10} - 2529)$	Cao	A1
			(4)
			Total 10

Question Number	Scheme	Notes	Marks
6(b) cont.	$\left(= (2\pi)4a^2 \int_{(0)}^{(2\pi)} \left(\sin \frac{\theta}{2} - \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} \right) d\theta \right)$ $S = 8\pi a^2 \left[-2 \cos \frac{\theta}{2} + \frac{2}{3} \cos^3 \frac{\theta}{2} \right]_{(0)}^{(2\pi)}$ or e.g., $\pi a^2 \left[-16 \cos \frac{\theta}{2} + \frac{16}{3} \cos^3 \frac{\theta}{2} \right]_{(0)}^{(2\pi)}$	A correct expression for the surface area ignoring limits ft their numerical k , i.e., $S = 2k\pi a^2 \left[-2 \cos \frac{\theta}{2} + \frac{2}{3} \cos^3 \frac{\theta}{2} \right]_{(0)}^{(2\pi)}$ oe If they integrate in a piecemeal fashion, award this mark if they have a correct expression for their k when integration is completed – any partial evaluations must be correct for their k	A1ft
	$= 8\pi a^2 \left[\left(-2 \cos \frac{2\pi}{2} + \frac{2}{3} \cos^3 \frac{2\pi}{2} \right) - \left(-2 \cos 0 + \frac{2}{3} \cos^3 0 \right) \right]$	Substitutes correct limits and attempts to subtract either way round following a completed attempt at integration with a numerical k . Requires previous M marks and must have used 2π . Look for evidence of correct limit substitution and subtraction. There may be slips but insist on limits being applied on all integrations if they have been carried out separately. Algebraic results of integration must be seen	ddM1
	$= \frac{64}{3} \pi a^2$	Correct exact answer. Accept equivalent fractions.	A1
	All marks available regardless of how $k = 4$ was obtained		(5)
			Total 9
	Other integration methods: Allow the second M mark to be available before any attempt at integration is made. Otherwise the second M is only awarded if they complete integration without any loss of the required forms (i.e., sign and coefficient errors only and just sign errors only with any trig identities). The first A (ft) mark is for a fully correct expression ignoring limits for their k . The last two marks are the same as the main scheme. For information: Applying parts to $\int \sin \frac{\theta}{2} \cos \theta \, d\theta$ gives $\frac{2}{3} \left(\cos \frac{\theta}{2} + 2 \sin \frac{\theta}{2} \right)$ Using addition formulae: $\int \sin \frac{\theta}{2} \cos \theta \, d\theta = \frac{1}{2} \int \left(\sin \frac{3\theta}{2} - \sin \frac{\theta}{2} \right) d\theta = \frac{1}{2} \left(2 \cos \frac{\theta}{2} - \frac{2}{3} \cos \frac{3\theta}{2} \right)$		

Question Number	Scheme	Notes	Marks
7(a)	$\begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$	M1: Attempts vector product of two vectors in the plane. Unless there is a full clear method they must achieve two correct components A1: $\pm(8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k})$ or multiple	M1 A1
	Allow any vector notation throughout this question		(2)
(b)	l has direction vector $\pm(2\mathbf{j} + 2\mathbf{k})$	Correct direction for l	B1
	$(\cos \alpha \text{ or } \sin \theta =)$ $\frac{ (8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} + 2\mathbf{k}) }{ \sqrt{8^2 + 2^2 + 3^2} \sqrt{2^2 + 2^2} } = \frac{ (8)(0) + (-2)(2) + (-3)(2) }{ \sqrt{8^2 + 2^2 + 3^2} \sqrt{0^2 + 2^2 + 2^2} } \left(= \left \frac{-10}{\sqrt{77} \times \sqrt{8}} \right \text{ or } \left \frac{-5\sqrt{154}}{154} \right \right)$ M1: For the scalar product of their normal and direction vector divided by the product of the magnitudes of their vectors. The first expression above or is sufficient. There must have been a valid attempt at both vectors. Allow copying errors/slips if intention is clear. Modulus not required. A1ft: A correct ft numerical expression with scalar product calculated as shown by second expression or better. Allow a decimal correct to 2sf. Modulus not required. Ignore labelling. Actual decimal is 0.40291148... Implied by awrt 24 or 66 or 114 provided some work and nothing incorrect seen. Allow awrt 0.41, 1.16 or 1.99 if working in radians.		M1 A1ft
	Acute angle between l and P $= 90 - \alpha = 90 - 66.23968409...$ or $\theta = 23.76031591... \Rightarrow 24^\circ$ to the nearest degree	awrt 24 from correct work which could be minimal. Degrees symbol not required. Mark final answer.	A1
			(4)
	Note that a vector product could be used: M1: $\frac{ (8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) \times (2\mathbf{j} + 2\mathbf{k}) }{ \sqrt{8^2 + 2^2 + 3^2} \sqrt{2^2 + 2^2} }$ A1: $\frac{ \sqrt{2^2 + 16^2 + 16^2} }{ \sqrt{8^2 + 2^2 + 3^2} \sqrt{2^2 + 2^2} } \left(= \frac{2\sqrt{129}}{\sqrt{77}\sqrt{8}} = 0.9152389511... \right)$ The modulus of the numerator is required for any marks		
(c)	Way 1 Parallel planes $(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) = -5$ or $(6\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}) \cdot (8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) = 72$	M1: Finds a value for the scalar product of a position vector of a point in the plane or the given point and their normal. A1: -5 or 72 (or 5 or -72 if normal is in the opposite direction). May be seen as a distance e.g., $\frac{-5}{\sqrt{77}}$	M1 A1
		dM1: Having attempted both scalar products, obtains a numerical expression for the distance. Award for $\frac{\pm 5 \pm 72}{\sqrt{8^2 + 2^2 + 3^2}}$ Dependent on previous M mark. A1: Correct exact distance. Isw	dM1 A1
			(4)

Question Number	Scheme	Notes	Marks
7(c)			
Way 2	$(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) = -5$	M1: Finds a value for the scalar product of a position vector to a point the plane and their normal. A1: -5 (or 5 if normal is in the opposite direction)	M1 A1
Perp. distance formula	$\frac{8x - 2y - 3z + 5 = 0}{\sqrt{8^2 + 2^2 + 3^2}}$ Shortest distance is $\frac{ (8)(6) + (-2)(-3) + (-3)(-6) + 5 }{\sqrt{8^2 + 2^2 + 3^2}}$ $= \frac{77}{\sqrt{77}} \text{ or } \sqrt{77}$	dM1: Uses distance formula with their normal and plane equation to reach a numerical expression for the distance. Condone sign slip on their -5 and their d must not be zero. Dependent on previous M mark. A1: Correct exact distance. Isw	dM1 A1
			(4)
Way 3	Let Q be the point on the plane $(1, 2, 3)$ then $\overrightarrow{PQ} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - (6\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}) = -5\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}$	M1: Attempts vector from given point to a point on the plane A1: Correct vector $(\pm)$	M1 A1
Projection/resolving formula	Shortest distance is $ \overrightarrow{PQ} \cdot \mathbf{n} =$ $\frac{ (-5\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}) \cdot (8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) }{\sqrt{8^2 + 2^2 + 3^2}} = \dots$ $= \frac{77}{\sqrt{77}} \text{ or } \sqrt{77}$	dM1: Uses formula with their vectors to reach a numerical expression for the distance Dependent on previous M mark. A1: Correct exact distance. Isw	dM1 A1
			(4)
Way 4	Line through given point in direction of normal is $r = (6\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}) + \lambda(8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k})$ & meets plane " $8x - 2y - 3z + 5 = 0$ " when $8(6 + 8\lambda) - 2(-3 - 2\lambda) - 3(-6 - 3\lambda) + 5 = 0$ $\Rightarrow \lambda = -1$	M1: Uses line through given point in the direction of their normal and substitutes into their plane to find a value for λ . The d in their plane equation must not be zero A1: Correct value	M1 A1
Example of method involving the point where the line meets plane	$ (-1)(8\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) = \sqrt{8^2 + 2^2 + 3^2}$ Or point of intersection is $(6 - 8, -3 - (-2), -6 - (-3)) = (-2, -1, -3)$ and distance is $\sqrt{(6 - (-2))^2 + (-3 - (-1))^2 + (-6 - (-3))^2}$ $\Rightarrow \sqrt{77}$	dM1: Attempts $ \lambda \mathbf{n} $ or finds point on the plane and obtains numerical expression for distance between this point and the given point Dependent on previous M mark. A1: Correct exact distance. Isw	dM1 A1
			(4)
	Marks are scored through the way which is the best overall match for the attempt. Credit for work done in (b) is only available for part (c) if it is used in part (c).		
			Total 10

