



GuideMyGrades

— YOUR GOALS. OUR EXPERTISE. —

EDEXCEL INTERNATIONAL A LEVEL MATHS: QUESTIONS BY TOPIC

Vectors (FP3.5)

International A Level Mathematics · FP3: Further Pure Mathematics 3 (WFM03)

13 questions · 137 marks · Mark schemes included

Past papers October 2020 - June 2025 · 1 sub-topic

 guidemygrades.com

 [@Guidemygrades](https://www.instagram.com/Guidemygrades)

 [@Guidemygrades](https://twitter.com/Guidemygrades)

 [@Guidemygrades](https://www.youtube.com/Guidemygrades)

 info@guidemygrades.com

 Piccadilly Business Centre, Unit C, Aldow Enterprise Park
Blackett Street, Manchester, M12 6AE

HOW TO USE THIS BOOKLET

Questions are grouped by sub-topic and run from the lowest to the highest mark tariff, then oldest paper first.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme; < QUESTION brings you back.

A gold bar in the left margin marks exactly where the question sits on the original page.

Diagrams, graphs, grids and tables are printed exactly as they appear in the original papers.

Each question is filed under its main topic; longer questions may also test skills from other topics.

QUESTION	SOURCE	MARKS	PAGE
FP3.5a VECTOR PRODUCTS, LINES & PLANES		13 Qs · 137 marks	2
1. Q1	IAL FP3 (WFM03) January 2021	6	3
2. Q8	IAL FP3 (WFM03) October 2020	10	7
3. Q5	IAL FP3 (WFM03) October 2021	10	12
4. Q7	IAL FP3 (WFM03) January 2023	10	16
5. Q9	IAL FP3 (WFM03) June 2024	10	20
6. Q6	IAL FP3 (WFM03) January 2025	10	23
7. Q8	IAL FP3 (WFM03) June 2025	10	27
8. Q7	IAL FP3 (WFM03) January 2021	11	31
9. Q7	IAL FP3 (WFM03) January 2022	11	35
10. Q8	IAL FP3 (WFM03) June 2022	12	39
11. Q4	IAL FP3 (WFM03) June 2023	12	43
12. Q6	IAL FP3 (WFM03) January 2024	12	47
13. Q6	IAL FP3 (WFM03) June 2021	13	51



QUESTIONS

13 questions · 137 marks · 1 sub-topic

SUB-TOPIC FP3.5a

VECTOR PRODUCTS, LINES & PLANES

Sub-topic FP3.5a · 13 questions · 137 marks · lowest tariff first



Leave
blank

1. Relative to a fixed origin O , the points A, B, C and D have coordinates $(0, 4, 1)$, $(4, 0, 0)$, $(3, 5, 2)$ and $(2, 2, k)$ respectively, where k is a constant.

(a) Determine the exact area of triangle ABC .

(3)

(b) Determine in terms of k , the volume of the tetrahedron $ABCD$, simplifying your answer.

(3)



Leave
blank

Question 1 continued

Leave
blank

Lined area for writing the answer to Question 1 continued.

Leave
blank

Q1

Leave
blank

Leave
blank

Leave
blank

Question 8 continued

Leave
blank

Lined area for writing the answer to Question 8.

Q8

(Total 10 marks)

END

TOTAL FOR PAPER: 75 MARKS



Question 5 continued

Leave
blank

Handwriting practice area with 30 horizontal lines.



Leave
blank

Question 5 continued

Leave
blank

Lined area for writing the answer to Question 5.

(Total 10 marks)

Q5



Question 7 continued

Lined area for writing the answer to Question 7 continued.



Question 7 continued

Lined area for writing the answer to Question 7 continued.



(Total for Question 7 is 10 marks)



Question 9 continued

Lined area for writing the answer to Question 9 continued.



TOTAL FOR PAPER IS 75 MARKS



Question 6 continued

Lined area for writing the answer to Question 6 continued.





(Total for Question 6 is 16 marks)



Question 8 continued

Lined area for writing the answer to Question 8 continued.



Question 8 continued

Lined area for writing the answer to Question 8 continued.



Question 8 continued

Lined area for writing the answer to Question 8.

(Total for Question 8 is 10 marks)



Leave
blank

Leave
blank

Question 7 continued

Leave
blank

Lined area for writing the answer to Question 7.

(Total 11 marks)

Q7	
----	--



Leave
blank

Question 7 continued

Leave
blank

Lined area for writing the answer to Question 7 continued.



Question 7 continued

Leave
blank

Handwriting practice area with 25 horizontal lines.

(Total 11 marks)

Q7

Grading box with two empty cells.



Question 8 continued

Lined area for writing the answer to Question 8 continued.



Question 8 continued

Lined area for writing the answer to Question 8 continued.



Question 8 continued

Lined area for writing the answer to Question 8.

(Total for Question 8 is 12 marks)



Question 4 continued

Lined area for writing the answer to Question 4 continued.



Question 4 continued

Lined area for writing the answer to Question 4 continued.



(Total for Question 4 is 12 marks)



Question 6 continued

Lined area for writing the answer to Question 6 continued.



Question 6 continued

Lined area for writing the answer to Question 6 continued.



Question 6 continued

Lined area for writing the answer to Question 6.

(Total for Question 6 is 12 marks)



Question 6 continued

Leave
blank



Question 6 continued

Leave
blank

Lined area for writing the answer to Question 6 continued.



Question 6 continued

Leave
blank

Lined area for writing the answer to Question 6.

(Total 13 marks)

Q6



MARK SCHEMES

official mark schemes, in the same order as the questions



General Principles for Further Pure Mathematics Marking

(But note that specific mark schemes may sometimes override these general principles)

Method mark for solving 3 term quadratic:**1. Factorisation**

$(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to $x = \dots$

$(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to $x = \dots$

2. Formula

Attempt to use the correct formula (with values for a , b and c).

3. Completing the square

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0$, $q \neq 0$, leading to $x = \dots$

Method marks for differentiation and integration:**1. Differentiation**

Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)

2. Integration

Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)



Question Number	Scheme	Notes	Marks
8(a)	$\mathbf{n} = \begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -10+6 \\ -(2-9) \\ -2+15 \end{pmatrix}$	Attempt vector product between normal vectors	M1
	$= \begin{pmatrix} -4 \\ 7 \\ 13 \end{pmatrix}$	Correct vector	A1
	$x=0 \Rightarrow -5y+3z=11, -2y+2z=7$ $\Rightarrow y=-\frac{1}{4}, z=\frac{13}{4}$ or $y=0 \Rightarrow x+3z=11, 3x+2z=7$ $\Rightarrow x=-\frac{1}{7}, z=\frac{26}{7}$ or $z=0 \Rightarrow x-5y=11, 3x-2y=7$ $\Rightarrow x=1, y=-2$	Correct strategy to find a point on l	M1
		Correct position vector of point on l	A1
	$\mathbf{r} = \mathbf{i} - 2\mathbf{j} + \lambda(-4\mathbf{i} + 7\mathbf{j} + 13\mathbf{k})$	Correct equation. (follow through their position and direction vectors but must be “ $\mathbf{r} =$ ”)	A1ft
			(5)
ALT	$x=11+5y-3z$		
	$3x-2y+2z=7 \Rightarrow 3(11+5y-3z)-2y+2z=7$ $\Rightarrow y-\frac{7z}{13}=-\frac{26}{13} \quad \left(z=\frac{13y+26}{7} \right)$ Eliminate one variable		M1
	$x=11+5\left(-\frac{26}{13}+\frac{7z}{13}\right) \Rightarrow z=\frac{13-13x}{4}$	Obtain 2 correct expressions for one of the variables	A1
	$\frac{x-1}{-\frac{4}{13}} = \frac{y+2}{\frac{7}{13}} = z$	M1 Obtain a Cartesian equation for l A1 Correct equation	M1A1
	$\mathbf{r} = (\mathbf{i} - 2\mathbf{j}) + \lambda\left(-\frac{4}{13}\mathbf{i} + \frac{7}{13}\mathbf{j} + \mathbf{k}\right) \text{ oe}$	Deduce a vector equation for l Follow through their Cartesian equation	A1ft
			(5)

(b)	$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$	Correct vector joining P to Q	B1	
	$\begin{pmatrix} -4 \\ 7 \\ 13 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -40 \\ 5 \\ -15 \end{pmatrix}$	Attempt vector product between the direction of l and their $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$	M1	
		Correct vector	A1	
	$\sin \theta = \frac{ -40\mathbf{i} + 5\mathbf{j} - 15\mathbf{k} }{ -4\mathbf{i} + 7\mathbf{j} + 13\mathbf{k} \mathbf{i} + 2\mathbf{j} - 2\mathbf{k} }$	Angle between PQ and line n		
	$d = \overline{PQ} \sin \theta$			
	$d = \frac{ -40\mathbf{i} + 5\mathbf{j} - 15\mathbf{k} }{ -4\mathbf{i} + 7\mathbf{j} + 13\mathbf{k} } = \frac{1}{\sqrt{234}} \sqrt{40^2 + 5^2 + 15^2}$	Fully correct method for the distance	M1	
	$d = \frac{5\sqrt{481}}{39}$	Cao Allow equivalent exact forms e.g. $d = \frac{5\sqrt{74}}{\sqrt{234}}$	A1	
			(5)	
ALT 1	$\mathbf{r}_m = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \\ 13 \\ 7 \end{pmatrix} \text{ or } \mathbf{r}_n = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 1 \\ 13 \\ 7 \end{pmatrix}$	Vector equation for either line with their direction vector from (a)	B1ft	
	$\overline{OP} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \quad \overline{ON} = \begin{pmatrix} 3 - \frac{4}{7}\mu \\ 2 + \mu \\ 1 + \frac{13}{7}\mu \end{pmatrix} \quad \overline{NP} = \begin{pmatrix} -1 + \frac{4}{7}\mu \\ -2 - \mu \\ 2 - \frac{13}{7}\mu \end{pmatrix}$	Uses either P and the parametric form of a point on n OR Q and the parametric form of a point on m		
	$\begin{pmatrix} -1 + \frac{4}{7}\mu \\ -2 - \mu \\ 2 - \frac{13}{7}\mu \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 1 \\ 13 \\ 7 \end{pmatrix} = 0$	M1: Forms scalar product of vector NP and direction vector of l and equates to zero A1: Correct vectors	M1A1	
	$\Rightarrow \mu = \frac{56}{117}$	Solves	M1	
	$\Rightarrow d = \sqrt{\left(-\frac{85}{117}\right)^2 + \left(-\frac{290}{117}\right)^2 + \left(\frac{10}{9}\right)^2} = \frac{5\sqrt{481}}{39}$	Obtains the correct distance	A1	
				(5)
	Alternative for M1A1M1			
	$\overline{NP} = \begin{pmatrix} -1 + \frac{4}{7}\mu \\ -2 - \mu \\ 2 - \frac{13}{7}\mu \end{pmatrix} \Rightarrow d = \sqrt{\left(-1 + \frac{4}{7}\mu\right)^2 + (-2 - \mu)^2 + \left(2 - \frac{13}{7}\mu\right)^2} \Rightarrow d \text{ is min when } \Rightarrow \mu = \frac{56}{117}$ M1: Find d in terms of a parameter A1: correct expression M1: use calculus (or simplify and complete the square) to find the parameter corresponding to the min d			

ALT 2	Correct vector PQ		B1
	$\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 7 \\ 13 \end{pmatrix} = \begin{vmatrix} 1 & -4 \\ 2 & 7 \\ -2 & 13 \end{vmatrix} \cos \theta$	Forms the scalar product and attempts to evaluate the LHS	M1
	$\cos \theta = \frac{-16}{3\sqrt{234}}$	Correct value for $\cos \theta$ exact or decimal	A1
	$d = PQ \sin \theta = 3\sqrt{1 - \left(\frac{-16}{3\sqrt{234}}\right)^2} = \frac{5\sqrt{74}}{\sqrt{234}}$	M1: Correct method for the distance. A1: Correct EXACT distance	M1A1
			(5)
			Total 10



Question	Scheme	Marks
4(a)	$\begin{vmatrix} 2 & 0 & -1 \\ k & 3 & 2 \\ -2 & 1 & k \end{vmatrix} = 2 \begin{vmatrix} 3 & 2 \\ 1 & k \end{vmatrix} - 0 \begin{vmatrix} k & 2 \\ -2 & k \end{vmatrix} + (-1) \begin{vmatrix} k & 3 \\ -2 & 1 \end{vmatrix} = 2(3k-2) - (k+6) = \dots$	M1
	$= 6k - 4 - k - 6 = 5k - 10^*$	A1*
		(2)
(b)	$\mathbf{M}^T = \begin{pmatrix} 2 & k & -2 \\ 0 & 3 & 1 \\ -1 & 2 & k \end{pmatrix} \text{ or minors } \begin{pmatrix} 3k-2 & k^2+4 & k+6 \\ 1 & 2k-2 & 2 \\ 3 & 4+k & 6 \end{pmatrix} \text{ or}$ $\text{cofactors } \begin{pmatrix} 3k-2 & -k^2-4 & k+6 \\ -1 & 2k-2 & -2 \\ 3 & -4-k & 6 \end{pmatrix}$	M1
	$\text{Adjugate matrix is } \begin{pmatrix} 3k-2 & -1 & 3 \\ -k^2-4 & 2k-2 & -4-k \\ k+6 & -2 & 6 \end{pmatrix} (\geq 6 \text{ entries correct})$	M1
	$\text{Hence } \mathbf{M}^{-1} = \frac{1}{5k-10} \begin{pmatrix} 3k-2 & -1 & 3 \\ -k^2-4 & 2k-2 & -4-k \\ k+6 & -2 & 6 \end{pmatrix}$	dM1A1
		(4)
(c) Alt method	Images of A, B and C are $(5, 4k-18, 3k-16)$, $(0, 7-2k, 9-4k)$ and $(0, 4k-2, 8k-14)$	M1 A1
	$(\pm) 50 = \frac{1}{6} \begin{vmatrix} 5 & 4k-18 & 3k-16 \\ 0 & 7-2k & 9-4k \\ 0 & 4k-2 & 8k-14 \end{vmatrix} \Rightarrow (\pm) 300 = 5(\dots) (= 200k - 400) \Rightarrow k = \dots$	M1
	$(300 = 200k - 400 \Rightarrow) k = \frac{7}{2} \quad \text{or} \quad (-300 = 200k - 400 \Rightarrow) k = \frac{1}{2}$	A1
	$k = \frac{1}{2} \text{ and } k = \frac{7}{2}$	A1
		(5)
	Using volume scale factor. Attempts $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 4 & -8 & 3 \\ -2 & 5 & -4 \\ 4 & -6 & 8 \end{vmatrix} = 4(40-24) + 8(-16+16) + 3(12-20) = \dots$	M1
	$\text{Volume of } T \text{ is } \frac{1}{6} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \left \frac{1}{6} \begin{vmatrix} 4 & -8 & 3 \\ -2 & 5 & -3 \\ 4 & 6 & -8 \end{vmatrix} \right = \dots \frac{20}{3}$	A1
	$\text{Volume image of } T = \det \mathbf{M} \times \frac{20}{3} \Rightarrow \frac{20}{3} 5k-10 = 50 \Rightarrow k = \dots$	M1