Question Number	Scheme	Notes	Marks
5(a) Way 1 From LHS	$(1 - \operatorname{sech}^2 x) = 1 - \left(\frac{2}{e^x + e^{-x}} \right)^2$	Replaces $\operatorname{sech} x$ with correct expression in terms of exponentials	B1
	$= \frac{(e^x + e^{-x})^2 - 4}{(e^x + e^{-x})^2} = \frac{e^{2x} + 2 + e^{-2x} - 4}{(e^x + e^{-x})^2}$	Expresses as a single fraction (or 2 fractions with the same denominator) and expands numerator	M1
	$= \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \tanh^2 x$	Fully correct proof	A1*
Way 2 Diff. of 2 squares	$1 - \operatorname{sech}^2 x = (1 + \operatorname{sech} x)(1 - \operatorname{sech} x) = \left(1 + \frac{2}{e^x + e^{-x}} \right) \left(1 - \frac{2}{e^x + e^{-x}} \right)$ Uses difference of two squares and replaces $\operatorname{sech} x$ with correct expression in terms of exponentials		B1
	$= \left(\frac{e^x + e^{-x} + 2}{e^x + e^{-x}} \right) \left(\frac{e^x + e^{-x} - 2}{e^x + e^{-x}} \right) = \frac{e^{2x} + 1 - 2e^x + 1 + e^{-2x} - 2e^{-x} + 2e^x + 2e^{-x} - 4}{(e^x + e^{-x})^2}$ Expresses as a single fraction and expands numerator		M1
	$= \frac{e^{2x} - 2 + e^{-2x}}{(e^x + e^{-x})^2} = \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \tanh^2 x$	Fully correct proof	A1*
Way 3 From RHS	$(\tanh^2 x) = \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2}$	Replaces $\tanh x$ with correct expression in terms of exponentials	B1
	$= \frac{e^{2x} - 2 + e^{-2x}}{(e^x + e^{-x})^2} = \frac{e^{2x} + 2 + e^{-2x}}{(e^x + e^{-x})^2} - \frac{4}{(e^x + e^{-x})^2}$ Expands numerator and splits into two fractions		M1
	$= \frac{(e^x + e^{-x})^2}{(e^x + e^{-x})^2} - \left(\frac{2}{e^x + e^{-x}} \right)^2 = 1 - \operatorname{sech}^2 x$	Fully correct proof	A1*
			(3)

Allow “meet in the middle” approaches as long as a conclusion is given e.g. lhs = rhs

Example:

$$rhs = \tanh^2 x = \frac{(e^{2x} - 1)^2}{(e^{2x} + 1)^2} \quad \text{or} \quad lhs = 1 - \operatorname{sech}^2 x = 1 - \left(\frac{2}{e^x + e^{-x}} \right)^2$$

B1: Replaces $\tanh x$ or $\operatorname{sech} x$ with a correct expression in terms of exponentials

$$\frac{(e^{2x} - 1)^2}{(e^{2x} + 1)^2} = \frac{e^{4x} - 2e^{2x} + 1}{e^{4x} + 2e^{2x} + 1} \quad \text{and} \quad 1 - \left(\frac{2}{e^x + e^{-x}} \right)^2 = \frac{(e^x + e^{-x})^2 - 4}{(e^x + e^{-x})^2} = \frac{e^{2x} - 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}}$$

M1: Makes progress by e.g. removing brackets on rhs and expressing lhs as a single fraction and expands numerator

$$\frac{e^{2x} - 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = \frac{e^{4x} - 2e^{2x} + 1}{e^{4x} + 2e^{2x} + 1} \Rightarrow 1 - \operatorname{sech}^2 x = \tanh^2 x$$

A1: Correct proof and (minimal) conclusion e.g. “= rhs” etc.



$$1 - \operatorname{sech}^2 x = 1 - \left(\frac{2}{e^x + e^{-x}} \right)^2 = \frac{e^{2x} + 2 + e^{-2x} - 4}{(e^x + e^{-x})^2} = \frac{e^{2x} + e^{-2x} - 2}{e^{2x} + e^{-2x} + 2} = \frac{\sinh^2 x}{\cosh^2 x} = \tanh^2 x$$

$$1 - \operatorname{sech}^2 x = 1 - \left(\frac{2}{e^x + e^{-x}} \right)^2 = \frac{e^{2x} + 2 + e^{-2x} - 4}{(e^x + e^{-x})^2} = \frac{e^{2x} + e^{-2x} - 2}{e^{2x} + e^{-2x} + 2} = \tanh^2 x$$

Both score B1M1A0 as we would need to see numerators and denominators factorised.

Note that we will allow an equivalent identity to be proved by exponentials and the given identity deduced e.g.

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2$$

B1: Correct exponential form seen for cosh or sinh used

$$= \frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4} - \frac{e^{2x}}{4} + \frac{1}{2} - \frac{e^{-2x}}{4} = 1$$

M1: Expands and collects terms

$$\Rightarrow \cosh^2 x - \sinh^2 x = 1 \Rightarrow 1 - \operatorname{sech}^2 x = \tanh^2 x$$

A1*: Fully correct work leading to the correct identity

(b)	$\int \tanh^n 3x \, dx = \int \tanh^{n-2} 3x \tanh^2 3x \, dx$ $= \int \tanh^{n-2} 3x (1 - \operatorname{sech}^2 3x) \, dx$	Splits $\tanh^n 3x$ into $\tanh^{n-2} 3x \tanh^2 3x$ and applies $\tanh^2 3x = 1 - \operatorname{sech}^2 3x$	M1
	Do not condone $\int \tanh^n 3x \, dx = \int \tanh^{n-2} 3x \tanh^2 3x \, dx$ unless it is clear that $3x$ was intended and is therefore recovered in subsequent work. $= \int \tanh^{n-2} 3x (1 - \operatorname{sech}^2 x) \, dx$		
	$= \int \tanh^{n-2} 3x \, dx - \int \tanh^{n-2} 3x \operatorname{sech}^2 3x \, dx$ $\int \tanh^{n-2} 3x \operatorname{sech}^2 3x \, dx = \frac{1}{3(n-1)} \tanh^{n-1} 3x$ Expands and integrates $\tanh^{n-2} 3x \operatorname{sech}^2 3x$ to obtain $\alpha \tanh^{n-1} 3x$ Or it is possible to use parts for $\int \tanh^{n-2} 3x \operatorname{sech}^2 3x \, dx$: $\int \tanh^{n-2} 3x \operatorname{sech}^2 3x \, dx = \frac{1}{3} \tanh 3x \tanh^{n-2} 3x - \frac{1}{3} \int 3(n-2) \tanh 3x \tanh^{n-3} 3x \operatorname{sech}^2 3x \, dx$ $= \frac{1}{3} \tanh^{n-1} 3x - (n-2) \int \tanh^{n-2} 3x \operatorname{sech}^2 3x \, dx$ $\Rightarrow \int \tanh^{n-2} 3x \operatorname{sech}^2 3x \, dx = \frac{1}{3(n-1)} \tanh^{n-1} 3x$ To score it must be a complete method leading to $\alpha \tanh^{n-1} 3x$ as above	dM1	
	$I_n = I_{n-2} - \frac{1}{3(n-1)} \left[\tanh^{n-1} 3x \right]_0^{\frac{1}{3} \ln 2} = I_{n-2} - \frac{1}{3(n-1)} \left(\frac{e^{2 \ln 2} - 1}{e^{2 \ln 2} + 1} \right)^{n-1}$ Introduces I_{n-2} and applies $x = \frac{1}{3} \ln 2$ using a correct exponential definition of $\tanh$ or accept use of a calculator if work is correct e.g. $\tanh(\ln 2) = \frac{3}{5}$		ddM1
	$I_n = I_{n-2} - \frac{\left(\frac{3}{5}\right)^{n-1}}{3(n-1)}$ but condone $I_n = I_{n-2} - \frac{\frac{3^{n-1}}{5}}{3(n-1)}$ Fully correct proof. Allow recovery from slips e.g. $\tanh \rightarrow \tan \rightarrow \tanh$ or e.g. $3x$ becoming x and then reverting to $3x$ again If there are clear errors that are not recovered score A0.		A1
			(4)

(c)	$I_5 = I_3 - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)} = I_1 - \frac{\left(\frac{3}{5}\right)^{3-1}}{3(3-1)} - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)}$ Uses their reduction formula to obtain I_5 in terms of I_1 Note that there may have already been an attempt at I_1 Condone the use of the letter p for the $\frac{3}{5}$ and allow a “made up” p for this mark. This may be implied by e.g. $I_5 = I_3 - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)}$, $I_3 = I_1 - \frac{\left(\frac{3}{5}\right)^{3-1}}{3(3-1)}$	M1
	$\int \tanh 3x \, dx = \frac{1}{3} \ln(\cosh 3x)$	M1
	Integrates to obtain $q \ln(\cosh rx)$ oe e.g. $q \ln(\operatorname{sech} rx)$ Condone q and/or $r = 1$	
	$I_5 = \frac{1}{3} \ln \left(\frac{e^{\ln 2} + e^{-\ln 2}}{2} \right) - \frac{\left(\frac{9}{25}\right)}{6} - \frac{\left(\frac{81}{625}\right)}{12}$ Applies $x = \frac{1}{3} \ln 2$ using correct exponential definition of cosh or uses a calculator if work is correct e.g. $\cosh(\ln 2) = \frac{5}{4}$ to obtain a numerical expression for I_5 Must not be in terms of p now and must be using a value of p obtained in part (b)	ddM1
	$\frac{1}{3} \ln \frac{5}{4} - \frac{177}{2500}$	A1
		(4)
		Total 11

Note that part (c) is “Hence” so they need to be using the given reduction formula, however, it is possible to find I_3 directly e.g. :

$$I_5 = I_3 - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)}$$

$$\int \tanh^3 3x \, dx = \int (\tanh 3x - \tanh 3x \operatorname{sech}^2 3x) \, dx = \left[\frac{1}{3} \ln(\cosh 3x) + \frac{1}{6} \operatorname{sech}^2 3x \right]$$

Score **M1** for using the reduction formula to obtain I_5 in terms of I_3 (allow the letter p for the $\frac{3}{5}$ and allow a “made up” p for this mark) **and** then integrating $\tanh^3 3x$ to the correct form e.g.

$$\alpha \ln(\cosh 3x) + \beta \operatorname{sech}^2 3x \text{ (oe)}$$

The second **M** mark would also score at this point as in the main scheme for integrating $\tanh 3x$ to obtain $q \ln(\cosh rx)$ oe e.g. $q \ln(\operatorname{sech} rx)$

$$\left[\frac{1}{3} \ln(\cosh 3x) + \frac{1}{6} \operatorname{sech}^2 3x \right]_0^{\frac{1}{3} \ln 2} - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)} = \frac{1}{3} \ln \frac{5}{4} + \frac{1}{6} \times \frac{16}{25} - \frac{1}{6} - \frac{27}{2500}$$

ddM1 for a complete method **using both limits** to obtain a numerical expression for I_5 using the correct exponential definitions or via a calculator.

$$\mathbf{A1:} \quad \frac{1}{3} \ln \frac{5}{4} - \frac{177}{2500}$$

Correct answer in correct form
(allow $a = \dots$, $b = \dots$, $c = \dots$) Allow -0.0708 for c

General Principles for Further Pure Mathematics Marking*(But note that specific mark schemes may sometimes override these general principles)***Method mark for solving 3 term quadratic:****1. Factorisation** $(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$ $(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$ **2. Formula**Attempt to use the correct formula (with values for a , b and c).**3. Completing the square**Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$ **Method marks for differentiation and integration:****1. Differentiation**Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)**2. Integration**Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)

General Principles for Further Pure Mathematics Marking
(But note that specific mark schemes may sometimes override these general principles).

Method mark for solving 3 term quadratic:

1. Factorisation

$(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$

$(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$

2. Formula

Attempt to use the correct formula (with values for a, b and c).

3. Completing the square

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$

Method marks for differentiation and integration:

1. Differentiation

Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)

2. Integration

Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)



Question Number	Scheme	Notes	Marks
1(a)	$(\cosh A \cosh B + \sinh A \sinh B) = \left(\frac{e^A + e^{-A}}{2} \right) \left(\frac{e^B + e^{-B}}{2} \right) + \left(\frac{e^A - e^{-A}}{2} \right) \left(\frac{e^B - e^{-B}}{2} \right)$ $= \frac{e^{A+B} + e^{A-B} + e^{B-A} + e^{-A-B} + e^{A+B} - e^{A-B} - e^{B-A} + e^{-A-B}}{4}$ Expresses the lhs in terms of exponentials correctly, combines terms and combines fractions with common denominator (Brackets not needed due to fraction lines)		M1
	$= \frac{2e^{A+B} + 2e^{-(A+B)}}{4} = \frac{e^{A+B} + e^{-(A+B)}}{2} = \cosh(A+B)^*$ Fully correct proof with no errors		A1*
			(2)
(b)	$\cosh(x + \ln 2) = \cosh x \cosh(\ln 2) + \sinh x \sinh(\ln 2)$ $= \left(\frac{2 + \frac{1}{2}}{2} \right) \cosh x + \left(\frac{2 - \frac{1}{2}}{2} \right) \sinh x$ Applies the result from part (a) and evaluates both $\cosh(\ln 2)$ and $\sinh(\ln 2)$ Use of (a) must be seen		M1
	$\frac{5}{4} \cosh x + \frac{3}{4} \sinh x = 5 \sinh x$ $\Rightarrow \frac{5}{4} \cosh x = \frac{17}{4} \sinh x$	Collects terms and reaches $a \cosh x = b \sinh x$ oe Depends on the first M mark	dM1
	$5 \cosh x = 17 \sinh x$ oe	Correct equation	A1
	$x = \frac{1}{2} \ln \left(\frac{1 + \frac{5}{17}}{1 - \frac{5}{17}} \right)$ Or $\frac{e^{2x} - 1}{e^{2x} + 1} = \frac{5}{17} \Rightarrow x = \dots$	Moves to $\tanh x$ and uses the correct logarithmic form for $\operatorname{artanh} x$ or reverts to exponential forms and solves for x Depends on both M marks	ddM1
	$x = \frac{1}{2} \ln \left(\frac{11}{6} \right)$	Cao (Accept integer multiples of $\frac{11}{6}$)	A1
			(5)
			Total 7

Question Number	Scheme	Notes	Marks
2	$\frac{dx}{d\theta} = \frac{\sec \theta \tan \theta + \sec^2 \theta}{\sec \theta + \tan \theta} - \cos \theta$ Correct derivative. Do not condone missing brackets e.g. $\frac{dx}{d\theta} = \frac{1}{\sec \theta + \tan \theta} \times \sec \theta \tan \theta + \sec^2 \theta - \cos \theta$ unless a correct expression is implied by subsequent work. Award when a correct expression is seen but note that other forms are possible e.g. $\sec \theta - \cos \theta$, $\tan \theta \sin \theta$		B1
	$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{\sec \theta \tan \theta + \sec^2 \theta}{\sec \theta + \tan \theta} - \cos \theta\right)^2 + (-\sin \theta)^2$ Attempts $\frac{dy}{d\theta}$ and then $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$		M1
	$S = (2\pi) \int \cos \theta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$ $= (2\pi) \int \cos \theta \sqrt{\left(\frac{\sec \theta \tan \theta + \sec^2 \theta}{\sec \theta + \tan \theta} - \cos \theta\right)^2 + (-\sin \theta)^2} d\theta$ Applies a correct surface area formula using their $\frac{dx}{d\theta}$ and their $\frac{dy}{d\theta}$ with or without the 2π For reference: $\sqrt{\left(\frac{\sec \theta \tan \theta + \sec^2 \theta}{\sec \theta + \tan \theta} - \cos \theta\right)^2 + (-\sin \theta)^2} = \tan \theta$ Allow π in front of the integral but must be an integral		M1
	$(2\pi) \int \sin \theta d\theta$	Fully correct simplified integral with or without the 2π	A1
	$= (2\pi)[- \cos \theta](+c)$	Correct integration with or without the 2π	A1
	$(2\pi)[- \cos \theta]_0^{\frac{\pi}{4}} = (2\pi)\left(-\frac{1}{\sqrt{2}} + 1\right)$ Applies the limits 0 and $\frac{\pi}{4}$. Must see evidence of both limits if necessary but condone e.g. $(2\pi)\left(-\frac{1}{\sqrt{2}} - 1\right)$ Depends on both previous method marks.		dM1
	TSA = $2\pi\left(-\frac{1}{\sqrt{2}} + 1\right) + \pi \times 1^2 + \pi \times \left(\frac{1}{\sqrt{2}}\right)^2$	Correct expressions for the 2 “ends” and adds these to their curved surface area. Depends on the previous method mark.	dM1
	$= \frac{\pi}{2}(7 - 2\sqrt{2})$	Correct answer in the required form or correct values for p and q .	A1
	Note: The final answer should follow correct work. The final mark should be withheld following e.g. $\frac{dy}{d\theta}$ clearly seen as $+\sin \theta$ or $\int \sin \theta d\theta = +\cos \theta$ Note: Without the “ends” the answer is $\frac{\pi}{2}(4 - 2\sqrt{2})$ (usually scores 6/8)		
			(8)
			Total 8