Question Number	Scheme	Notes	Marks
5(b)(ii)	$(2-\lambda)(4-\lambda)(5-\lambda)+4(5-\lambda)+(-4-16+4\lambda)=0$ $\Rightarrow (5-\lambda)[(2-\lambda)(4-\lambda)+4-4]=0$ $\Rightarrow (5-\lambda)(2-\lambda)(4-\lambda)=0 \Rightarrow \lambda = \dots$	Uses their value of a in a recognisable attempt at a characteristic equation and achieves a real non-zero eigenvalue $\neq 2$. There must be some algebra but it may be poor.	M1
	4 and 5	Both correct (no extra) and from correct work	A1
	For information the cubic is $\pm(\lambda^3 - 11\lambda^2 + 38\lambda - 40) = 0$		(2)
(c)	$\mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = "4" \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ or } (\mathbf{A} - "4"\mathbf{I}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow$ $\begin{array}{rcl} 2x+2y+z=4x & & -2x+2y+z=0 \\ -2x+4y=4y & \text{or} & -2x=0 \\ 4x+2y+5z=4z & & 4x+2y+z=0 \end{array}$ OR $\mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = "5" \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ or } (\mathbf{A} - "5"\mathbf{I}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow$ $\begin{array}{rcl} 2x+2y+z=5x & & -3x+2y+z=0 \\ -2x+4y=5y & \text{or} & -2x-y=0 \\ 4x+2y+5z=5z & & 4x+2y+z=0 \end{array}$ Uses $\mathbf{Ax} = \lambda\mathbf{x}$ or $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ with their value of a and a real non-zero value of $\lambda \neq 2$ to obtain three simultaneous equations (allow if given as two equal vectors) Alternatively attempts vector product of two rows of $\mathbf{A} - "4"\mathbf{I}$		M1
	$\pm \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \text{ or } \pm \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix}$	One correct eigenvector. As shown or multiple or with components multiplied by e.g. " k " Accept e.g., $x=0, y=-1, z=2$	A1
	$\pm \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \text{ and } \pm \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix}$	Both correct eigenvectors. As shown or multiple or with components multiplied by e.g. k Accept $x=..., y=..., z=...$ Both these 2 A marks could be implied by their normalised eigenvectors	A1
	$\pm \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \pm \frac{1}{\sqrt{54}} \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix} \text{ oe}$	M1: A correct method to normalise at least one of their eigenvectors A1: Both correct. Allow any exact equivalents. Isw	M1 A1
	All marks available regardless of how $a = 2, \lambda_2 = 4$ & $\lambda_3 = 5$ have been obtained		(5)
			Total 12



Question Number	Scheme	Notes	Marks
6(a)	$\frac{dx}{d\theta} = \begin{cases} a(1 - \cos \theta) \\ \text{or} \\ a - a \cos \theta \end{cases} \quad \text{or} \quad \frac{dy}{d\theta} = a \sin \theta$	At least one correct derivative	B1
	$\begin{aligned} & a^2(1 - \cos \theta)^2 + (a \sin \theta)^2 \\ & = a^2(1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta) \\ & = 2a^2(1 - \cos \theta) \end{aligned}$	Squares and adds their derivatives and uses $\cos^2 \theta + \sin^2 \theta = 1$ to obtain an expression in $\cos \theta$ only (not $\cos^2 \theta$) Could be implied	M1
	$= 2a^2 \left(1 - \left(1 - 2 \sin^2 \left(\frac{\theta}{2} \right) \right) \right) = 4a^2 \sin^2 \frac{\theta}{2}$	dM1: Replaces $\cos \theta$ with $\pm 1 \pm 2 \sin^2 \frac{\theta}{2}$ or equivalent trig work (sign errors only on identities) to obtain an expression in $\sin^2 \frac{\theta}{2}$ only Requires previous M mark. Can be implied. A1: Achieves $4a^2 \sin^2 \frac{\theta}{2}$ or $k = 4$ from correct work	dM1 A1
			(4)
(b)	$\begin{aligned} \text{S.A.} &= (2\pi) \int y \sqrt{\left(\frac{dx}{d\theta} \right)^2 + \left(\frac{dy}{d\theta} \right)^2} d\theta \\ &= (2\pi) \int_{(0)}^{(2\pi)} a(1 - \cos \theta) \left(2a \sin \frac{\theta}{2} \right) d\theta \end{aligned}$	Applies $y \sqrt{\left(\frac{dx}{d\theta} \right)^2 + \left(\frac{dy}{d\theta} \right)^2}$ with their $ka^2 \sin^2 \frac{\theta}{2}$ and square roots. The result of the square root may be incorrect but must be of the form $p \sin \frac{\theta}{2}$ Allow a slip replacing y but they must not have used x , $\frac{dx}{d\theta}$ or $\frac{dy}{d\theta}$ for y Allow the letter k or an invented value. 2π may be absent or wrong. Integral not required.	M1
	$\begin{aligned} &= (2\pi) 2a^2 \int_{(0)}^{(2\pi)} \left(\sin \frac{\theta}{2} - \sin \frac{\theta}{2} \cos \theta \right) d\theta \\ &\Rightarrow (2\pi) 2a^2 \int_{(0)}^{(2\pi)} \left(\sin \frac{\theta}{2} - \sin \frac{\theta}{2} \left(2 \cos^2 \frac{\theta}{2} - 1 \right) \right) d\theta \\ &\text{or e.g., } \Rightarrow (2\pi) 2a^2 \int_{(0)}^{(2\pi)} 2 \sin^3 \frac{\theta}{2} d\theta \end{aligned}$	Uses trig identity/identities (condoning sign errors) to obtain an expression with arguments of $\frac{\theta}{2}$ only. Allow the letter k or an invented value. 2π may be absent or wrong. Integral not required. Dependent on previous M mark.	dM1
	Scheme continues...		

Question Number	Scheme	Notes	Marks	
6(a)	$\frac{x^2}{25} + \frac{y^2}{9} = 1 \quad P(5\cos\theta, 3\sin\theta)$			
	$\left\{ \frac{dx}{d\theta} = -5\sin\theta \quad \frac{dy}{d\theta} = 3\cos\theta \right\}$ $\frac{dy}{dx} = -\frac{3\cos\theta}{5\sin\theta}$	$\frac{2x}{25} + \frac{2y}{9} \frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{9x}{25y} \left\{ = -\frac{45\cos\theta}{75\sin\theta} \right\}$	$y = \left(9 - \frac{9}{25} x^2 \right)^{\frac{1}{2}}$ $\frac{dy}{dx} = \frac{-\frac{18}{25} x}{2\sqrt{9 - \frac{9}{25} x^2}} \left\{ = \frac{-\frac{18}{25} \times 5\cos\theta}{2\sqrt{9 - 9\cos^2\theta}} \right\}$	B1
	Any correct expression for $\frac{dy}{dx}$ in terms of θ , or x and y , or x . Allow for a correct $\frac{dx}{dy}$ or $-\frac{dx}{dy}$			
	$m_T = -\frac{3\cos\theta}{5\sin\theta} \Rightarrow m_N = \frac{5\sin\theta}{3\cos\theta}$		Correct perpendicular gradient rule for their $\frac{dy}{dx}$ in terms of θ May see $m_T = -\frac{3}{5} \cot\theta \Rightarrow m_N = \frac{5}{3} \tan\theta$	M1
	$y - 3\sin\theta = \frac{5\sin\theta}{3\cos\theta}(x - 5\cos\theta)$ OR $y = mx + c \Rightarrow 3\sin\theta = \frac{5\sin\theta}{3\cos\theta} \times 5\cos\theta + c \Rightarrow c = -\frac{16}{3}\sin\theta$ Correct straight line method with a changed gradient in terms of θ			M1
	$3y\cos\theta - 9\sin\theta\cos\theta = 5x\sin\theta - 25\sin\theta\cos\theta$ $\Rightarrow 5x\sin\theta - 3y\cos\theta = 16\sin\theta\cos\theta^*$	Reaches given answer with intermediate line of working and no errors. Allow this equation written in reverse, x and y terms in different order provided they are together with the third term on the other side and allow the products in a different order provided the numerical coefficients “5”, “-3” and “16” are at the front of the terms.		A1*
	The last three marks require $P(5\cos\theta, 3\sin\theta)$ to be substituted but condone using e.g. $\frac{25y}{9x}$ as the normal gradient when forming the straight line <u>provided</u> appropriate substitution is seen before the given answer.			
(4)				

Question Number	Scheme	Notes	Marks
6(b)	At Q, $x = 0 \Rightarrow y = -\frac{16}{3} \sin \theta$	Correct y coordinate of Q. Accept unsimplified	B1
	M is $\left(\frac{5 \cos \theta + 0}{2}, \frac{3 \sin \theta + (-\frac{16}{3} \sin \theta)}{2} \right)$ Accept $x = \frac{5}{2} \cos \theta$, $y = -\frac{7}{6} \sin \theta$	Correct method for midpoint for both coordinates with their y_Q . Could be implied. Alternatively , award for $\Delta OPM = \frac{1}{2} \Delta OPQ = \frac{1}{2} \times \frac{1}{2} \times \frac{16}{3} \sin \theta \times 5 \cos \theta$ (see area examples below)	M1
	e.g., PQ meets x-axis at $R \left(\frac{16}{5} \cos \theta, 0 \right)$ $\Rightarrow$ Area $\Delta OPM = \Delta OPR + \Delta OMR$ $= \frac{1}{2} \times \frac{16}{5} \cos \theta \left(3 \sin \theta + \frac{7}{6} \sin \theta \right)$	Correct unsimplified expression for area of ΔOPM Do not allow recovery from a negative area. Can only follow incorrect work i.e., an incorrect midpoint if $\Delta OPM = \frac{1}{2} \Delta OPQ$ is used. Please see below for alternatives	M1
	If shoelace method is used, score for a correct “extracted” expression for the area (allow with modulus if correct) e.g., $\frac{1}{2} \begin{vmatrix} 0 & 5 \cos \theta & \frac{5}{2} \cos \theta & 0 \\ 0 & 3 \sin \theta & -\frac{7}{6} \sin \theta & 0 \end{vmatrix}$ $\Rightarrow \frac{1}{2} \left (5 \cos \theta) \left(-\frac{7}{6} \sin \theta \right) - \left(\frac{5}{2} \cos \theta \right) (3 \sin \theta) \right $ or $\frac{1}{2} \left[(5 \cos \theta) \left(\frac{7}{6} \sin \theta \right) + \left(\frac{5}{2} \cos \theta \right) (3 \sin \theta) \right]$		
	$\left\{ = \frac{20}{3} \sin \theta \cos \theta = \frac{10}{3} \sin 2\theta \right\} \Rightarrow (\text{area}) = \frac{10}{3}$ Correct area <u>following a correct expression</u>		A1
	$\frac{10}{3}$ <u>and justification</u> : From $\frac{10}{3} \sin 2\theta$: max (value) of $\sin 2\theta$ is 1 or e.g., $-1 \leq \sin 2\theta \leq 1$ or states $\theta = \frac{\pi}{4}$ or 45° or obtains this using differentiation: $\left\{ \frac{10}{3} \right\} \sin 2\theta \Rightarrow \left\{ \frac{20}{3} \right\} \cos 2\theta = 0 \Rightarrow \dots$ Do not accept if there is any wrong statement e.g., $\sin 2\theta \leq 1$, $-1 < \sin 2\theta < 1$ but we will condone the ambiguous “ $\sin 2\theta$ is between 1 and -1” From any other expression : Must <u>differentiate</u> (unless rewrites as $\frac{10}{3} \sin 2\theta$) e.g., $\frac{20}{3} \sin \theta \cos \theta \Rightarrow \frac{20}{3} (\cos^2 \theta - \sin^2 \theta) \Rightarrow \frac{20}{3} \cos 2\theta = 0$ or $\tan^2 \theta = 1 \Rightarrow \theta = \frac{\pi}{4}$ or 45° Ignore any further differentiation to justify maximum		A1
(5)			
Total 9			
		May see: $\Delta OPM = \frac{1}{2} \Delta OPQ = \frac{1}{2} \times \frac{1}{2} \times \frac{16}{3} \sin \theta \times 5 \cos \theta$ (Scores the first 2 M marks together since M is not required – ignore an absent or wrong M) $\Delta OPM = \Delta OPQ - \Delta OMQ$ $\frac{1}{2} \times \frac{16}{3} \sin \theta \times 5 \cos \theta - \frac{1}{2} \times \frac{16}{3} \sin \theta \times \frac{5}{2} \cos \theta$ $\Delta OPM = \Delta PQS - \Delta OMQ - \Delta PSO$ $= \frac{1}{2} \times \left(\frac{16}{3} \sin \theta + 3 \sin \theta \right) \times 5 \cos \theta - \frac{1}{2} \times \frac{16}{3} \sin \theta \times \frac{5}{2} \cos \theta - \frac{1}{2} \times 3 \sin \theta \times 5 \cos \theta$ $\left\{ = \frac{125}{6} \sin \theta \cos \theta - \frac{20}{3} \sin \theta \cos \theta - \frac{15}{2} \sin \theta \cos \theta \right\}$ $\Delta OPM = \Delta PSTU - \Delta PSO - \Delta OMT - \Delta PMU$ $= 5 \cos \theta \times \left(3 \sin \theta + \frac{7}{6} \sin \theta \right) - \frac{1}{2} \times 3 \sin \theta \times 5 \cos \theta$ $- \frac{1}{2} \times \frac{5}{2} \cos \theta \times \frac{7}{6} \sin \theta - \frac{1}{2} \times \left(5 \cos \theta - \frac{5}{2} \cos \theta \right) \left(3 \sin \theta + \frac{7}{6} \sin \theta \right)$ $\left\{ = \left(\frac{125}{6} - \frac{15}{2} - \frac{35}{24} - \frac{125}{24} \right) \sin \theta \cos \theta \right\}$	
Note that attempts that start by using Pythagoras for PM will also require the perpendicular distance from O to the line			