Alternative for first 4 marks:

	$\frac{dx}{d\theta} = \frac{\sec \theta \tan \theta + \sec^2 \theta}{\sec \theta + \tan \theta} - \cos \theta$ Correct derivative. Do not condone missing brackets e.g. $\frac{dx}{d\theta} = \frac{1}{\sec \theta + \tan \theta} \times \sec \theta \tan \theta + \sec^2 \theta - \cos \theta$ unless a correct expression is implied by subsequent work. Award when a correct expression is seen but note that other forms are possible e.g. $\sec \theta - \cos \theta$, $\tan \theta \sin \theta$	B1
	$1 + \left(\frac{dy}{dx} \right)^2 = 1 + \left(\frac{-\sin \theta}{\sec \theta - \cos \theta} \right)^2$ Attempts $1 + \left(\frac{dy}{dx} \right)^2$ with $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$	M1
	$S = (2\pi) \int \cos \theta \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \frac{dx}{d\theta} d\theta$ $= (2\pi) \int \cos \theta \sqrt{1 + \left(\frac{-\sin \theta}{\sec \theta - \cos \theta} \right)^2} (\sec \theta - \cos \theta) d\theta$ Applies a correct surface area formula using their $\frac{dx}{d\theta}$ and their $\frac{dy}{dx}$ with or without the 2π For reference: $\sqrt{1 + \left(\frac{-\sin \theta}{\sec \theta - \cos \theta} \right)^2} (\sec \theta - \cos \theta) = \tan \theta$ Allow π in front of the integral but must be an integral	M1
	$(2\pi) \int \sin \theta d\theta$	Fully correct simplified integral with or without the 2π A1

Question Number	Scheme/Notes	Marks
4(c)	There is no credit for attempts that do not use part (b) so e.g., do not award marks for attempts that apply exponential definitions to $10\sinh x + 8\cosh x = 18\sqrt{7}$ but note that it is acceptable to use exponential definitions with $6\sinh(x + \ln 3) = 18\sqrt{7}$. Allow work with “made up” values for R and p provided $R > 0$, $p \in \mathbb{Z}$, $p > 1$	
	$6\sinh(x + \ln 3) = 18\sqrt{7}$ $\Rightarrow x = \operatorname{arsinh}(3\sqrt{7}) - \ln 3$ $\Rightarrow x = \ln\left(3\sqrt{7} + \sqrt{(3\sqrt{7})^2 + 1}\right) - \ln 3$ Obtains $x = \operatorname{arsinh}\left(\frac{18\sqrt{7}}{6}\right) \pm \ln 3$ or $x \pm \ln 3 = \operatorname{arsinh}\left(\frac{18\sqrt{7}}{6}\right)$ from “6”$\sinh(x \pm \ln 3) = 18\sqrt{7}$ and uses the correct logarithmic form to obtain an expression for, or equation in x in “ln”s only but condone loss of the $-\ln 3$ or $+\ln 3$ after it has been seen. If the $-\ln 3$ or $+\ln 3$ is immediately incorporated to make a single logarithm the subtraction/addition law must be applied correctly. Work must be exact and not in decimals. If e.g., $C = \operatorname{arsinh}(3\sqrt{7})$ is found using $\frac{e^C - e^{-C}}{2} = 3\sqrt{7}$, the exponential definition must be correct and they must solve a 3TQ in e^C satisfying usual rules (or one root correct if no working) and proceed to a valid $C = \dots$ (e.g., not $\ln(\text{negative})$). This also applies to attempts via $6 \frac{e^{x+\ln 3} - e^{-x-\ln 3}}{2} = 18\sqrt{7} \quad \left\{ \Rightarrow 3e^x - \frac{1}{3}e^{-x} = 6\sqrt{7} \Rightarrow 9e^{2x} - 18\sqrt{7}e^x - 1 = 0 \Rightarrow x = \ln\left(\frac{8+3\sqrt{7}}{3}\right) \right\}$ Note that $e^{2(x+\ln 3)} - 6\sqrt{7}e^{x+\ln 3} - 1 = 0 \Rightarrow e^{x+\ln 3} = 8+3\sqrt{7} \Rightarrow x = \ln\left(\frac{8+3\sqrt{7}}{3}\right)$ is also possible and in such cases the $x + \ln 3$ must be handled correctly	M1
	$\left\{ x = \ln\left(\frac{3\sqrt{7}+8}{3}\right) \right\} = \ln\left(\sqrt{7} + \frac{8}{3}\right)$ Correct answer in correct form. Accept e.g., $\ln\left(2\frac{2}{3} + \sqrt{7}\right)$. Must be fully bracketed correctly. Accept $q = \frac{8}{3}$ if $\ln(\sqrt{7} + q)$ is seen. No additional answers.	A1
		(2)
		Total 9

Question Number	Scheme	Notes	Marks
7(a)	$5\mathbf{i} + 3\mathbf{j} - 8\mathbf{k}$ and $2\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$ lie in Π_1	Identifies 2 correct vectors lying in Π_1	B1
	$\mathbf{n} = \begin{pmatrix} 5 \\ 3 \\ -8 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ -6 \end{pmatrix} = \begin{pmatrix} -18-24 \\ -(-30+16) \\ -15-6 \end{pmatrix}$ Attempts the vector product between 2 correct vectors in Π_1 If no working is shown, look for at least 2 correct elements. Or e.g. Let $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ then $(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (5\mathbf{i} + 3\mathbf{j} - 8\mathbf{k}) = 0$, $(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}) = 0$ $\Rightarrow 5a + 3b - 8c = 0$, $2a - 3b - 6c = 0 \Rightarrow a = 2c$, $3b = -2c \Rightarrow \mathbf{n} = \dots$		M1
	$= \begin{pmatrix} -42 \\ 14 \\ -21 \end{pmatrix} \text{ or e.g. } \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$	Correct normal vector	A1
	$(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \dots$ Attempts scalar product between their normal vector and position vector of a point in Π_1. Do not allow this mark if the “5” (or equivalent) just ‘appears’. There must be some evidence for its origin e.g. $\mathbf{a} \cdot \mathbf{n} = \dots$ where $\mathbf{a}$ and $\mathbf{n}$ have been defined earlier. Depends on the first method mark.		dM1
	$6x - 2y + 3z = 5^*$	Correct proof	A1*
			(5)
	Alternative 1 for (a):		
	E.g. Let equation of Π_1 be $ax + by + z = c$		B1
	3 points on Π_1 are $(1, 2, 1)$, $(3, -1, -5)$ and e.g. $(8, 2, -13)$		
	$a + 2b + 1 = c$, $3a - b - 5 = c$, $8a + 2b - 13 = c \Rightarrow a = \dots, b = \dots, c = \dots$		M1
	Solves simultaneously for a , b and c using correct points		
	$\Rightarrow a = 2, b = -\frac{2}{3}, c = \frac{5}{3}$	Correct values	A1
	$2x - \frac{2}{3}y + z = \frac{5}{3}$	Forms Cartesian equation	dM1
	$6x - 2y + 3z = 5^*$	Correct proof	A1*
	Alternative 2 for (a):		
	$(1, 2, 1) \rightarrow 6x - 2y + 3z = 6 - 4 + 3 = 5$		B1
	Shows $(1, 2, 1)$ lies on Π_1		
	$\frac{x-3}{5} = \frac{y+1}{3} = \frac{z+5}{-8} \rightarrow \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \\ -8 \end{pmatrix} \text{ or equivalent}$ M1: Converts l to correct parametric form <u>seen as part of an attempt at this alternative</u> allow 1 slip with one of the elements A1: Correct form		M1A1
	$6(3 + 5\lambda) - 2(-1 + 3\lambda) + 3(-5 - 8\lambda) = 5$		dM1
	Shows l lies in Π_1		
	P lies in Π_1 and l lies in Π_1 so $6x - 2y + 3z = 5^*$		A1*
	All correct with conclusion		

(b) Way 1	$d = \frac{ 6(2) - 2k + 3(-7) - 5 }{\sqrt{6^2 + 2^2 + 3^2}}$	Correct method for the shortest distance	M1
	$= \frac{1}{7} -2k - 14 = \frac{2}{7} k + 7 *$	Correct completion	A1*
	(2)		
(b) Way 2	Distance O to Π_1 is $\frac{5}{\sqrt{6^2 + 2^2 + 3^2}}$. Distance O to parallel plane containing Q is $\frac{(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot (2\mathbf{i} + k\mathbf{j} - 7\mathbf{k})}{\sqrt{6^2 + 2^2 + 3^2}} = \frac{-9 - 2k}{7}$ $d = \left \frac{5}{7} - \frac{-9 - 2k}{7} \right $ Correct method for the shortest distance		M1
	$= \frac{1}{7} 2k + 14 = \frac{2}{7} k + 7 *$	Correct completion	A1*
(b) Way 3	$d = \frac{ \overrightarrow{PQ} \cdot \mathbf{n} }{ \mathbf{n} } = \frac{ (\mathbf{i} + (k - 2)\mathbf{j} - 8\mathbf{k}) \cdot (-42\mathbf{i} + 14\mathbf{j} - 21\mathbf{k}) }{\sqrt{42^2 + 14^2 + 21^2}}$ Correct method for the shortest distance		M1
	$= \frac{ -42 + 14k - 28 + 168 }{49} = \frac{ 14k + 98 }{49} = \frac{2}{7} k + 7 *$	Correct completion	A1*
(c)	$\frac{2}{7} k + 7 = \frac{ 8(2) - 4k - 7 + 3 }{\sqrt{8^2 + 4^2 + 1^2}}$ Correctly attempts the distance between $(2, k, -7)$ and Π_2 and sets equal to the result from (a). May see alternative methods here for the distance between $(2, k, -7)$ and Π_2 e.g. finds the coordinates of a point on Π_2 e.g. $R(1, 1, -7)$ and then finds $d = \frac{ \overrightarrow{RQ} \cdot (8\mathbf{i} - 4\mathbf{j} + \mathbf{k}) }{ 8\mathbf{i} - 4\mathbf{j} + \mathbf{k} } = \frac{ (\mathbf{i} + (k - 1)\mathbf{j}) \cdot (8\mathbf{i} - 4\mathbf{j} + \mathbf{k}) }{\sqrt{8^2 + 4^2 + 1^2}} = \left \frac{8 - 4k + 4}{9} \right = \left \frac{12 - 4k}{9} \right $		M1
	$\frac{2}{7}(k + 7) = \frac{1}{9}(12 - 4k) \Rightarrow k = \dots$ or $\frac{2}{7}(k + 7) = \frac{1}{9}(4k - 12) \Rightarrow k = \dots$ Attempts to solve one of these equations where their distance from Q to Π_2 is of the form $ak + b$ where a and b are non-zero. or $\frac{2}{7}(k + 7) = \frac{1}{9}(12 - 4k) \Rightarrow \frac{4}{49}(k + 7)^2 = \frac{1}{81}(12 - 4k)^2$ $\Rightarrow 23k^2 - 462k - 441 = 0 \Rightarrow k = \dots$ Squares both sides and attempts to solve resulting quadratic. Condone poor attempts at squaring the brackets and there is no requirement to follow the usual guidance for solving the quadratic		dM1
	$k = -\frac{21}{23}$ or $k = 21$	One correct value. Must be 21 but allow equivalent exact fractions for $-\frac{21}{23}$	A1
	$k = -\frac{21}{23}$ and $k = 21$	Both correct values. Must be 21 but allow equivalent exact fractions for $-\frac{21}{23}$ and no other values.	A1
	(4)		
	Total 11		

Question Number	Scheme	Notes	Marks
3 Way 1 Converts to sinh and cosh	$4 \tanh x - \operatorname{sech} x = 1$ $4 \frac{\sinh x}{\cosh x} - \frac{1}{\cosh x} = 1$ $4 \sinh x - 1 - \cosh x = 0$ $4 \frac{e^x - e^{-x}}{2} - 1 - \frac{e^x + e^{-x}}{2} = 0$	Replaces one hyperbolic function with its correct exponential equivalent. Allow for correct replacement of just e.g., $\sinh x$ after using $\tanh x = \frac{\sinh x}{\cosh x}$. May follow errors but do not allow any further marks if the original equation was reduced to one in a single hyperbolic function.	M1
	$3e^{2x} - 2e^x - 5 = 0$	M1: Obtains an equation which if terms are collected is a 3TQ (or 2TQ with no constant) in e^x A1: Correct 3TQ	M1 A1
	$e^x = \frac{2 \pm \sqrt{4+60}}{6} \left(\Rightarrow \frac{2+8}{6} = \frac{5}{3} \right)$	M1: Solves 3TQ (or 2TQ with no constant) in e^x . Apply usual rules. If no working seen they must achieve one correct root of their equation to 3sf which may be complex. If 2TQ must get a correct non-zero root of their equation. A1: Any correct unsimplified expression for e^x that includes the positive root. Must be exact	M1 A1
	$x = \ln \frac{5}{3}$	$\ln \frac{5}{3}, \ln 1\frac{2}{3}, \ln 1.\dot{6}$ only but allow $k = \dots$ No unrejected extra solutions	A1
			Total 6
Way 2 Straight to e^x	$4 \frac{e^x - e^{-x}}{e^x + e^{-x}} - \frac{2}{e^x + e^{-x}} = 1$	Replaces one hyperbolic function with its correct exponential equivalent	M1
	$3e^{2x} - 2e^x - 5 = 0$	M1: Obtains an equation which if terms are collected is a 3TQ (or 2TQ with no constant) in e^x A1: Correct 3TQ	M1 A1
	$e^x = \frac{2 \pm \sqrt{4+60}}{6} \left(\Rightarrow \frac{2+8}{6} = \frac{5}{3} \right)$	M1: Solves 3TQ (or 2TQ with no constant) in e^x . Apply usual rules. If no working seen they must achieve one correct root of their equation to 3sf which may be complex. If 2TQ must get a correct non-zero root of their equation. A1: Any correct unsimplified expression for e^x that includes the positive root. Must be exact	M1 A1
	$x = \ln \frac{5}{3}$	$\ln \frac{5}{3}, \ln 1\frac{2}{3}, \ln 1.\dot{6}$ only but allow $k = \dots$ No unrejected extra solutions	A1
			Total 6
	In Ways 1 & 2, if they form an equation which is not a quadratic in e^x they must achieve the correct exact root of $\frac{5}{3}$ to access the middle four marks		

Question Number	Scheme	Notes	Marks
3 Way 3a Squaring (sinh)	$4 \sinh x - 1 = \cosh x$ $16 \sinh^2 x - 8 \sinh x + 1 = \cosh^2 x$ $16 \sinh^2 x - 8 \sinh x + 1 = 1 + \sinh^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in $\sinh x$	M1
	$15 \sinh^2 x - 8 \sinh x = 0$	M1: Obtains a 2TQ with no constant or 3TQ in $\sinh x$ A1: Correct 2TQ	M1 A1
	$\sinh x = \frac{8}{15}$	Solves 2TQ (with no constant) or 3TQ in $\sinh x$. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x = \operatorname{arsinh} \frac{8}{15} = \ln \left(\frac{8}{15} + \sqrt{\left(\frac{8}{15}\right)^2 + 1} \right)$ or $15e^{2x} - 16e^x - 15 = 0 \Rightarrow$ $e^x = \frac{16 \pm \sqrt{256 + 900}}{30}$	A correct unsimplified expression for x as a $\ln$ (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x = \ln \frac{5}{3}$	$\ln \frac{5}{3}, \ln 1\frac{2}{3}, \ln 1.\dot{6}$ only but allow $k = \dots$ No unrejected extra solutions	A1
			Total 6
Way 3b Squaring (sech)	$4 \tanh x = 1 + \operatorname{sech} x$ $16 \tanh^2 x = 1 + 2 \operatorname{sech} x + \operatorname{sech}^2 x$ $16(1 - \operatorname{sech}^2 x) = 1 + 2 \operatorname{sech} x + \operatorname{sech}^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in $\operatorname{sech} x$	M1
	$17 \operatorname{sech}^2 x + 2 \operatorname{sech} x - 15 = 0$	M1: Obtains a 2TQ (with no constant) or 3TQ in $\operatorname{sech} x$ A1: Correct 3TQ	M1 A1
	$(17 \operatorname{sech} x - 15)(\operatorname{sech} x + 1) = 0$ $\operatorname{sech} x = \frac{15}{17}$	Solves 2TQ with no constant or 3TQ in $\operatorname{sech} x$. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x = \operatorname{arcosh} \frac{17}{15} = \ln \left(\frac{17}{15} + \sqrt{\left(\frac{17}{15}\right)^2 - 1} \right)$ or $15e^{2x} - 34e^x + 15 = 0 \Rightarrow$ $e^x = \frac{34 \pm \sqrt{1156 - 900}}{30}$	A correct unsimplified expression for x as a $\ln$ (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x = \ln \frac{5}{3}$	$\ln \frac{5}{3}, \ln 1\frac{2}{3}, \ln 1.\dot{6}$ only but allow $k = \dots$ No unrejected extra solutions	A1
			Total 6