Question Number	Scheme	Notes	Marks
7	$y = \ln\left(\tanh \frac{x}{2}\right) \quad 1 \leq x \leq 2$		
(a)	$\frac{dy}{dx} = \frac{1}{\tanh \frac{x}{2}} \times \frac{1}{2} \operatorname{sech}^2 \frac{x}{2} \text{ or e.g., } \frac{1}{2} \coth \frac{x}{2} \left(1 - \tanh^2 \frac{x}{2}\right)$ $\text{or } e^y = \tanh \frac{x}{2} \Rightarrow \left(\tanh \frac{x}{2}\right) \frac{dy}{dx} = \frac{1}{2} \operatorname{sech}^2 \frac{x}{2}$ $\text{or } \Rightarrow \operatorname{artanh}(e^y) = \frac{x}{2} \Rightarrow \left(\frac{e^y}{1 - e^{2y}}\right) \frac{dy}{dx} = \frac{1}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \coth \frac{x}{2} \left(1 - \tanh^2 \left(\frac{x}{2}\right)\right)$ Obtains an expression for (or equation involving) $\frac{dy}{dx}$ of appropriate form. Condone sign/coefficient errors only and any $\frac{x}{2}$s written as x but no “y”s. Do not condone missing “h”s in hyperbolic functions unless they are recovered		M1
	$\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \Rightarrow \int \sqrt{1 + \left(\frac{\operatorname{sech}^2 \frac{x}{2}}{2 \tanh \frac{x}{2}}\right)^2} (dx) \text{ or e.g., } \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Rightarrow \sqrt{1 + \left(\frac{\cosh \frac{x}{2}}{2 \sinh \frac{x}{2} \cosh^2 \frac{x}{2}}\right)^2}$ Applies arc length formula (with or without the integration sign) with their $\frac{dy}{dx}$ which may have been simplified incorrectly before substitution. Do not condone attempts that clearly have worked backwards to deduce that the derivative is cosech x. Also condone incorrect work processing $1 + \left(\frac{dy}{dx}\right)^2$ provided the expression is shown as square rooted afterwards. Not dependent. Ignore any multiplier such as π or 2π or k but forming $y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ is M0		M1
	$\sqrt{1 + \left(\frac{1}{2 \sinh \frac{x}{2} \cosh \frac{x}{2}}\right)^2} \rightarrow \sqrt{1 + \left(\frac{1}{\sinh x}\right)^2}$ Uses identity/identities (sign errors only) to obtain $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ in terms of x and not $\frac{x}{2}$. Attempts that square the derivative and add the 1 first before attempting to convert to x must be convincing. Requires both previous M marks.		ddM1
	$\sqrt{1 + \left(\frac{1}{\sinh x}\right)^2} = \sqrt{1 + \operatorname{cosech}^2 x} \Rightarrow s = \int_1^2 \coth x \, dx \text{ or e.g., } = \int \sqrt{\frac{\sinh^2 x + 1}{\sinh^2 x}} dx \Rightarrow s = \int_1^2 \coth x \, dx$ Obtains given answer with no errors and at least one non-trivial intermediate line following the first expression in terms of x. Allow without “s = ” but RHS must be exactly as printed. Allow a “meet in the middle” approach but it must be convincing and $\int_1^2 \coth x \, dx$ must be seen. No missing “h”s in hyperbolic functions or missing arguments even if recovered. We will condone the occasional notational error e.g., $\operatorname{sech} \frac{x}{2}$ if recovered		A1*
(4)			

Question Number	Scheme	Notes	Marks	
7(b)	$\int \coth x \, dx = \ln(\sinh x)$ Correct integration. May see $-\ln(\operatorname{cosech} x)$ May see the $\sinh x$ in exponentials without the “2” which may come from the substitution $u = e^x - e^{-x}$ i.e., $\ln(e^x - e^{-x})$		B1	
	1, 2 & 3. $\ln\left(\frac{e^2 - e^{-2}}{2}\right) - \ln\left(\frac{e - e^{-1}}{2}\right) = \ln\left(\frac{e^2 - \frac{1}{e^2}}{e - \frac{1}{e}}\right)$ or 4. $\ln(\sinh 2) - \ln(\sinh 1) = \ln\left(\frac{\sinh 2}{\sinh 1}\right)$ Following replacement of $\int \coth x \, dx$ with $\pm \ln(\sinh x)$, $\pm \ln(\cosh x)$, $\pm \ln(\operatorname{cosech} x)$ or $\pm \ln(\operatorname{sech} x)$, substitutes given limits, subtracts and writes as a single logarithm. Condone sign errors if exponential forms used and may use negative powers of e.		M1	
	1. $\ln\left(\frac{e^2 - \frac{1}{e^2}}{e - \frac{1}{e}}\right) = \ln\left(\frac{\frac{e^4 - 1}{e^2}}{\frac{e^2 - 1}{e}}\right) = \ln\left(\frac{e^4 - 1}{e^3 - e}\right)$ or 2. $\Rightarrow \ln\left(\frac{\left(e + \frac{1}{e}\right)\left(e - \frac{1}{e}\right)}{e - \frac{1}{e}}\right)$ or $\ln\left(\frac{(e + e^{-1})(e - e^{-1})}{(e - e^{-1})}\right)$ or 3. $\Rightarrow \ln\left(\frac{e^2 - e^{-2}}{e - e^{-1}} \times \frac{e + e^{-1}}{e + e^{-1}}\right) = \ln\left(\frac{(e^2 - e^{-2})(e + e^{-1})}{e^2 - e^{-2}}\right)$ or 4. $\ln\left(\frac{\sinh 2}{\sinh 1}\right) = \ln\left(\frac{2\sinh 1 \cosh 1}{\sinh 1}\right)$ Following use of correct exponential form for $\sinh/\cosh$: 1. Obtains a correct $\ln$ of a single fraction (or product of single fractions) with no negative powers of e or 2. Uses difference of two squares to correctly factorise numerator or 3. Applies correct multiplier to achieve expression shown or 4. Correctly replaces $\sinh 2$ with $2\sinh 1 \cosh 1$ allowing equivalent work e.g., $\frac{\sinh 2}{\sinh 1} = \sqrt{\frac{(2\cosh^2 1 - 1)^2 - 1}{\cosh^2 1 - 1}} = \sqrt{\frac{4\cosh^4 1 - 4\cosh^2 1}{\cosh^2 1 - 1}} \Rightarrow s = \ln \sqrt{4\cosh^2 1}$ Requires previous M mark.		dM1	
	1. $s = \ln\left(\frac{(e^2 + 1)(e^2 - 1)}{e(e^2 - 1)}\right) = \ln\left(e + \frac{1}{e}\right)$ or 2 & 3. $s = \ln\left(e + \frac{1}{e}\right)$ or 4. $s = \ln(2\cosh 1)$ or $\ln\left(2\left(\frac{e + e^{-1}}{2}\right)\right) = \ln\left(e + \frac{1}{e}\right)$		Obtains given answer from complete and correct work. Minimum for each route shown. Allow $\ln(e^{-1} + e)$	A1*
	Algebraic integration must be used			
	Note that there are potentially other ways e.g., factorising followed by log laws: $\ln\left(\frac{e^2 - e^{-2}}{2}\right) - \ln\left(\frac{e - e^{-1}}{2}\right) = \ln\left(\frac{1}{2}\left(e + \frac{1}{e}\right)\left(e - \frac{1}{e}\right)\right) - \ln\left(\frac{1}{2}\left(e - \frac{1}{e}\right)\right) \text{ M1}$ $= \ln\left(e + \frac{1}{e}\right) + \ln\left(\frac{1}{2}\left(e - \frac{1}{e}\right)\right) - \ln\left(\frac{1}{2}\left(e - \frac{1}{e}\right)\right) \text{ dM1} = \ln\left(e + \frac{1}{e}\right) \text{ A1*}$			
(4)				
Total 8				

Question Number	Scheme	Notes	Marks
8	$I_n = \int_0^k x^n (k-x)^{\frac{1}{2}} dx \quad n \geq 0$ If d(...) notation is used marks are only scored when it is removed. Please see overleaf if the split is done first		
(a)	$u = x^n \quad u' = nx^{n-1} \quad v' = (k-x)^{\frac{1}{2}} \quad v = -\frac{2}{3}(k-x)^{\frac{3}{2}}$ $I_n = \left[-\frac{2}{3}x^n (k-x)^{\frac{3}{2}} \right]_0^k - \int_0^k -\frac{2}{3}nx^{n-1} (k-x)^{\frac{3}{2}} dx$ M1: Uses parts in the correct direction to obtain an expression of the form $\pm \dots x^n (k-x)^{\frac{3}{2}} \pm \int \dots x^{n-1} (k-x)^{\frac{3}{2}} (dx)$ A1: Correct expression (limits not required on either part and 'dx' may be missing)		M1 A1
	$(I_n =) 0 + \frac{2}{3}n \int_0^k x^{n-1} (k-x)(k-x)^{\frac{1}{2}} dx$	Applies $(k-x)^{\frac{3}{2}} = (k-x)(k-x)^{\frac{1}{2}}$ to integral. Could be implied if work correct but do not accept going straight to $\frac{2}{3}nkI_{n-1} - \frac{2}{3}nI_n$ Requires previous M mark.	dM1
	$\frac{2}{3}n \int_0^k \left(kx^{n-1} (k-x)^{\frac{1}{2}} - x^n (k-x)^{\frac{1}{2}} \right) dx$ $\Rightarrow \frac{2}{3}n(kI_{n-1} - I_n) \text{ or } \frac{2}{3}knI_{n-1} - \frac{2}{3}nI_n \text{ or}$ $\frac{2}{3}kn \int_0^k x^{n-1} (k-x)^{\frac{1}{2}} (dx) - \frac{2}{3}n \int_0^k x^n (k-x)^{\frac{1}{2}} (dx)$	Expands and writes RHS in terms of both I_n and I_{n-1} i.e., RHS = $\dots I_{n-1} \pm \dots I_n$ with no other terms. This mark is not available until the $\left[\dots x^n (k-x)^{\frac{3}{2}} \right]_0^k$ disappears. Allow if actual integrals are used for both I_n and/or I_{n-1} and allow going straight to $\frac{2}{3}knI_{n-1} - \frac{2}{3}nI_n$ provided the split was seen. Requires both previous M marks.	ddM1
	$\Rightarrow \left(1 + \frac{2}{3}n \right) I_n = \frac{2}{3}knI_{n-1} \text{ or } \Rightarrow \frac{3+2n}{3} I_n = \frac{2}{3}knI_{n-1}$ $\Rightarrow I_n = \frac{2kn}{3+2n} I_{n-1} *$ Reaches given answer with no mathematical errors seen. Allow poor bracketing if it is recovered. There must be at least one non-trivial intermediate line where the LHS = f(n) I_n allowing e.g., $I_n + \frac{2}{3}I_n = \dots$ Allow minor variations in given answer e.g., $I_n = \frac{2nkI_{n-1}}{2n+3}$ Condone missing 'dx's and allow if limits only seen once but $\left[-\frac{2}{3}x^n (k-x)^{\frac{3}{2}} \right]_0^k$ must be replaced by "0" or better		A1*
(5)			

Question Number	Scheme/Notes	Marks
8(a) Alt Split first	$I_n = \int_0^k x^n (k-x)^{\frac{1}{2}} dx = \int_0^k x^n (k-x)(k-x)^{-\frac{1}{2}} dx = \int_0^k kx^n (k-x)^{-\frac{1}{2}} dx - \int_0^k x^{n+1} (k-x)^{-\frac{1}{2}} dx$ $= \left[-2kx^n (k-x)^{\frac{1}{2}} \right]_0^k + \int_0^k 2knx^{n-1} (k-x)^{\frac{1}{2}} dx + \left[2x^{n+1} (k-x)^{\frac{1}{2}} \right]_0^k - \int_0^k 2(n+1)x^n (k-x)^{\frac{1}{2}} dx$ $\Rightarrow 0 + 2knI_{n-1} + 0 - 2(n+1)I_n \Rightarrow (3+2n)I_n = 2knI_{n-1} \Rightarrow I_n = \frac{2kn}{3+2n} I_{n-1}^*$ For attempts like this award the first 2 method marks together for applying the split, expanding and applying parts to achieve a correct form. The first accuracy mark can be awarded for a correct expression (limits not required on either part and 'dx's may be missing). As main scheme for the following two marks (note that in this case the first and third terms must both be replaced by "0" or better). There is no mark for just applying the split.	(5)