Question Number	Scheme	Notes	Marks
3 Way 3c Squaring (tanh)	$4 \tanh x - 1 = \operatorname{sech} x$ $16 \tanh^2 x - 8 \tanh x + 1 = \operatorname{sech}^2 x$ $16 \tanh^2 x - 8 \tanh x + 1 = 1 - \tanh^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in $\tanh x$	M1
	$17 \tanh^2 x - 8 \tanh x = 0$	M1: Obtains a 2TQ with no constant or 3TQ in $\tanh x$ A1: Correct 2TQ	M1 A1
	$\tanh x = \frac{8}{17}$	Solves 2TQ with no constant or 3TQ in $\tanh x$. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x = \operatorname{artanh} \frac{8}{17} = \frac{1}{2} \ln \left(\frac{1 + \frac{8}{17}}{1 - \frac{8}{17}} \right)$ or $9e^{2x} - 25 = 0 \Rightarrow$ $e^x = \frac{5}{3}$	A correct unsimplified expression for x as a $\ln$ (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x = \ln \frac{5}{3}$	$\ln \frac{5}{3}, \ln 1\frac{2}{3}, \ln 1.\dot{6}$ only but allow $k = \dots$ No unrejected extra solutions	A1
			Total 6
Way 3d Squaring (cosh)	$4 \sinh x = 1 + \cosh x$ $16 \sinh^2 x = 1 + 2 \cosh x + \cosh^2 x$ $16 \cosh^2 x - 16 = 1 + 2 \cosh x + \cosh^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in $\cosh x$	M1
	$15 \cosh^2 x - 2 \cosh x - 17 = 0$	M1: Obtains a 2TQ with no constant or 3TQ in $\cosh x$ A1: Correct 3TQ	M1 A1
	$(15 \cosh x - 17)(\cosh x + 1) = 0$ $\cosh x = \frac{17}{15}$	Solves 2TQ (with no constant) or 3TQ in $\cosh x$. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x = \operatorname{arcosh} \frac{17}{15} = \ln \left(\frac{17}{15} + \sqrt{\left(\frac{17}{15}\right)^2 - 1} \right)$ or $15e^{2x} - 34e^x + 15 = 0 \Rightarrow$ $e^x = \frac{34 \pm \sqrt{1156 - 900}}{30}$	A correct unsimplified expression for x as a $\ln$ (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x = \ln \frac{5}{3}$	$\ln \frac{5}{3}, \ln 1\frac{2}{3}, \ln 1.\dot{6}$ only but allow $k = \dots$ No unrejected extra solutions	A1
			Total 6

Question Number	Scheme	Notes	Marks
4(a)	$\int \frac{1}{\sqrt{9x^2+16}} dx = \frac{1}{3} \int \frac{1}{\sqrt{x^2+\frac{16}{9}}} dx$ $= \frac{1}{3} \operatorname{arsinh}\left(\frac{3x}{4}\right) \text{ or } \frac{1}{3} \operatorname{arsinh}\left(\frac{x}{\frac{4}{3}}\right) (+c)$ $\text{or } \frac{1}{3} \ln\left(x + \sqrt{x^2 + \left(\frac{4}{3}\right)^2}\right) (+c)$	M1: Obtains $p \operatorname{arsinh}(qx) \text{ or } r \ln\{x + \sqrt{x^2 + s}\}$ $\text{or } t \ln(ux + \sqrt{vx^2 + w})$ $p, q, r, s, t, u, v, w > 0$ A1: Any correct expression. Could be unsimplified and isw. The “+c” is not required. Allow $\sinh^{-1}$ and condone “arcsinh”. “arcsin” or “arsin” is M0	M1 A1
			(2)
(b)	$\int_{-2}^2 \frac{1}{\sqrt{9x^2+16}} dx$ $= \left[\frac{1}{3} \operatorname{arsinh}\left(\frac{3x}{4}\right) \right]_{-2}^2 \text{ or } \left[\frac{2}{3} \operatorname{arsinh}\left(\frac{3x}{4}\right) \right]_{-2}^2$ $= \frac{1}{3} \operatorname{arsinh}\left(\frac{3 \times 2}{4}\right) - \frac{1}{3} \operatorname{arsinh}\left(\frac{3 \times -2}{4}\right) \text{ or } \frac{2}{3} \operatorname{arsinh}\left(\frac{3}{2}\right)$ OR $\left[\frac{1}{3} \ln\left(x + \sqrt{x^2 + \frac{16}{9}}\right) \right]_{-2}^2$ $= \frac{1}{3} \ln\left(2 + \sqrt{2^2 + \frac{16}{9}}\right) - \frac{1}{3} \ln\left(-2 + \sqrt{(-2)^2 + \frac{16}{9}}\right)$ $\text{or } \frac{2}{3} \left(\ln\left(2 + \sqrt{2^2 + \frac{16}{9}}\right) - \ln\left(0 + \sqrt{0^2 + \frac{16}{9}}\right) \right)$	Substitutes the limits 2 and -2 into an expression of the form $p \operatorname{arsinh}(qx) \text{ or } r \ln\{x + \sqrt{x^2 + s}\}$ $\text{or } t \ln(ux + \sqrt{vx^2 + w})$ $p, q, r, s, t, u, v, w > 0$ and subtracts either way round or obtains an expression for $2[\dots]_0^{\pm 2}$ The expression does not have to be consistent with their answer to (a). No rounded decimals unless exact values recovered. Any $f(0) = 0$ can be implied by omission. Condone poor bracketing.	M1
	$\frac{1}{3} \ln\left(\frac{11}{2} + \frac{3\sqrt{13}}{2}\right) \text{ or } \frac{1}{3} \ln \frac{11+3\sqrt{13}}{2}$ $\text{or } \frac{2}{3} \ln\left(\frac{3}{2} + \frac{\sqrt{13}}{2}\right) \text{ or } \frac{2}{3} \ln \frac{3+\sqrt{13}}{2}$	dM1: Obtains an expression of the form $a \ln(b + c\sqrt{13}) \text{ or } a \ln\left(\frac{d + e\sqrt{13}}{f}\right)$ where a, b, c, d, e, f are exact and > 0. Condone poor bracketing. Requires previous M mark. A1: Any correct equivalent in an appropriate form (fractions may not be in simplest form) with correct bracketing if necessary and isw. Must come from correct work. Allow e.g., $a = \frac{2}{3}, b = \frac{3}{2}, c = \frac{1}{2}$	dM1 A1
	For information the decimal answer is 0.7965038115		(3)
			Total 5

Question Number	Scheme	Notes	Marks
5(a)	$\begin{vmatrix} a & a & 1 \\ -a & 4 & 0 \\ 4 & a & 5 \end{vmatrix}$ $= a(4 \times 5 - 0) - a(-5a - 0) + 1(-a^2 - (4 \times 4))$	Uses a correct method for $\det \mathbf{A}$ (implied by two correct parts) to obtain an expression in a	M1
	$\Rightarrow 20a + 5a^2 - a^2 - 16 = 0$ $\Rightarrow a^2 + 5a - 4 = 0$ $\Rightarrow a = \frac{-5 \pm \sqrt{41}}{2}$	Correct exact value of a Condone $\frac{-5 \pm \sqrt{41}}{2}$	A1
			(2)
(b)(i) Way 1 $ \mathbf{A} - \lambda \mathbf{I} $	$ \mathbf{A} - \lambda \mathbf{I} = \begin{vmatrix} a - \lambda & a & 1 \\ -a & 4 - \lambda & 0 \\ 4 & a & 5 - \lambda \end{vmatrix}$ $= (a - \lambda)(4 - \lambda)(5 - \lambda) - a \times a(5 - \lambda) + (-a^2 - 4(4 - \lambda))$ $\text{or } \mathbf{A} - 2\mathbf{I} = \begin{vmatrix} a - 2 & a & 1 \\ -a & 2 & 0 \\ 4 & a & 3 \end{vmatrix}$ $= 6(a - 2) - a \times -3a + (-a^2 - 8)$	Obtains an expression for $ \mathbf{A} - \lambda \mathbf{I} $ in terms of a and λ or just a if λ is replaced by 2. If method unclear insist on 2 out of 3 correct parts. May multiply along any row/column. Sarrus leads to the same expressions shown (or the expressions all multiplied by -1 if “=0”).	M1
	$\lambda = 2 \Rightarrow (a - 2) \times 2 \times 3 + 3a^2 - a^2 - 8 = 0$ $2a^2 + 6a - 20 = 0 \Rightarrow a^2 + 3a - 10 = 0$ $\Rightarrow (a - 2)(a + 5) = 0 \Rightarrow a = \dots$	Following use of $\lambda = 2$, forms and solves a 3TQ in a . Apply usual rules. If no working they must obtain one correct solution for their 3TQ which could be complex. Could be implied. Requires previous M mark.	dM1
	$(a > 0 \Rightarrow) a = 2$	Correct value of a from correct work. If -5 is offered imply its rejection if 2 alone is used in (ii)	A1
	If $a = 2$ is arrived at fortuitously, all marks are available for the remainder of the question		(3)
(b)(i) Way 2 $\mathbf{Ax} = 2\mathbf{x}$	$\mathbf{Ax} = 2\mathbf{x} \Rightarrow$ $ax + ay + z = 2x$ $-ax + 4y = 2y$ $4x + ay + 5z = 2z$	Uses $\mathbf{Ax} = 2\mathbf{x}$ [or $(\mathbf{A} - 2\mathbf{I})\mathbf{x} = 0$] to obtain three simultaneous equations. Allow if given as two equal vectors.	M1
	$\Rightarrow a^2 + 3a - 10 = 0$ $\Rightarrow (a - 2)(a + 5) = 0 \Rightarrow a = \dots$	Forms and solves a 3TQ in a . Apply usual rules. If calculator used must obtain one correct solution for their 3TQ which could be complex. Could be implied. Requires previous M mark.	dM1
	$(a > 0 \Rightarrow) a = 2$	Correct value of a from correct work. If -5 is offered imply its rejection if 2 alone is used in (ii)	A1
	If $a = 2$ is arrived at fortuitously, all marks are available for the remainder of the question		(3)