Question Number	Scheme	Notes	Marks
8(b)	$\int_0^k x^2 (k-x)^{\frac{1}{2}} dx = \frac{9\sqrt{3}}{280} \quad I_n = \frac{2kn}{3+2n} I_{n-1}$		
	$I_2 = \frac{4k}{7} I_1 = \frac{4k}{7} \left(\frac{2k}{5} I_0 \right)$ or $I_2 = \frac{4k}{7} I_1, \quad I_1 = \frac{2k}{5} I_0$	Attempts I_2 in terms of I_0 or I_2 in terms of I_1 and I_1 in terms of I_0 Accept with their I_0 substituted if I_0 attempted first. Allow $I_0 = 1$ to be used (i.e., I_0 lost) See note below if only see I_2 in terms of I_1	M1
	$I_0 = \int_0^k (k-x)^{\frac{1}{2}} dx = \left[-\frac{2}{3} (k-x)^{\frac{3}{2}} \right]_0^k$	$I_0 = \dots (k-x)^{\frac{3}{2}}$ Limits do not have to be seen or applied	M1
	$I_2 = \frac{8k^2}{35} \times \frac{2}{3} k^{\frac{3}{2}} \Rightarrow \frac{16}{105} k^{\frac{7}{2}} = \frac{9\sqrt{3}}{280} \Rightarrow k = \dots$ Solves an equation of the form $\frac{a}{b} k^{\frac{c}{2}} = \frac{9\sqrt{3}}{280}$ where $a, b \in \mathbb{Z}^+, \frac{a}{b} \notin \mathbb{Z}, c = 5 \text{ or } 7$ and where the LHS is their I_2. No processing or working requirements just look for a <u>value or numerical expression</u> for k from an appropriate equation. May see $k = e^{\frac{2}{7} \ln \left(\frac{27\sqrt{3}}{128} \right)}$ or other logarithmic work. Requires both previous M marks. Note that $\frac{16}{105} k^{\frac{5}{2}} = \frac{9\sqrt{3}}{280} \Rightarrow k = \sqrt[5]{\frac{2187}{16384}}$ or 0.668...		ddM1
	$k^{\frac{7}{2}} = \frac{27\sqrt{3}}{128} \Rightarrow k^7 = \frac{2187}{16384} \Rightarrow k = \frac{3}{4}$	Correct exact <u>value</u> for k from a correct equation. Not $\sqrt[7]{\frac{2187}{16384}}$ nor $\pm \frac{3}{4}$	A1
	Note that if I_2 is only found in terms of I_1 then award the first two marks together when a correct form for I_1 is achieved i.e., $\dots x(k-x)^{\frac{3}{2}} + \dots (k-x)^{\frac{5}{2}} \text{ or } \dots (\dots x + \dots k)(k-x)^{\frac{3}{2}}$ Using parts: $I_1 = \left[-\frac{2}{3} x(k-x)^{\frac{3}{2}} - \frac{4}{15} (k-x)^{\frac{5}{2}} \right]_0^k = \frac{4}{15} k^{\frac{5}{2}}$ Using substitution: $u = k-x \Rightarrow I_1 = \int_0^k x(k-x)^{\frac{1}{2}} dx = \left[-\frac{2}{15} (3x+2k)(k-x)^{\frac{3}{2}} \right]_0^k = \frac{4}{15} k^{\frac{5}{2}}$ There are no marks if the reduction formula is not used including direct attempts at I_2 or if $k = \frac{3}{4}$ is arrived at by purely solving the integral equation on a calculator		
			(4)
			Total 9

Question Number	Scheme	Notes	Marks
9	May use i, j, k notation		
9(a)	$\mathbf{n} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \dots \quad \left\{ \begin{pmatrix} 2 \\ -5 \\ -6 \end{pmatrix} \right\}$	Calculates the vector product of two vectors in Π_1 (two components correct)	M1
	$\begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ -6 \end{pmatrix} = \dots \quad \{-5\}$	Calculates the scalar product of a point in the plane and their normal. Not dependent but must follow an attempt at a vector product which could be poor, e.g., $3\mathbf{i} + 2\mathbf{k}$. Value must be correct if there is no indication of a correct method to evaluate the scalar product.	M1
	$\mathbf{r} \cdot \begin{pmatrix} 2 \\ -5 \\ -6 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ -6 \end{pmatrix} \Rightarrow 2x - 5y - 6z = -5$	Any correct Cartesian equation, e.g., $-2x + 5y + 6z = 5$ $2x - 5y - 6z + 5 = 0$	A1
(3)			
Alt Sim eqns	$\begin{aligned} x &= 5 + 3s + t \\ y &= 3 - 2t \quad \Rightarrow \text{e.g., } y + z = 3 + s \\ z &= s + 2t \end{aligned}$	Forms simultaneous equations in x, y, z, s and t and obtains an equation that eliminates at least one of s and t	M1
	$\begin{aligned} x &= 5 + 3(y + z - 3) + \frac{1}{2}z - \frac{1}{2}(y + z - 3) \\ x &= \frac{5}{2}y + 3z - \frac{5}{2} \end{aligned}$	M1: Proceeds to an equation in x, y and z only A1: Any correct equation with terms collected	M1 A1
(3)			

Question Number	Scheme	Notes	Marks
9(b) Way 1	$2x - 5y - 6z = -5, \quad 5x - 2y + 3z = 1$ $\Rightarrow$ e.g., $12x - 9y = -3$	Uses both plane equations to eliminate one variable. May see $21y + 36z = 27, \quad 21x + 27z = 15$	M1
	$\text{e.g., } 4x - 3y = -1 \Rightarrow x = \frac{3y-1}{4} \Rightarrow y = \frac{4x+1}{3}$ $3z = 1 - \frac{5(3y-1)}{4} + 2y = \frac{4-15y+5+8y}{4} \Rightarrow z = \frac{9-7y}{12} \Rightarrow y = \frac{12z-9}{-7}$ Expresses one variable in terms of the other two (single underlining) or expresses two variables in terms of the other one (double underlining). This work may be seen by setting a variable equal to a parameter to find the other variables in terms of the parameter (or the parameter in terms of the other two variables) e.g., $y = \lambda, \quad x = f(\lambda), \quad z = g(\lambda) \quad \left\{ \Rightarrow x = \frac{-1+3\lambda}{4}, y = \lambda, z = \frac{9-7\lambda}{12} \right\}$ $y = \lambda, \quad \lambda = f(x), \quad \lambda = g(z) \quad \left\{ \Rightarrow \lambda = \frac{4x+1}{3}, y = \lambda, \lambda = \frac{12z-9}{-7} \right\}$ See examples below. Requires previous M mark.		ddM1
	$\text{e.g., } \frac{4x+1}{3} = y = \frac{12z-9}{-7} \Rightarrow \frac{x+\frac{1}{4}}{\frac{3}{4}} = \frac{y-0}{1} = \frac{z-\frac{3}{4}}{-\frac{7}{12}}$ or e.g., $x = \frac{-1+3\lambda}{4}, y = \lambda, z = \frac{9-7\lambda}{12} \Rightarrow$ $\Rightarrow \mathbf{r} = \begin{pmatrix} -\frac{1}{4} \\ 0 \\ \frac{3}{4} \end{pmatrix} + \lambda \begin{pmatrix} \frac{3}{4} \\ 1 \\ -\frac{7}{12} \end{pmatrix} \text{ or e.g., } \mathbf{r} = \begin{pmatrix} -\frac{1}{4} \\ 0 \\ \frac{3}{4} \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ 12 \\ -7 \end{pmatrix}$	ddM1: Correct method to form RHS of vector equation. Allow slips but must not be a clearly incorrect method (e.g., point and direction confused, all non-zero point coordinates the wrong sign, no attempt seen or implied to obtain single coefficients for the variables in the numerator where necessary). Allow this mark if the point is later changed by multiplication e.g., $(-\frac{1}{4}, 0, \frac{3}{4})$ becomes $(-1, 0, 3)$ Condone missing $\mathbf{r} = \dots$ Allow this mark if $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} (=0)$ or $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$ are appropriately used. Requires both previous M marks. A1: Any correct equation (with any parameter). Do not condone e.g., $l = \dots$ Do not isw if the point is changed by multiplication.	ddM1 A1
examples	$x = \frac{3y-1}{4} = \frac{5-9z}{7} \Rightarrow \frac{x-0}{1} = \frac{y-\frac{1}{3}}{\frac{4}{3}} = \frac{z-\frac{5}{9}}{-\frac{7}{9}} \text{ or } x = \lambda, y = \frac{4\lambda+1}{3}, z = \frac{5-7\lambda}{9} \Rightarrow \mathbf{r} = \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{5}{9} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ \frac{4}{3} \\ -\frac{7}{9} \end{pmatrix}$ $\frac{5-7x}{9} = \frac{9-7y}{12} = z \Rightarrow \frac{x-\frac{5}{7}}{-\frac{9}{7}} = \frac{y-\frac{9}{7}}{-\frac{12}{7}} = \frac{z-0}{1} \text{ or } x = \frac{5-9\lambda}{7}, y = \frac{12z-9}{-7}, z = \lambda \Rightarrow \mathbf{r} = \begin{pmatrix} \frac{5}{7} \\ \frac{9}{7} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -\frac{9}{7} \\ -\frac{12}{7} \\ 1 \end{pmatrix}$		(4)

Question Number	Scheme	Notes	Marks
9(b) Way 2 Finds point and takes vector product of normals	Work may be minimal if they obtain a correct point. But do not accept just sight of an incorrect point without some evidence of an appropriate method to obtain it.		
	$2x - 5y - 6z = -5, \quad 5x - 2y + 3z = 1$ Let $y = 0 \Rightarrow 2x - 6z = -5, \quad 5x + 3z = 1$ or $\Rightarrow$ e.g., $12x - 9y = -3$	Assigns a value to one variable to obtain two equations in the other variables or eliminates one variable as in Way 1.	M1
	$\Rightarrow 12x = -3 \Rightarrow x = -\frac{1}{4}, y = 0, z = \frac{3}{4}$ May see $(0, \frac{1}{3}, \frac{5}{9})$ or $(\frac{5}{7}, \frac{9}{7}, 0)$	Solves or assigns a value to one variable to find values for the other variables. There is no need to check a point that arises from no working provided it is clear that the previous M mark has been scored. Requires previous M mark.	dM1
	Note that a point could be obtained via substituting the given form of Π_1 into Π_2 and expanding (M1) and then finding values of s and t that satisfy the equation and then finding a point (dM1)		
	$\begin{pmatrix} 2 \\ -5 \\ -6 \end{pmatrix} \times \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -27 \\ -36 \\ 21 \end{pmatrix} \Rightarrow \begin{pmatrix} -\frac{1}{4} \\ 0 \\ \frac{3}{4} \end{pmatrix} + \lambda \begin{pmatrix} -27 \\ -36 \\ 21 \end{pmatrix}$	Calculates vector product of normals (two components correct) and forms RHS of vector equation (allowing for copying slips but must not confuse point and direction). Allow this mark if the point is later changed by multiplication. Condone missing $\mathbf{r} = \dots$ Allow this mark if $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} (= 0)$ or $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$ are appropriately used. Requires both previous M marks.	ddM1
	$\Rightarrow \mathbf{r} = \begin{pmatrix} -\frac{1}{4} \\ 0 \\ \frac{3}{4} \end{pmatrix} + \lambda \begin{pmatrix} -27 \\ -36 \\ 21 \end{pmatrix}$ or e.g., $\mathbf{r} = \begin{pmatrix} -\frac{1}{4} \\ 0 \\ \frac{3}{4} \end{pmatrix} + \lambda \begin{pmatrix} -9 \\ -12 \\ 7 \end{pmatrix}$	Any correct equation in this form (with any parameter). Do not condone e.g., $l = \dots$ Do not isw if the point is changed by multiplication. Correct points will have the form $(\frac{3\alpha-1}{4}, \alpha, \frac{9-7\alpha}{12})$	A1
(4)			
Way 3 2 points	Finding 2 points on the line and subtract for direction e.g., Finds $(-\frac{1}{4}, 0, \frac{3}{4})$ (M1dM1 as Way 2) Then finds $(0, \frac{1}{3}, \frac{5}{9}) \Rightarrow$ direction $= (\frac{1}{4}, \frac{1}{3}, -\frac{7}{36}) \Rightarrow$ forms RHS of vector equation (ddM1) Then A1 for a correct equation		
			(4)
	Correct points/positions include:		
	$\begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{5}{9} \end{pmatrix} \quad \begin{pmatrix} -\frac{1}{4} \\ 0 \\ \frac{3}{4} \end{pmatrix} \quad \begin{pmatrix} \frac{5}{7} \\ \frac{9}{7} \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ \frac{5}{3} \\ -\frac{2}{9} \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{6} \end{pmatrix} \quad \begin{pmatrix} -\frac{4}{7} \\ -\frac{3}{7} \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ -1 \\ \frac{4}{3} \end{pmatrix} \quad \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$		

Question Number	Scheme	Notes	Marks
9(c)	Note that use of their line from part (b) must be seen to score any marks in (c)		
	$\mathbf{r} = \begin{pmatrix} -\frac{1}{4} \\ 0 \\ \frac{3}{4} \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ 12 \\ -7 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} + 9\lambda \\ 12\lambda \\ \frac{3}{4} - 7\lambda \end{pmatrix}$ $4\left(-\frac{1}{4} + 9\lambda\right) - 3(12\lambda) - \left(\frac{3}{4} - 7\lambda\right) = 0 \Rightarrow 7\lambda = \frac{7}{4} \Rightarrow \lambda = \frac{1}{4}$	Substitutes the parametric form of their line (allow slips but must not clearly confuse position and direction) from (b) into Π_3 and solves for λ The “=0” could be implied by a solution.	M1
	$\Rightarrow \left(9\left(\frac{1}{4}\right) - \frac{1}{4}, 12\left(\frac{1}{4}\right), -7\left(\frac{1}{4}\right) + \frac{3}{4}\right) = \dots$ Substitutes their λ into their line and obtains a point/position vector with values for all coordinates/components. If there is no working at least two coordinates/components should be consistent with their equation or parametric form. Isw if the point/position is altered by multiplication. Requires previous M mark.		dM1
	(2, 3, -1)	Correct point. No others. Allow $x = \dots$, $y = \dots$, $z = \dots$ and condone as a position vector. Do not isw.	A1
(3)			
Total 10			
PAPER TOTAL 75			

Question	Scheme	Marks
6(a)	$\begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 - (-14) \\ -(6 - 21) \\ -4 - 0 \end{pmatrix} = \begin{pmatrix} 14 \\ 15 \\ -4 \end{pmatrix}$	M1; A1
		(2)
(b)	Direction of l is $\pm \left(\begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} \right) = \pm \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$	B1
	Angle between line and normal to plane is given by $(\cos \beta) = \frac{14 \times 2 + 15 \times 2 - 4 \times 1}{\sqrt{437} \sqrt{9}} = \frac{18}{\sqrt{437}}$	M1
	So angle between plane and line is $\alpha = 90^\circ - \beta = \dots$	M1
	$= 59^\circ$	A1
		(4)
(c)	$\left\{ \begin{array}{l} \text{Equation of plane is } \mathbf{r} \cdot \begin{pmatrix} 14 \\ 15 \\ -4 \end{pmatrix} = \end{array} \right\} \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ 15 \\ -4 \end{pmatrix} = 53$	M1 A1
	So perpendicular distance is $\frac{ 14 \times 1 + 15 \times 7 - 4 \times 3 - 53 }{\sqrt{437}}$	M1
	$= \frac{54}{\sqrt{437}}$	A1
		(4)
(10 marks)		
Notes		
(a) M1: Attempts the vector product of the two directions of the plane. If no method shown, at least two correct coordinates implies the method. Alternatively, sets up and solves a pair of simultaneous equations using dot products of a general vector with the directions. E.g. $\left. \begin{array}{l} (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (2\mathbf{i} + 7\mathbf{k}) = 0 \\ (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 0 \end{array} \right\} \Rightarrow \begin{array}{l} 2a + 7c = 0 \\ 3a - 2b + 3c = 0 \end{array} \Rightarrow a = \dots c, b = \dots c$ A1: Correct normal to the plane. (b) B1: Correct direction vector for l, either direction.		