Question Number	Scheme	Notes	Marks
5(b)(ii)	$(2-\lambda)(4-\lambda)(5-\lambda)+4(5-\lambda)+(-4-16+4\lambda)=0$ $\Rightarrow (5-\lambda)[(2-\lambda)(4-\lambda)+4-4]=0$ $\Rightarrow (5-\lambda)(2-\lambda)(4-\lambda)=0 \Rightarrow \lambda = \dots$	Uses their value of a in a recognisable attempt at a characteristic equation and achieves a real non-zero eigenvalue $\neq 2$. There must be some algebra but it may be poor.	M1
	4 and 5	Both correct (no extra) and from correct work	A1
	For information the cubic is $\pm(\lambda^3 - 11\lambda^2 + 38\lambda - 40) = 0$		(2)
(c)	$\mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = "4" \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ or } (\mathbf{A} - "4"\mathbf{I}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow$ $\begin{array}{rcl} 2x+2y+z=4x & & -2x+2y+z=0 \\ -2x+4y=4y & \text{or} & -2x=0 \\ 4x+2y+5z=4z & & 4x+2y+z=0 \end{array}$ OR $\mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = "5" \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ or } (\mathbf{A} - "5"\mathbf{I}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow$ $\begin{array}{rcl} 2x+2y+z=5x & & -3x+2y+z=0 \\ -2x+4y=5y & \text{or} & -2x-y=0 \\ 4x+2y+5z=5z & & 4x+2y+z=0 \end{array}$ Uses $\mathbf{Ax} = \lambda\mathbf{x}$ or $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ with their value of a and a real non-zero value of $\lambda \neq 2$ to obtain three simultaneous equations (allow if given as two equal vectors) Alternatively attempts vector product of two rows of $\mathbf{A} - "4"\mathbf{I}$		M1
	$\pm \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \text{ or } \pm \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix}$	One correct eigenvector. As shown or multiple or with components multiplied by e.g. " k " Accept e.g., $x=0, y=-1, z=2$	A1
	$\pm \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \text{ and } \pm \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix}$	Both correct eigenvectors. As shown or multiple or with components multiplied by e.g. k Accept $x=..., y=..., z=...$ Both these 2 A marks could be implied by their normalised eigenvectors	A1
	$\pm \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \pm \frac{1}{\sqrt{54}} \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix} \text{ oe}$	M1: A correct method to normalise at least one of their eigenvectors A1: Both correct. Allow any exact equivalents. Isw	M1 A1
	All marks available regardless of how $a = 2, \lambda_2 = 4$ & $\lambda_3 = 5$ have been obtained		(5)
			Total 12

Question Number	Scheme	Marks	
7(a)	$y = \cos 2x \Rightarrow \frac{dy}{dx} = -2 \sin 2x \text{ and then also:}$ $S = 2\pi \int_{(0)}^{\left(\frac{\pi}{4}\right)} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} (dx) \text{ and } \left(\frac{dy}{dx}\right)^2 = \dots \Rightarrow S = \dots$ or $S = 2\pi \int_{(0)}^{\left(\frac{\pi}{4}\right)} \cos 2x \sqrt{1 + (-2 \sin 2x)^2} (dx) \Rightarrow S = \dots$	M1	
	$S = 2\pi \int_0^{\frac{\pi}{4}} \cos 2x \sqrt{1 + 4 \sin^2 2x} dx *$	A1*	
		(2)	
(b)	$2 \sin 2x = \sinh \theta \Rightarrow 4 \cos 2x \frac{dx}{d\theta} = \cosh \theta \quad \text{o.e.}$	B1	
	$S = 2\pi \int \cos 2x \sqrt{1 + 4 \sin^2 2x} dx = 2\pi \int \cancel{\cos 2x} \sqrt{1 + \sinh^2 \theta} \frac{\cosh \theta}{4 \cancel{\cos 2x}} d\theta$	M1	
	$= \frac{\pi}{2} \int \cosh \theta \sqrt{1 + \sinh^2 \theta} d\theta = \frac{\pi}{2} \int \cosh^2 \theta d\theta$	A1	
ALT for first 3 marks	$x = \frac{1}{2} \arcsin\left(\frac{1}{2} \sinh \theta\right) \Rightarrow \frac{dx}{d\theta} = \frac{1}{2} \frac{\frac{1}{2} \cosh \theta}{\sqrt{1 - \frac{1}{4} \sinh^2 \theta}}$	$\theta = \operatorname{ar} \sinh (2 \sin 2x) \Rightarrow \frac{d\theta}{dx} = \frac{4 \cos 2x}{\sqrt{1 + 4 \sin^2 2x}}$	B1
	$S = 2\pi \int \cos 2x \sqrt{1 + 4 \sin^2 2x} dx$ $= 2\pi \int \sqrt{1 - \frac{1}{4} \sinh^2 \theta} \sqrt{1 + \sinh^2 \theta} \frac{1}{2} \frac{\frac{1}{2} \cosh \theta}{\sqrt{1 - \frac{1}{4} \sinh^2 \theta}} d\theta$	$S = 2\pi \int \cos 2x \sqrt{1 + 4 \sin^2 2x} dx$ $= 2\pi \int \cancel{\cos 2x} \sqrt{1 + 4 \sin^2 2x} \frac{\sqrt{1 + 4 \sin^2 2x}}{4 \cancel{\cos 2x}} d\theta$ $= 2\pi \int \cancel{\cos 2x} \frac{1 + \sinh^2 \theta}{4 \cancel{\cos 2x}} d\theta$	M1
	$= \frac{\pi}{2} \int \cosh^2 \theta d\theta$	A1	
	$= \frac{\pi}{2} \int \left(\frac{1}{2} + \frac{1}{2} \cosh 2\theta\right) d\theta \text{ or } = \frac{\pi}{2} \int \left(\frac{e^\theta + e^{-\theta}}{2}\right)^2 d\theta = \frac{\pi}{2} \int \left(\frac{e^{2\theta} + 2 + e^{-2\theta}}{4}\right) d\theta$	M1	
	$= \frac{\pi}{4} \left[\theta + \frac{1}{2} \sinh 2\theta\right] \text{ or } \frac{\pi}{2} \left[\frac{1}{8} e^{2\theta} + \frac{1}{2} \theta - \frac{1}{8} e^{-2\theta}\right]$	A1	
	$\frac{\pi}{4} \left[\theta + \frac{1}{2} \sinh 2\theta\right]_0^{\operatorname{arsinh} 2} = \frac{\pi}{4} \left(\operatorname{arsinh} 2 + \frac{1}{2} \sinh (2 \operatorname{arsinh} 2)\right)$ or $\frac{\pi}{2} \left[\frac{1}{8} e^{2\theta} + \frac{1}{2} \theta - \frac{1}{8} e^{-2\theta}\right]_0^{\operatorname{ar} \sinh 2} = \frac{\pi}{2} \left(\frac{1}{8} e^{2 \ln (2 + \sqrt{5})} + \frac{1}{2} \ln (2 + \sqrt{5}) - \frac{1}{8} e^{-2 \ln (2 + \sqrt{5})}\right)$	M1	
	$= \frac{\pi}{4} (\ln (2 + \sqrt{5}) + 2 \sqrt{5})$	A1	
		(7)	
	Total 9		