M1: Attempts a relevant dot product of the direction of l and normal to the plane. If no method shown the value given must be correct for their vectors. Need not be named correctly, so accept if they think the angle between these is what they need for this mark.

M1: Correct method to find the required angle. Must be using the correct vectors, but may be scored if the previous M is withheld for incorrect method for the scalar product if they have stated this is what they were attempting to find. Look for use of $\cos \beta$ and subtracting from 90° , or for use of $\sin \alpha$ with the dot product.

A1: Correct final angle (awrt). You may isw incorrect subsequent rounding following a correct answer.

(c)

M1: Attempts to find “ $\pm d$ ” for the equation of the plane with their normal from (a) and the point B .

A1: Correct value or equation for plane.

M1: Attempts the perpendicular distance formula or any other full method for the distance, allowing

minor slips (e.g. confusion in sign of d). E.g. $\frac{|\vec{OA} \cdot \vec{n} - \vec{OB} \cdot \vec{n}|}{|\vec{n}|} = \dots$ is an equivalent method.

A1: Correct answer.

Alt (c)	$ \vec{AB} = \sqrt{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}} = \sqrt{4+4+1} = 3$	M1 A1
	So perpendicular distance is $ \vec{AB} \sin \alpha = \vec{AB} \cos(90 - \alpha) = \vec{AB} \cos \beta = 3 \times \frac{18}{\sqrt{437}}$	M1
	$= \frac{54}{\sqrt{437}}$	A1
		(4)

M1: Correct method for the distance AB

A1: Correct AB .

M1: Attempts correct trig ratio with the exact answer for the relevant cosine from part (b).

A1: Correct answer.

Alt 2 (c)	$\hat{n} = \frac{\vec{n}}{ \vec{n} } = \frac{14\mathbf{i} + 15\mathbf{j} - 4\mathbf{k}}{\sqrt{437}}$ or $\vec{AB} \cdot \hat{n} = (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot (14\mathbf{i} + 15\mathbf{j} - 4\mathbf{k}) = 54$	M1 A1
	So perpendicular distance is $ \vec{AB} \cdot \hat{n} = \frac{ (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot (14\mathbf{i} + 15\mathbf{j} - 4\mathbf{k}) }{\sqrt{437}} = \dots$	M1
	$= \frac{54}{\sqrt{437}}$	A1
		(4)

M1: Attempts the unit normal vector or correct attempt at $\vec{AB} \cdot \hat{n}$.

A1: Correct vector or correct value

M1: Applies formula for distance between planes.

A1: Correct answer.



Alt 3 (c)	$\left\{ \begin{array}{l} \text{Equation of plane is } \mathbf{r} \cdot \begin{pmatrix} 14 \\ 15 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ 15 \\ -4 \end{pmatrix} = 53 \end{array} \right.$	M1 A1
	Let X be where shortest distance to plane occurs. Then $\overrightarrow{AX} = \mathbf{i} + 7\mathbf{j} + 3\mathbf{k} - t(14\mathbf{i} + 15\mathbf{j} - 4\mathbf{k})$ So X is where $((1+14t)\mathbf{i} + (7+15t)\mathbf{j} + (3-4t)\mathbf{k}) \cdot (14\mathbf{i} + 15\mathbf{j} - 4\mathbf{k}) = 53$ $\Rightarrow 437t + 107 = 53 \Rightarrow t = -\frac{54}{437}$ $\overrightarrow{AX} = -\frac{54}{437}(14\mathbf{i} + 15\mathbf{j} - 4\mathbf{k}) \Rightarrow \overrightarrow{AX} = \frac{54}{437} \times \sqrt{437} = \dots$	M1
	$= \frac{54}{\sqrt{437}}$	A1
		(4)

M1: Attempts to find “ d ” for the equation of the plane with their normal from (a) and the point B .
A1: Correct value / equation of plane implied.
M1: Sets up equation for line through A and perpendicular to the plane and forms and solves with the equation of the plane to find the parameter for which this line intersects the plane and attempts the length of AX .
A1: Correct answer.



Question Number	Scheme	Marks
7(a)	$y = \cos 2x \Rightarrow \frac{dy}{dx} = -2 \sin 2x$ and then also: $S = 2\pi \int_{(0)}^{\left(\frac{\pi}{4}\right)} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} (dx)$ and $\left(\frac{dy}{dx}\right)^2 = \dots \Rightarrow S = \dots$ or $S = 2\pi \int_{(0)}^{\left(\frac{\pi}{4}\right)} \cos 2x \sqrt{1 + (-2 \sin 2x)^2} (dx) \Rightarrow S = \dots$	M1
	$S = 2\pi \int_0^{\frac{\pi}{4}} \cos 2x \sqrt{1 + 4 \sin^2 2x} dx^*$	A1*
		(2)
(b)	$2 \sin 2x = \sinh \theta \Rightarrow 4 \cos 2x \frac{dx}{d\theta} = \cosh \theta$ o.e.	B1
	$S = 2\pi \int \cos 2x \sqrt{1 + 4 \sin^2 2x} dx = 2\pi \int \cancel{\cos 2x} \sqrt{1 + \sinh^2 \theta} \frac{\cosh \theta}{4 \cancel{\cos 2x}} d\theta$	M1
	$= \frac{\pi}{2} \int \cosh \theta \sqrt{1 + \sinh^2 \theta} d\theta = \frac{\pi}{2} \int \cosh^2 \theta d\theta$	A1
ALT for first 3 marks	$x = \frac{1}{2} \arcsin\left(\frac{1}{2} \sinh \theta\right) \Rightarrow \frac{dx}{d\theta} = \frac{1}{2} \frac{\frac{1}{2} \cosh \theta}{\sqrt{1 - \frac{1}{4} \sinh^2 \theta}}$	B1
	$S = 2\pi \int \cos 2x \sqrt{1 + 4 \sin^2 2x} dx$ $= 2\pi \int \sqrt{1 - \frac{1}{4} \sinh^2 \theta} \sqrt{1 + \sinh^2 \theta} \frac{1}{2} \frac{\frac{1}{2} \cosh \theta}{\sqrt{1 - \frac{1}{4} \sinh^2 \theta}} d\theta$	M1
	$= 2\pi \int \cancel{\cos 2x} \sqrt{1 + 4 \sin^2 2x} \frac{\sqrt{1 + 4 \sin^2 2x}}{4 \cancel{\cos 2x}} d\theta$ $= 2\pi \int \cancel{\cos 2x} \frac{1 + \sinh^2 \theta}{4 \cancel{\cos 2x}} d\theta$	M1
	$= \frac{\pi}{2} \int \cosh^2 \theta d\theta$	A1
	$= \frac{\pi}{2} \int \left(\frac{1}{2} + \frac{1}{2} \cosh 2\theta\right) d\theta$ or $= \frac{\pi}{2} \int \left(\frac{e^\theta + e^{-\theta}}{2}\right)^2 d\theta = \frac{\pi}{2} \int \left(\frac{e^{2\theta} + 2 + e^{-2\theta}}{4}\right) d\theta$	M1
	$= \frac{\pi}{4} \left[\theta + \frac{1}{2} \sinh 2\theta\right]$ or $\frac{\pi}{2} \left[\frac{1}{8} e^{2\theta} + \frac{1}{2} \theta - \frac{1}{8} e^{-2\theta}\right]$	A1
	$\frac{\pi}{4} \left[\theta + \frac{1}{2} \sinh 2\theta\right]_{\text{ar sinh } 2}^{\text{ar sinh } 2} = \frac{\pi}{4} \left(\text{ar sinh } 2 + \frac{1}{2} \sinh(2 \text{ar sinh } 2)\right)$ or $\frac{\pi}{2} \left[\frac{1}{8} e^{2\theta} + \frac{1}{2} \theta - \frac{1}{8} e^{-2\theta}\right]_{\text{ar sinh } 2}^{\text{ar sinh } 2} = \frac{\pi}{2} \left(\frac{1}{8} e^{2 \ln(2+\sqrt{5})} + \frac{1}{2} \ln(2+\sqrt{5}) - \frac{1}{8} e^{-2 \ln(2+\sqrt{5})}\right)$	M1
	$= \frac{\pi}{4} (\ln(2+\sqrt{5}) + 2\sqrt{5})$	A1
		(7)
		Total 9

Notes

(a)

Note: for part (a) there are two elements that must be combined to score the M mark

- Differentiation of y to the correct form
- Showing the substitution of their derivative and y into a correct formula (which could be implied by a derivative of the correct form, a correct $\left(\frac{dy}{dx}\right)^2$ for their derivative and a correct SA formula quoted).

M1: Differentiates y to obtain $y = \cos 2x \Rightarrow \frac{dy}{dx} = -2 \sin 2x$ and substitutes into the correct surface area formula.

Must see evidence of using the correct formula, which could be implied by a correct substitution with their derivative and y substituted- simply stating the result after differentiation (even if correct differentiation) is M0A0 without some evidence of application of their derivative to the correct SA formula. Differentiation must be seen, and the 2π but condone the omission of the limits and/or the dx for this mark.

A1*: Reaches the printed answer with no errors or omissions. Fully correct working shown, including limits, and the dx etc. Must be fully shown. Some exemplars are below:

$$y = \cos 2x \Rightarrow \frac{dy}{dx} = -2 \sin 2x \Rightarrow S = 2\pi \int_0^{\frac{\pi}{4}} \cos 2x \sqrt{1 + 4 \sin^2 2x} \, dx \quad \text{M0A0*}$$

(Correct derivative but no substitution shown, no use of formula seen, states given answer with no justification)

$$y = \cos 2x \Rightarrow \frac{dy}{dx} = -2 \sin 2x \Rightarrow S = 2\pi \int_0^{\frac{\pi}{4}} \cos 2x \sqrt{1 + (-2 \sin 2x)^2} \, dx = 2\pi \int_0^{\frac{\pi}{4}} \cos 2x \sqrt{1 + 4 \sin^2 2x} \, dx \quad \text{M1A1*}$$

(Correct derivative, use of correct formula seen - implied by a correct substitution with their derivative and y)

$$y = \cos 2x \Rightarrow \frac{dy}{dx} = 2 \sin 2x \Rightarrow S = 2\pi \int_0^{\frac{\pi}{4}} \cos 2x \sqrt{1 + 4 \sin^2 2x} \, dx \quad \text{M0A0*}$$

(Incorrect derivative, no use of formula seen by correct substitution, states given result with no justification)

$$y = \cos 2x \Rightarrow \frac{dy}{dx} = 2 \sin 2x \Rightarrow S = 2\pi \int_0^{\frac{\pi}{4}} \cos 2x \sqrt{1 + (2 \sin 2x)^2} \, dx = 2\pi \int_0^{\frac{\pi}{4}} \cos 2x \sqrt{1 + 4 \sin^2 2x} \, dx \quad \text{M1A0*}$$

(Incorrect derivative, use of correct formula implied by a correct substitution with their derivative and y)

$$y = \cos 2x \Rightarrow \frac{dy}{dx} = -2 \sin 2x \text{ and } S = 2\pi \int_0^{\frac{\pi}{4}} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \Rightarrow S = 2\pi \int_0^{\frac{\pi}{4}} \cos 2x \sqrt{1 + 4 \sin^2 2x} \, dx \quad \text{M0A0*}$$

(Correct derivative and formula seen but no evidence of using the correct formula to obtain the given result)

$$y = \cos 2x \Rightarrow \frac{dy}{dx} = -2 \sin 2x \text{ and } \left(\frac{dy}{dx}\right)^2 = 4 \sin^2 2x \Rightarrow S = 2\pi \int_0^{\frac{\pi}{4}} \cos 2x \sqrt{1 + 4 \sin^2 2x} \, dx \quad \text{M0A0*}$$

(Correct derivative and square seen but no evidence of using this in the correct formula to obtain the given result)

$$y = \cos 2x \Rightarrow \frac{dy}{dx} = -2 \sin 2x, \left(\frac{dy}{dx}\right)^2 = 4 \sin^2 2x, S = 2\pi \int_0^{\frac{\pi}{4}} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = 2\pi \int_0^{\frac{\pi}{4}} \cos 2x \sqrt{1 + 4 \sin^2 2x} \, dx \quad \text{M1A1*}$$

(Correct derivative and square seen and use of this in the correct formula is implied by the correct formula being quoted.

(b)

B1: Correct differentiation. Accept equivalent forms if the substitution is rearranged first. Must be a fully correct differentiation to score this mark.**M1:** Substitutes fully into the given integral and makes progress to eliminate the x terms. Must substitute fully, including the dx and arrive at an integral involving only θ . Limits not needed for this mark and condone if they are not changed yet.**A1:** Correct integral in terms of $\cosh \theta$ (can ignore limits for this mark).**M1:** Applies $\cosh^2 \theta = \pm \frac{1}{2} \pm \frac{1}{2} \cosh 2\theta$ or uses the correct exponential form and expands to obtain an integrable form for the integral (can ignore limits for this mark)**A1:** Fully correct integration (can ignore limits for this mark)**M1:** Applies the limits 0 and $\operatorname{arsinh} 2$ or e.g. 0 and $\ln(2 + \sqrt{5})$ to an integrated expression of the form $a\theta + b\sinh 2\theta$ or $ae^{2\theta} + b\theta + ce^{-2\theta}$. Must see clear evidence of the limits being applied, but condone the omission of the lower limit as it gives 0**A1:** Cao. Must be in the exact form requested.

Question Number	Scheme	Marks
	Accept i, j, k notation or column vector notation throughout this question	
8(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 5 & -6 \\ -3 & 1 & 2 \end{vmatrix} = \begin{pmatrix} 16 \\ 4 \\ 22 \end{pmatrix}$	B1
		(1)
(b)	$\begin{pmatrix} 16 \\ 4 \\ 22 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \dots \text{ or } \begin{pmatrix} 8 \\ 2 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \dots$	M1
	$\mathbf{r} \cdot (8\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}) = 25$	A1
		(2)
ALT Using another point on the plane	e.g. $\lambda = 0, \mu = 1 \Rightarrow$ point $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ on the plane $\Rightarrow \begin{pmatrix} 16 \\ 4 \\ 22 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \dots$ or $\Rightarrow \begin{pmatrix} 8 \\ 2 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \dots$	M1
	$\mathbf{r} \cdot (8\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}) = 25$	A1
		(2)
(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 2 & 11 \\ 1 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 13 \\ 3 \\ -10 \end{pmatrix}$ Or Points on the line are (see below) $\begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} \frac{52}{3} \\ 0 \\ -\frac{31}{3} \end{pmatrix}$ so direction is $\begin{pmatrix} \frac{52}{3} \\ 4 \\ -\frac{40}{3} \end{pmatrix}$ or any multiple e.g. $\begin{pmatrix} 52 \\ 12 \\ -40 \end{pmatrix}$ etc	M1
	$x = \dots(0) \Rightarrow \begin{cases} 2y + 11z = 25 \\ -y + z = 7 \end{cases} \Rightarrow z = \dots(3), y = \dots(-4)$ $y = \dots(0) \Rightarrow \begin{cases} 8x + 11z = 25 \\ x + z = 7 \end{cases} \Rightarrow z = \dots\left(-\frac{31}{3}\right), x = \dots\left(\frac{52}{3}\right)$ $z = \dots(0) \Rightarrow \begin{cases} 8x + 2y = 25 \\ x - y = 7 \end{cases} \Rightarrow x = \dots\left(\frac{39}{10}\right), y = \dots\left(-\frac{31}{10}\right)$ Note: points will have the form $\left(\frac{52+13\alpha}{3}, \alpha, \frac{-31-10\alpha}{3}\right)$	M1



	Common points $(0, -4, 3) \left(\frac{52}{3}, 0, -\frac{31}{3} \right) \left(\frac{39}{10}, -\frac{31}{10}, 0 \right) \left(1, -\frac{49}{13}, \frac{29}{13} \right) \left(\frac{65}{3}, 1, -\frac{41}{3} \right) \left(\frac{13}{5}, -\frac{17}{5}, 1 \right)$	
	$(\mathbf{r} - (-4\mathbf{j} + 3\mathbf{k})) \times (13\mathbf{i} + 3\mathbf{j} - 10\mathbf{k}) = \mathbf{0}$ or $\left(\mathbf{r} - \left(\frac{52}{3}\mathbf{i} - \frac{31}{3}\mathbf{k} \right) \right) \times (13\mathbf{i} + 3\mathbf{j} - 10\mathbf{k}) = \mathbf{0}$ or $\left(\mathbf{r} - \left(\frac{39}{10}\mathbf{i} - \frac{31}{10}\mathbf{j} \right) \right) \times (13\mathbf{i} + 3\mathbf{j} - 10\mathbf{k}) = \mathbf{0}$ Or equivalent for their correct point on the line and correct direction vector	A1
		(3)
Alt to find a point and/or direction vector for either M1 mark	Combining cartesian equations together (other eliminations are possible): $\begin{cases} x - y + z = 7 \\ 8x + 2y + 11z = 25 \end{cases} \Rightarrow 3x - 13y = 52 \Rightarrow y = \frac{3x - 52}{13} \quad x = \frac{52 + 13y}{3}$ $z = 7 - x + y = \frac{-31 - 10y}{3} \Rightarrow y = \frac{-3z - 31}{10}$ So $\frac{3x - 52}{13} = y = \frac{-3z - 31}{10} \Rightarrow \frac{x - \frac{52}{3}}{\frac{13}{3}} = \frac{y - 0}{1} = \frac{z + \frac{31}{3}}{-\frac{10}{3}}$ so point is $\begin{pmatrix} \frac{52}{3} \\ 0 \\ -\frac{31}{3} \end{pmatrix}$ Or Direction is $\begin{pmatrix} \frac{13}{3} \\ 1 \\ -\frac{10}{3} \end{pmatrix}$ or any multiple	M1
	point is $\begin{pmatrix} \frac{52}{3} \\ 0 \\ -\frac{31}{3} \end{pmatrix}$ and direction is $\begin{pmatrix} \frac{13}{3} \\ 1 \\ -\frac{10}{3} \end{pmatrix}$ or any multiple	M1
	$(\mathbf{r} - (-4\mathbf{j} + 3\mathbf{k})) \times (13\mathbf{i} + 3\mathbf{j} - 10\mathbf{k}) = \mathbf{0}$ Or $\left(\mathbf{r} - \left(\frac{52}{3}\mathbf{i} - \frac{31}{3}\mathbf{k} \right) \right) \times (13\mathbf{i} + 3\mathbf{j} - 10\mathbf{k}) = \mathbf{0}$ Or $\left(\mathbf{r} - \left(\frac{39}{10}\mathbf{i} - \frac{31}{10}\mathbf{j} \right) \right) \times (13\mathbf{i} + 3\mathbf{j} - 10\mathbf{k}) = \mathbf{0}$ Or equivalent for their correct point on the line and correct direction vector	A1
		(3)



(d)	$\pm \left(\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right) = \pm \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$	M1
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 13 & 3 & -10 \\ 1 & -1 & -1 \end{vmatrix} = \begin{pmatrix} -13 \\ 3 \\ -16 \end{pmatrix} \text{ or } \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 13 & 3 & -10 \end{vmatrix} = \begin{pmatrix} 13 \\ -3 \\ 16 \end{pmatrix}$	M1
	$d = \left \frac{(-4\mathbf{j} + 3\mathbf{k}) \cdot (-13\mathbf{i} + 3\mathbf{j} - 16\mathbf{k})}{\sqrt{13^2 + 3^2 + 16^2}} \right = \dots$	M1
	$d = \frac{11}{\sqrt{434}}$	A1
		(4)
Alt using general points on the lines	$\pm \left(\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right) = \pm \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$	M1
	Let the general points on the lines be X and Y $\overrightarrow{XY} = \begin{pmatrix} 2 + \lambda \\ 1 - \lambda \\ 3 - \lambda \end{pmatrix} - \begin{pmatrix} 0 + 13\mu \\ -4 + 3\mu \\ 3 - 10\mu \end{pmatrix} = \begin{pmatrix} 2 + \lambda - 13\mu \\ 5 - \lambda - 3\mu \\ -\lambda + 10\mu \end{pmatrix} \Rightarrow$ $\begin{pmatrix} 2 + \lambda - 13\mu \\ 5 - \lambda - 3\mu \\ -\lambda + 10\mu \end{pmatrix} \cdot \begin{pmatrix} 13 \\ 3 \\ -10 \end{pmatrix} = 0 \text{ and } \begin{pmatrix} 2 + \lambda - 13\mu \\ 5 - \lambda - 3\mu \\ -\lambda + 10\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 0 \Rightarrow \begin{cases} 41 + 20\lambda - 278\mu = 0 \\ -3 + 3\lambda - 20\mu = 0 \end{cases}$ $\Rightarrow \lambda = \dots \left(\frac{827}{217} \right), \mu = \dots \left(\frac{183}{434} \right)$	M1
	$\overrightarrow{XY} = \begin{pmatrix} 2 + \lambda - 13\mu \\ 5 - \lambda - 3\mu \\ -\lambda + 10\mu \end{pmatrix} = \begin{pmatrix} \frac{143}{434} \\ -\frac{33}{434} \\ \frac{88}{217} \end{pmatrix}$ $ \overrightarrow{XY} = \sqrt{\left(\frac{143}{434} \right)^2 + \left(\frac{33}{434} \right)^2 + \left(\frac{88}{217} \right)^2} = \dots \left(\frac{121}{434} \right)$	M1
	$d = \frac{11}{\sqrt{434}}$	A1
		(4)
		Total 10



Notes

(a)

B1: Correct vector found. ISW once a correct vector product is seen e.g. if the components are divided by 2 etc.

(b)

M1: Attempts $\mathbf{r} \bullet \mathbf{n} = p$ form for the plane by finding the scalar product between their part (a) or a multiple of their part (a) and $4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ (or another point on the line, correctly obtained by substituting values of the parameters into the line). If the scalar product is found with a point not on the line, then M0.**A1:** Correct equation

(c)

M1: Attempts the vector product between $8\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$ or e.g. finds 2 points on the line and subtracts to find the direction. If method not directly shown, then two correct components correct implies method.**M1:** Attempts to find a point on the line of intersection, either by substituting a value for x, y, z into the cartesian equations of the planes, and solving to find the other two values or by combining the cartesian equations together. M0 if they find a correct point then multiply by a scalar.**A1:** Any correct equation in the required form e.g. $\left(\mathbf{r} - \left(\frac{39}{10}\mathbf{i} - \frac{31}{10}\mathbf{j}\right)\right) \times \left(-\frac{13}{10}\mathbf{i} - \frac{3}{10}\mathbf{j} + \mathbf{k}\right) = \mathbf{0}$ **ALT****M1:** Combines the cartesian equations together by eliminating one variable and finding one in terms of the other two or expressing two variables in terms of the other one, and making progress to write the line in the form $\frac{x-a}{d} = \frac{y-b}{e} = \frac{z-c}{f}$ and then correctly extracting either their point or their direction correctly from

their line equation. Allow minor slips if the line is in the correct form.

M1: Correctly extracts both the point and the direction from their line equation in cartesian form. Allow minor slips if the line is in the correct form.**A1:** Forms the correct equation of the line in the correct form. Condone the ' $\mathbf{0}$ ' and/or the ' $\mathbf{r}$ ' without an underscore to indicate that it is a vector.

(d)

M1: Attempts the direction of l_2 **M1:** Attempts the vector product between their direction from part (c) and their direction of l_2 . If method not directly shown, then two correct components implies the correct method.**M1:** A full method to find the required distance using formula $\frac{|(\mathbf{a} - \mathbf{c}) \bullet \mathbf{n}|}{|\mathbf{n}|}$ where $\mathbf{a}$ and $\mathbf{c}$ are the positionvectors of individual points on the lines l_1 and l_2 and $\mathbf{n}$ is the vector product of the direction vectors for theirlines. $\left(\text{e.g. } \mathbf{r}_1 = \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 13 \\ 3 \\ -10 \end{pmatrix} \text{ and } \mathbf{r}_2 = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}\right)$. Note that the point A could also be used in the line $\mathbf{r}_2$ so that $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ isreplaced with $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ (or another point). The numerator may be seen as a scalar triple product e.g. $\begin{vmatrix} -3 & -4 & 1 \\ 13 & 3 & -10 \\ 1 & -1 & -1 \end{vmatrix}$ etc.**A1:** For $\frac{11}{\sqrt{434}}$ or awrt 0.528**ALT****M1:** Attempts the direction of l_2 **M1:** Forms the general point between the two lines, attempts the dot product between the general point and both direction vectors of the lines l_1 and l_2 to form and then solve two simultaneous equations in their parameters. Must find a value for both.**M1:** Substitutes their parameters to find their $\overline{XY}$ and then finds the magnitude of their $\overline{XY}$ **A1:** For $\frac{11}{\sqrt{434}}$ or awrt 0.528