Question	Scheme	Marks
5(a)	$x = (2t+3)^{\frac{3}{2}} \quad y = \frac{3}{2}t^2 + 3t + 6 \quad -\frac{3}{2} \leq t \leq 3$	
	$\frac{dx}{dt} = \frac{3}{2}(2t+3)^{\frac{1}{2}} \times 2 \quad \frac{dy}{dt} = 3t + 3$	M1A1
	$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9(2t+3) + (3t+3)^2 = \dots$	M1
	$[= 9(t^2 + 4t + 4)] = 9(t+2)^2$	A1
		(4)
(b)	Arc length $= \int_{-\frac{3}{2}}^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = 3 \int_{-\frac{3}{2}}^3 t + 2 dt = 3 \left[\frac{t^2}{2} + 2t \right]_{-\frac{3}{2}}^3$	M1
	$= 3 \left[\left(\frac{9}{2} + 6 \right) - \left(\frac{9}{8} - 3 \right) \right] = \dots$	M1
	$= \frac{297}{8}$	A1
		(3)
(c)	Surface area $= 2\pi \int_{-\frac{3}{2}}^3 y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} (dt) = 6\pi \int_{-\frac{3}{2}}^3 \left(\frac{3}{2}t^2 + 3t + 6 \right) (t+2) (dt)$	B1ft
	$= 6\pi \int_{-\frac{3}{2}}^3 \frac{3}{2}t^3 + 6t^2 + 12t + 12 dt = 6\pi \left[\frac{3}{8}t^4 + 2t^3 + 6t^2 + 12t \right]_{-\frac{3}{2}}^3$	M1
	$= 6\pi \left[\left(\frac{3}{8}3^4 + 2 \times 3^3 + 6 \times 3^2 + 12 \times 3 \right) - \left(\frac{3}{8} \left(\frac{81}{16} \right) + 2 \left(-\frac{27}{8} \right) + 6 \left(\frac{9}{4} \right) + 12 \left(-\frac{3}{2} \right) \right) \right]$	dM1
	$= 6\pi \left(\frac{1395}{8} - \frac{-1197}{128} \right) = \dots$	
	$= \frac{70551\pi}{64}$	A1
		(4)
(11 marks)		

Notes
(a)

M1: Attempts both derivatives with correct form for each, $\frac{dx}{dt} = A(2t+3)^{\frac{1}{2}}$, $\frac{dy}{dt} = Bt+3$.

A1: Both correct.

M1: Squares and adds their two derivatives and proceeds to simplify to a 3TQ

A1: Correct answer obtained with no incorrect working seen. Allow $a = 9$ stated as answer if they reach at least $9(t^2 + 4t + 4)$ but do not show the factorisation.

(b)

M1: Applies the arc length formula and integrates the square root of their answer to (a) achieving a quadratic answer. (Allow if their a is not square rooted.) Limits not needed for this mark. Note they may integrate via reverse chain rule to $A(t+2)^2$ (which is equivalent).

M1: Applies the correct limits to their integral, which must have been from an attempt at the correct formula, but allow if $a(t+2)^2$ was used following a stated correct formula. Must have seen an increase in power in at least one term in the linear function – or quadratic if $a(t+2)^2$ was used per note above. Do not be concerned about checking if they just give values for each bracket (or indeed just an answer) – assume they have applied the correct limits unless there is contrary evidence (e.g. wrong limits stated or seen applied).

A1: Correct answer.

(c)

B1ft: Applies the surface area formula correctly with their answer to (a). The 2π and correct limits must be seen at some stage. The dt may be implied.

M1: Expands and integrates from a correct attempt at the area formula. Must be using

$$\int y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} (dt) \text{ but again allow here if the correct formula is quoted but they}$$

subsequently forget to take the square root of (a). May be missing limits and 2π for this mark.

dM1: Applies the correct limits and simplifies to a single term. Do not be concerned about checking if they just give values for each bracket (or indeed just an answer) – assume they have applied the correct limits unless there is contrary evidence (e.g. wrong limits stated or seen applied).

A1: Correct answer, but do not isw if they try and add “ends” to the surface and award A0 in such cases.

Note: If they neglect to square root the answer in both parts the second M in (b) and both M’s in (c) may be gained. This will require a cubic integral in (b) and integrating a quartic to quintic in (c).



Question Number	Scheme	Notes	Marks
7	$x = \cosh t + t, \quad y = \cosh t - t$		
(a)	$\frac{dx}{dt} = \sinh t + 1, \quad \frac{dy}{dt} = \sinh t - 1$	Correct derivatives	B1
	$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sinh^2 t + 2\sinh t + 1 + \sinh^2 t - 2\sinh t + 1$ $= 2\sinh^2 t + 2$ M1: Squares correctly, cancels and collects terms		M1
	$= 2(1 + \sinh^2 t) = 2\cosh^2 t^*$	Uses $\cosh^2 t = 1 + \sinh^2 t$ to complete the proof with no errors	A1*
			(3)
(b)	$S = 2\pi \int y \, ds = 2\pi \int (\cosh t - t)\sqrt{2} \cosh t \, dt$	Uses $S = 2\pi \int y \, ds$ with the given y and the result from part (a)	M1
	$= 2\sqrt{2}\pi \int_0^{\ln 3} (\cosh^2 t - t \cosh t) \, dt^*$	Correct proof with no errors	A1*
			(2)
(c)	$\int \cosh^2 t \, dt = \int \pm \frac{1}{2} \pm \frac{1}{2} \cosh 2t \, dt$	Uses $\cosh^2 t = \pm \frac{1}{2} \pm \frac{1}{2} \cosh 2t$	M1
	$\int t \cosh t \, dt = t \sinh t - \int \sinh t \, dt$	Attempts integration by parts the right way round on $t \cosh t$	M1
		Correct expression	A1
	$S = (2\sqrt{2}\pi) \int (\cosh^2 t - t \cosh t) \, dt = (2\sqrt{2}\pi) \left[\frac{1}{2}t + \frac{1}{4}\sinh 2t - t \sinh t + \cosh t \right]$ A1: 2 correct terms A1: All correct		A1A1
	$(S =) 2\sqrt{2}\pi \left\{ \left(\frac{1}{2}\ln 3 + \frac{10}{9} - \frac{4}{3}\ln 3 + \frac{5}{3} \right) - (1) \right\}$ dM1: Correct use of limits 0 and $\ln 3$ depends on both preceding M marks		dM1
	$S = \frac{1}{9}\sqrt{2}\pi (32 - 15 \ln 3)$	cao	A1 (7)
			Total 12
Alternative for (c)			
	$\int \cosh^2 t \, dt = \int \left(\frac{e^t + e^{-t}}{2} \right)^2 \, dt$ $= \frac{1}{4} \int (e^{2t} + 2 + e^{-2t}) \, dt$	Substitutes the exponential form and attempts to square	M1
	$\int t \cosh t \, dt = \frac{1}{2} \int t(e^t + e^{-t}) \, dt$ $= \frac{1}{2}te^t - \frac{1}{2} \int te^t \, dt - \left\{ \frac{1}{2}te^{-t} - \frac{1}{2} \int e^{-t} \, dt \right\}$	Substitutes the exponential form and attempts integration by parts the right way round Correct expression	M1 A1
	$(S =) (2\sqrt{2}\pi) \left\{ \frac{1}{4} \left(\frac{1}{2}e^{2t} + 2t - \frac{1}{2}e^{-2t} \right) - \frac{1}{2}te^t + \frac{1}{2}e^t + \frac{1}{2}te^{-t} - \frac{1}{2}e^{-t} \right\}$ A1: either integral correct A1: other integral correct but both must be in a complete expression for S		A1A1
	Depends on both M marks above	Correct use of limits 0 and $\ln 3$	dM1
	$S = \frac{1}{9}\sqrt{2}\pi (32 - 15 \ln 3)$	cao	A1

	Alternative for the first 3 marks of (c)		
	$= 2\sqrt{2}\pi \int (\cosh^2 t - t \cosh t) dt$ $= 2\sqrt{2}\pi \int \cosh t (\cosh t - t) dt$		
	$2\sqrt{2}\pi \left(\left[\sinh t (\cosh t - t) \right] - \int \sinh t (\sinh t - 1) dt \right)$		
	$2\sqrt{2}\pi \left(\left[\sinh t (\cosh t - t) \right] - \left[\cosh t (\sinh t - 1) \right] + \int \cosh^2 t dt \right)$ M1 (2 nd on e-PEN): Use parts twice A1 Correct expression		M1A1
	$\int \cosh^2 t dt = \int \pm \frac{1}{2} \pm \frac{1}{2} \cosh 2t dt$	Uses $\cosh^2 t = \pm \frac{1}{2} \pm \frac{1}{2} \cosh 2t$	M1 (1st on e-PEN)
	Rest as main scheme		