Question Number	Scheme	Notes	Marks
7(a)	$5\mathbf{i} + 3\mathbf{j} - 8\mathbf{k}$ and $2\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$ lie in Π_1	Identifies 2 correct vectors lying in Π_1	B1
	$\mathbf{n} = \begin{pmatrix} 5 \\ 3 \\ -8 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ -6 \end{pmatrix} = \begin{pmatrix} -18-24 \\ -(-30+16) \\ -15-6 \end{pmatrix}$ Attempts the vector product between 2 correct vectors in Π_1 If no working is shown, look for at least 2 correct elements. Or e.g. Let $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ then $(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (5\mathbf{i} + 3\mathbf{j} - 8\mathbf{k}) = 0$, $(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}) = 0$ $\Rightarrow 5a + 3b - 8c = 0$, $2a - 3b - 6c = 0 \Rightarrow a = 2c$, $3b = -2c \Rightarrow \mathbf{n} = \dots$		M1
	$= \begin{pmatrix} -42 \\ 14 \\ -21 \end{pmatrix} \text{ or e.g. } \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$	Correct normal vector	A1
	$(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \dots$ Attempts scalar product between their normal vector and position vector of a point in Π_1. Do not allow this mark if the “5” (or equivalent) just ‘appears’. There must be some evidence for its origin e.g. $\mathbf{a} \cdot \mathbf{n} = \dots$ where $\mathbf{a}$ and $\mathbf{n}$ have been defined earlier. Depends on the first method mark.		dM1
	$6x - 2y + 3z = 5^*$	Correct proof	A1*
			(5)
	Alternative 1 for (a):		
	E.g. Let equation of Π_1 be $ax + by + z = c$		B1
	3 points on Π_1 are $(1, 2, 1)$, $(3, -1, -5)$ and e.g. $(8, 2, -13)$		M1
	$a + 2b + 1 = c$, $3a - b - 5 = c$, $8a + 2b - 13 = c \Rightarrow a = \dots, b = \dots, c = \dots$		M1
	Solves simultaneously for a , b and c using correct points		M1
	$\Rightarrow a = 2, b = -\frac{2}{3}, c = \frac{5}{3}$	Correct values	A1
	$2x - \frac{2}{3}y + z = \frac{5}{3}$	Forms Cartesian equation	dM1
	$6x - 2y + 3z = 5^*$	Correct proof	A1*
	Alternative 2 for (a):		
	$(1, 2, 1) \rightarrow 6x - 2y + 3z = 6 - 4 + 3 = 5$		B1
	Shows $(1, 2, 1)$ lies on Π_1		B1
	$\frac{x-3}{5} = \frac{y+1}{3} = \frac{z+5}{-8} \rightarrow \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \\ -8 \end{pmatrix} \text{ or equivalent}$ M1: Converts l to correct parametric form <u>seen as part of an attempt at this alternative</u> allow 1 slip with one of the elements A1: Correct form		M1A1
	$6(3 + 5\lambda) - 2(-1 + 3\lambda) + 3(-5 - 8\lambda) = 5$		dM1
	Shows l lies in Π_1		dM1
	P lies in Π_1 and l lies in Π_1 so $6x - 2y + 3z = 5^*$		A1*
	All correct with conclusion		

Question Number	Scheme	Notes	Marks
7(a)	$\frac{x-3}{4} = \frac{y-5}{-2} = \frac{z-4}{7} \Rightarrow \mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} \pm \lambda \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix}$	Converts to parametric form. “ $\mathbf{r} =$ ” is not required	M1
	$2x + 4y - z = 1$ $\Rightarrow 2(3 + 4\lambda) + 4(5 - 2\lambda) - 4 - 7\lambda = 1$ $\Rightarrow \lambda = \dots(3) \Rightarrow P$ is ...	Correct strategy for finding P . Condone the use of $2x + 4y - z = 0$ for the plane equation.	M1
	$(15, -1, 25)$	Correct coordinates. Condone if given as a vector.	A1
			(3)
(a) Way 2	$\frac{x-3}{4} = \frac{y-5}{-2} \Rightarrow x = 13 - 2y$	Uses the Cartesian equation to find x in terms of y	M1
	$2x + 4y - z = 1 \Rightarrow 26 - 4y + 4y - z = 1$ $\Rightarrow z = \dots, x = \dots, y = \dots$	Correct strategy for finding P . Condone the use of $2x + 4y - z = 0$ for the plane equation.	M1
	$(15, -1, 25)$	Correct coordinates. Condone if given as a vector.	A1
(b)	$\begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = 8 - 8 - 7 = -7$	Applies the scalar product between the direction of l_1 and the normal to the plane	M1
	Examples: $\phi = \cos^{-1} \frac{\pm 7}{\sqrt{69}\sqrt{21}} = \dots \quad \phi = \sin^{-1} \frac{\pm 7}{\sqrt{69}\sqrt{21}} = \dots$ Attempts to find a relevant angle in degrees or radians. Depends on the first method mark.		dM1
	$\theta = 10.6^\circ$	Allow awrt 10.6 but do not isw and mark the final answer. For reference $\theta = 10.5965654^\circ$	A1
			(3)
(b) Way 2	$\begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 26 \\ -18 \\ -20 \end{pmatrix}$	Attempts vector product of normal to Π and direction of l_1	M1
	$\sqrt{26^2 + 18^2 + 20^2} = \sqrt{21}\sqrt{69} \sin \alpha$ $\sin \alpha = \frac{10\sqrt{46}}{69} \Rightarrow \alpha = \dots$	Attempts to find a relevant angle. Depends on the first method mark.	dM1
	$\theta = 10.6^\circ$	Allow awrt 10.6 but do not isw and mark the final answer. For reference $\theta = 10.5965654^\circ$	A1



(c)	$\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -1 \\ 4 & -2 & 7 \end{vmatrix} = \begin{pmatrix} 26 \\ -18 \\ -20 \end{pmatrix}$	Attempts vector product of normal to Π and direction of l_1 . If no method is seen expect at least 2 correct components.	M1
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 13 & -9 & -10 \\ 2 & 4 & -1 \end{vmatrix} = \begin{pmatrix} 49 \\ -7 \\ 70 \end{pmatrix}$	Attempts vector product of “a” with normal to Π to find direction of l_2	M1
		Correct direction for l_2	A1
	$\mathbf{r} = \begin{pmatrix} 15 \\ -1 \\ 25 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ -1 \\ 10 \end{pmatrix}$	Depends on both previous M marks Attempts vector equation using their direction vector and their P	ddM1
		Correct equation or any equivalent correct vector equation	A1
			(5)
(c) Way 2	$\lambda = 1 \Rightarrow (7, 3, 11) \text{ lies on } l_1$ $\mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ 11 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ $\Rightarrow 2(7 + 2t) + 4(3 + 4t) - 11 + t = 1$ $t = -\frac{2}{3} \Rightarrow \left(\frac{17}{3}, \frac{1}{3}, \frac{35}{3}\right) \text{ is on } l_2$	Complete method to find a point on l_2	M1
	Direction of l_2 is $\begin{pmatrix} 15 \\ -1 \\ 25 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 17 \\ 1 \\ 35 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 28 \\ -4 \\ 40 \end{pmatrix}$	Uses their point and their P to find direction of l_2	M1
		Correct direction for l_2	A1
	$\mathbf{r} = \begin{pmatrix} 15 \\ -1 \\ 25 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ -1 \\ 10 \end{pmatrix}$	Attempts vector equation using their direction vector and their point on l_2	ddM1
		Correct equation or any equivalent correct vector equation. Must have $\mathbf{r} =$ and not e.g. $l_2 = \dots$	A1
	(c) Way 3	Normal to plane from l_1 $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ $\Rightarrow 2(3 + 2t) + 4(5 + 4t) - (4 - t) = 1$ $t = -1 \Rightarrow (1, 1, 5) \text{ is on } l_2$	Complete method to find a point on l_2
Direction of l_2 is $\begin{pmatrix} 15 \\ -1 \\ 25 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 14 \\ -2 \\ 20 \end{pmatrix}$		Uses their point and their P to find direction of l_2	M1
		Correct direction for l_2	A1
$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ -1 \\ 10 \end{pmatrix}$		Attempts vector equation using their direction vector and their point on l_2	ddM1
		Correct equation or any equivalent correct vector equation. Must have $\mathbf{r} =$ and not e.g. $l_2 = \dots$	A1

Question Number	Scheme	Notes	Marks
7.	$I_n = \int \frac{x^n}{\sqrt{10-x^2}} dx \quad n \in \mathbb{N}, x < \sqrt{10}$		
(a)	$I_n = \int \frac{x^n}{\sqrt{10-x^2}} dx = \int \frac{x^{n-1} \times x}{\sqrt{10-x^2}} dx$	Writes x^n as $x \times x^{n-1}$	M1
	$\int \frac{x^{n-1} \times x}{\sqrt{10-x^2}} dx = -x^{n-1} (10-x^2)^{\frac{1}{2}} + (n-1) \int x^{n-2} (10-x^2)^{\frac{1}{2}} dx$ dM1: Uses integration by parts to obtain $\int \frac{x^{n-1} \times x}{\sqrt{10-x^2}} dx = \alpha x^{n-1} (10-x^2)^{\frac{1}{2}} + \beta \int x^{n-2} (10-x^2)^{\frac{1}{2}} dx$ A1: Correct expression		dM1A1
	$= \dots + (n-1) \int x^{n-2} (10-x^2) (10-x^2)^{-\frac{1}{2}} dx$ $= \dots + 10(n-1) \int x^{n-2} (10-x^2)^{-\frac{1}{2}} dx - (n-1) \int x^n (10-x^2)^{-\frac{1}{2}} dx$ Applies $(10-x^2)^{\frac{1}{2}} = (10-x^2)(10-x^2)^{-\frac{1}{2}}$ and splits into 2 integrals		dM1
	$= \dots + 10(n-1)I_{n-2} - (n-1)I_n \Rightarrow nI_n$	Introduces I_{n-2} and I_n and makes progress to the given result	dM1
	$nI_n = 10(n-1)I_{n-2} - x^{n-1} (10-x^2)^{\frac{1}{2}} *$ Fully correct proof with no errors (recovery of missing brackets counts as an error) as does missing dx		A1*
			(6)
(b)	$I_1 = \int_0^1 \frac{x}{\sqrt{10-x^2}} dx = \left[-(10-x^2)^{\frac{1}{2}} \right]_0^1 = (-3 + \sqrt{10})$ Correct method for I_1 Limits can be substituted later		M1
	$5I_5 = 10 \times 4I_3 + \dots$	Applies the reduction formula at least once Allow with 3 or $\left[-x^4 (10-x^2)^{\frac{1}{2}} \right]_0^1$	M1
	$I_5 = 8I_3 - \frac{3}{5} = 8 \left(\frac{20}{3} I_1 - 1 \right) - \frac{3}{5} = \frac{160}{3} I_1 - \frac{43}{5}$ $I_5 = \frac{160}{3} (\sqrt{10} - 3) - \frac{43}{5}$ Completes the process using their I_1 to obtain a numerical value for I_5 Limits must now be substituted		M1
	$= \frac{1}{15} (800\sqrt{10} - 2529)$	Cao	A1
			(4)
			Total 10

Question Number	Scheme	Notes	Marks
8(a)	$(\mathbf{r} =) \begin{pmatrix} -4 \\ -5 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$	Forms the parametric form of the line	M1
	$3(3t-4) + 4(4t-5) - (3-t) = 17$ $\Rightarrow t = (2)$	Substitutes the parametric form for the line into the plane equation and solves for "t". Depends on the first mark.	dM1
	$\begin{pmatrix} -4 \\ -5 \\ 3 \end{pmatrix} + "2" \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$	Uses their value of t correctly to find Q. Depends on the previous mark.	dM1
	$(2, 3, 1)$	Correct coordinates Accept if written as a column vector but not with i, j, k	A1 (4)
Way 2	$\frac{x+4}{3} = \frac{y+5}{4} = \frac{z-3}{-1}$ eg $x = f(y)$ $z = g(y)$	Forms the Cartesian equation of the line, rearranges twice to get 2 of x, y, z as functions of the third	M1
		Substitutes these into the plane equation and solves for one coordinate	dM1
		Obtains the other 2 coordinates	dM1
	$(2, 3, 1)$	Correct coordinates Accept if written as a column vector but not with i, j, k	A1
			(4)
(b)	$\mathbf{PQ} = \begin{pmatrix} 2+4 \\ 3+5 \\ 1-3 \end{pmatrix}, \mathbf{PR} = \begin{pmatrix} -1+4 \\ 6+5 \\ 4-3 \end{pmatrix}, \mathbf{RQ} = \begin{pmatrix} 2+1 \\ 3-6 \\ 1-4 \end{pmatrix}$	Attempts 2 vectors in plane PQR (Must use the given coordinates of P, R and their coordinates of Q)	M1
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 8 & -2 \\ 3 & 11 & 1 \end{vmatrix} = \begin{pmatrix} 30 \\ -12 \\ 42 \end{pmatrix}$	Attempt vector product between 2 vectors in PQR. Depends on the first mark.	dM1
	$\begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 11$	Uses any of P, Q or R to find constant. Depends on the previous mark.	dM1
	$5x - 2y + 7z = 11$	Any correct Cartesian equation	A1
			(4)
Way 2	$-4a - 5b - 3c = 1$ $2a + 3b + c = 1$ $-a + 6b + 4c = 1$	Uses the Cartesian form of the equation of a plane, $ax + by + cz = 1$, and substitutes the coordinates of each of the 3 points	M1
	Solves to get a value for any of a, b or c		dM1
	Obtains values for the other 2		dM1
	$\frac{5}{11}x - \frac{2}{11}y + \frac{7}{11}z = 1$	Any correct Cartesian equation	A1
			(4)

Question Number	Scheme	Notes	Marks
1	$7 \cosh x + 3 \sinh x = 2e^x + 7 \Rightarrow$ $7\left(\frac{e^x + e^{-x}}{2}\right) + 3\left(\frac{e^x - e^{-x}}{2}\right) = 2e^x + 7$ $\left\{\frac{7}{2}e^x + \frac{7}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} = 2e^x + 7\right\}$	Substitutes at least one correct exponential form for either of the hyperbolic terms and achieves an equation in exponentials and constants alone	M1
	$\Rightarrow 7(e^{2x} + 1) + 3(e^{2x} - 1) = 4e^{2x} + 14e^x$ $\{\Rightarrow 5e^{2x} + 2 = 2e^{2x} + 7e^x\}$	Multiplies through by e^x to obtain any equation that would form a 3TQ in e^x if like terms were collected	M1
	$\Rightarrow 6e^{2x} - 14e^x + 4 = 0 \quad \{3e^{2x} - 7e^x + 2 = 0\}$	A correct three term quadratic in e^x . Could be implied by a correct root even if terms have not been collected.	A1
	$\Rightarrow (3e^x - 1)(e^x - 2) = 0 \Rightarrow e^x = \dots$	Solves their 3TQ - usual rules. One correct root for their quadratic if no working. Ignore labelling of the roots even if e.g., “ x ” is used.	M1
	$x = \ln 2, \ln \frac{1}{3}$	Both correct and simplified but do not isw if there are other answers . Allow $-\ln \frac{1}{2}$ for $\ln 2$ and $-\ln 3$ or $\ln 3^{-1}$ for $\ln \frac{1}{3}$	A1
	Answer only is 0/5		Total 5
	Note that it is possible to multiply through by e^{-x} to form an equation in e^{-2x}, e^{-x} and constants. Score as main scheme, e.g., $\frac{7}{2}e^x + \frac{7}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} = 2e^x + 7$ $\Rightarrow \frac{7}{2} + \frac{7}{2}e^{-2x} + \frac{3}{2} - \frac{3}{2}e^{-2x} = 2 + 7e^{-x} \quad (\text{M1})$ $\Rightarrow 2e^{-2x} - 7e^{-x} + 3 = 0 \quad (\text{A1})$ $(2e^{-x} - 1)(e^{-x} - 3) = 0 \Rightarrow e^{-x} = \frac{1}{2}, 3 \quad (\text{M1})$ $\Rightarrow e^x = 2, \frac{1}{3} \Rightarrow x = \ln 2, \ln \frac{1}{3} \quad (\text{A1})$		

Question Number	Scheme	Notes	Marks
6(a)	$\pm \overrightarrow{AB} = \pm \left(\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \right) = \pm \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix}, \pm \overrightarrow{AC} = \pm \left(\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \right) = \pm \begin{pmatrix} -5 \\ 2 \\ 0 \end{pmatrix}, \pm \overrightarrow{BC} = \pm \left(\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right) = \pm \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$ Correct method to obtain two relevant vectors using subtraction. You can ignore labelling e.g. if they find $\overrightarrow{BA}$ but call it $\overrightarrow{AB}$		M1
	$\text{e.g., } \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -5 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ -13 \end{pmatrix}$ Correct method to find the vector product of two relevant vectors (if a correct method is not shown, two correct components for their vectors must be obtained)		dM1
	$\text{e.g., } \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 13 \end{pmatrix} = 6 + 10 + 26 = 42$ Attempts the scalar product between their normal vector and any of the position vectors of A, B or C.		ddM1
	$2x + 5y + 13z = 42$ oe e.g. $-2x - 5y - 13z + 42 = 0$	Any correct Cartesian equation.	A1
			(4)
(a) alt 1	$\pm \overrightarrow{AB} = \pm \left(\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \right) = \pm \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix}, \pm \overrightarrow{AC} = \pm \left(\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \right) = \pm \begin{pmatrix} -5 \\ 2 \\ 0 \end{pmatrix}, \pm \overrightarrow{BC} = \pm \left(\begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right) = \pm \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$ Correct method to obtain two relevant vectors using subtraction.		M1
	$\text{e.g., } \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 2 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x = 3 - 4\lambda - 5\mu \\ y = 2 - \lambda + 2\mu \\ z = 2 + \lambda \end{matrix} \Rightarrow \text{e.g. } \lambda = z - 2$ Attempts the parametric equation of the plane and uses components to eliminate at least one of their parameters.		dM1
	$x = 3 - 4\lambda - 5\mu$ $\text{e.g., } y = 2 - \lambda + 2\mu \Rightarrow \text{e.g. } \lambda = z - 2 \Rightarrow \text{e.g. } \mu = \frac{1}{2}(y - 4 + z)$ $z = 2 + \lambda$ Eliminates both of their parameters.		ddM1
	$\text{e.g. } x = 3 - 4(z - 2) - \frac{5}{2}(y - 4 + z)$	Any correct Cartesian equation.	A1
(a) alt 2	$3a + 2b + 2c = 1$ $ax + by + cz = 1 \rightarrow -a + b + 3c = 1 \Rightarrow a = \frac{1}{21}, b = \frac{5}{42}, c = \frac{13}{42}$ $-2a + 4b + 2c = 1$ $\Rightarrow \frac{1}{21}x + \frac{5}{42}y + \frac{13}{42}z = 1$ M1: Substitutes the given points to give 3 equations in 3 unknowns dM1: Solves simultaneously to find values for “a”, “b” and “c” ddM1: Substitutes back in to obtain a Cartesian equation A1: Any correct equation		



(b)	Line $DE : (\mathbf{r} =) \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \pm \lambda \begin{pmatrix} 2 \\ 5 \\ 13 \end{pmatrix}$	Obtains parametric form for line DE with their normal (or recalculated normal) seen or implied. Allow one slip only.	M1
	$14(2\lambda - 1) - (5\lambda + 1) - 17(13\lambda - 2) = -66 \Rightarrow \lambda = \dots$ Substitutes their parametric form into the equation of Π_2 and solves for λ – can follow M0 provided their parametric form was an attempt at $\overline{OD} \pm \lambda(\text{their } \mathbf{n})$		M1
	$\lambda = \frac{85}{198}$	A correct exact value for λ depending on their method e.g. use of $\mathbf{n} = -2\mathbf{i} - 5\mathbf{j} - 13\mathbf{k}$ gives $\lambda = -\frac{85}{198}$	A1
	$DE = \sqrt{(2 \times \frac{85}{198})^2 + (5 \times \frac{85}{198})^2 + (13 \times \frac{85}{198})^2}$ or e.g. $E = (-\frac{14}{99}, \frac{623}{198}, \frac{709}{198}) \Rightarrow DE = \sqrt{(-1 + \frac{14}{99})^2 + (1 - \frac{623}{198})^2 + (-2 - \frac{709}{198})^2}$ Correct method to find a numerical expression for distance DE Requires previous method mark Note $DE = -\frac{85}{198}\sqrt{(2)^2 + (5)^2 + (13)^2} = \dots$ is ok for this mark		dM1
	$DE = \frac{85\sqrt{22}}{66}$	Correct exact answer in the required form or $p = \frac{85}{66}$ or $1\frac{19}{66}$ Not $DE = -\frac{85\sqrt{22}}{66}$	A1
			(5)

Beware – Special Case!

An incorrect sign of λ may fortuitously give the correct length DE .

E.g. $\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 13 \end{pmatrix}$ leading incorrectly to $\lambda = -\frac{85}{198}$ would lead in both dM1 cases above to $DE = \frac{85\sqrt{22}}{66}$

E.g. $\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -5 \\ -13 \end{pmatrix}$ leading incorrectly to $\lambda = \frac{85}{198}$ would lead in both dM1 cases above to $DE = \frac{85\sqrt{22}}{66}$

In such cases score as M1M1A0M1A1ft i.e. we will only penalise it once.



Way 2 Sim. eqns For first three marks	$(\pm)\left(\frac{x+1}{2} = \frac{y-1}{5} = \frac{z+2}{13}\right)$ $\Rightarrow y = \frac{5}{2}x + \frac{7}{2}, z = \frac{13}{2}x + \frac{9}{2}$	Obtains Cartesian form for line DE with their normal (or recalculated normal) allowing one slip only and attempts to find two variables in terms of the other variable	M1	
	$14x - \left(\frac{5}{2}x + \frac{7}{2}\right) - 17\left(\frac{13}{2}x + \frac{9}{2}\right) = -66$ $\Rightarrow x = -\frac{14}{99}, y = \frac{623}{198}, z = \frac{709}{198}$	M1: Substitutes into the plane equation and finds $x = \dots, y = \dots, z = \dots$ A1: Correct exact values $\Rightarrow$ Way 1 for last two marks	M1 A1	
(c)	e.g. $\overrightarrow{AF} \cdot \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 1 \\ 1 \\ q-2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 13 \end{pmatrix} = 2 + 5 + 13q - 26$ or e.g. $\begin{vmatrix} -4 & -1 & 1 \\ -5 & 2 & 0 \\ 1 & 1 & q-2 \end{vmatrix} = -4(2(q-2)) - 5(q-2) - 5 - 2$ or e.g. rule of Sarrus: $\begin{vmatrix} -4 & -1 & 1 & -4 & -1 \\ -5 & 2 & 0 & -5 & 2 \\ 1 & 1 & q-2 & 1 & 1 \end{vmatrix} = -4(2(q-2)) - 5 - 5(q-2) - 2$ Correct method for vector between F and A, B or C and finds scalar product with their normal or attempts the scalar triple product to obtain a linear expression in q . For the scalar triple product look for at least 2 correct “elements”.		M1	
	$\frac{1}{6}(13q - 19) = \pm 12 \Rightarrow q = \dots$ Sets $\frac{1}{6}$ of their expression in q equal to one or both of ± 12 (or equivalent work e.g. their expression in q equal to one or both of ± 72) and proceeds to a value for q		dM1	
	$q = 7, -\frac{53}{13}$	Correct values. Allow exact equivalents for $-\frac{53}{13}$ e.g. $-4\frac{1}{13}$		A1
				(3)
			Total 12	

Question Number	Scheme	Notes	Marks
3.	$\mathbf{M} = \begin{pmatrix} 3 & 1 & p \\ 1 & 1 & 2 \\ -1 & p & 2 \end{pmatrix}$		
(a)	$\det \mathbf{M} = \begin{vmatrix} 3 & 1 & p \\ 1 & 1 & 2 \\ -1 & p & 2 \end{vmatrix}$ $= 3(2 - 2p) - 1(2 + 2) + p(p + 1)$	Attempts determinant. Requires at least 2 correct “terms”. May use other rows/columns or rule of Sarrus.	M1
	$= p^2 - 5p + 2$	Correct simplified determinant.	A1
	$p^2 - 5p + 2 = 0 \Rightarrow p = \dots$	Solves 3TQ	M1
	$\frac{5 \pm \sqrt{17}}{2}$	Correct values.	A1
			(4)
(b)	Minors $\begin{pmatrix} 2-2p & 4 & p+1 \\ (2-p^2) & 6+p & (3p+1) \\ 2-p & (6-p) & 2 \end{pmatrix}$	Attempts the matrix of minors. If there is any doubt look for at least 6 correct elements. May be implied by their matrix of cofactors.	M1 (B1 on EPEN)
	Cofactors $\begin{pmatrix} 2-2p & -4 & p+1 \\ -(2-p^2) & 6+p & -(3p+1) \\ 2-p & -(6-p) & 2 \end{pmatrix}$	Attempts cofactors.	M1
		Correct matrix	A1
	$\mathbf{M}^{-1} = \frac{1}{p^2 - 5p + 2} \begin{pmatrix} 2-2p & p^2 - 2 & 2-p \\ -4 & 6+p & p-6 \\ p+1 & -3p-1 & 2 \end{pmatrix}$	Transposes matrix of cofactors and divides by determinant.	M1
		Follow though their det $\mathbf{M}$ from part (a) but the adjoint matrix must be correct.	A1ft
			(5)
			Total 9

Question Number	Scheme	Notes	Marks
-----------------	--------	-------	-------

4(i)	$f(x) = x \arccos x, -1 \leq x \leq 1,$		
	$f'(x) = \arccos x - \frac{x}{\sqrt{1-x^2}}$ M1: Differentiates using the product rule to obtain an expression of the form: $\arccos x \pm \frac{x}{\sqrt{1-x^2}}$ A1: Correct derivative		M1A1
	$f'(0.5) = \arccos 0.5 - \frac{0.5}{\sqrt{1-0.5^2}} = \frac{\pi - \sqrt{3}}{3}$	$\frac{\pi - \sqrt{3}}{3} \text{ oe e.g. } \frac{\pi}{3} - \frac{1}{\sqrt{3}}$	A1
			(3)
(ii)	$g(x) = \arctan(e^{2x})$		
	$g'(x) = \frac{2e^{2x}}{e^{4x} + 1}$ M1: Differentiates using the chain rule to obtain an expression of the form: $\frac{ke^{2x}}{(e^{2x})^2 + 1}$ A1: Correct derivative in any form		M1A1
	$g'(x) = \frac{2}{e^{2x} + e^{-2x}} = \operatorname{sech}(2x)$	Introduces $\operatorname{sech}(2x)$. Depends on previous M.	dM1
	$g''(x) = -2 \operatorname{sech}(2x) \tanh(2x)$	Differentiates $\operatorname{sech}(u) \rightarrow \pm \operatorname{sech} u \tanh u$ Depends on both previous M's.	dM1
		Correct expression.	A1
			(5)
(ii) ALT 1	$g'(x) = \frac{2e^{2x}}{e^{4x} + 1}$ M1: Differentiates using the chain rule to obtain an expression of the form: $\frac{ke^{2x}}{(e^{2x})^2 + 1}$ A1: Correct derivative in any form		M1A1
	$g''(x) = \frac{4e^{2x}(1 + e^{4x}) - 4e^{4x} \times 2e^{2x}}{(e^{4x} + 1)^2}$	Differentiates using quotient or product rule. Depends on first M.	dM1
	$= \frac{4e^{2x} - 4e^{6x}}{(e^{4x} + 1)^2} = \frac{-4(e^{2x} - e^{-2x})}{(e^{2x} + e^{-2x})^2}$	Multiply through by e^{-4x} . Depends on both previous M's.	dM1
	$= -2 \frac{2}{e^{2x} + e^{-2x}} \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$ $= -2 \operatorname{sech} 2x \tanh 2x$	Correct expression.	A1
	Note that the first derivative may be obtained implicitly in either method e.g. $y = \arctan(e^{2x}) \Rightarrow \tan y = e^{2x} \Rightarrow \sec^2 y \frac{dy}{dx} = 2e^{2x} \Rightarrow \frac{dy}{dx} = \frac{2e^{2x}}{1 + (e^{2x})^2}$		
			Total 8

Question Number	Scheme	Notes	Marks
5.	$I_n = \int \sec^n x \, dx,$	$n \geq 0$	

5(a)	$\int \sec^n x \, dx = \int \sec^{n-2} x \sec^2 x \, dx$	Splits $\sec^n x$ into $\sec^{n-2} x \sec^2 x$	M1
	$\int \sec^n x \, dx = \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x \tan^2 x \, dx$ Depends on previous M mark dM1: Uses integration by parts to obtain $\sec^{n-2} x \tan x - k \int \sec^{n-2} x \tan^2 x \, dx$ A1: Correct integration		dM1A1
	$\int \sec^n x \, dx = \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x (\sec^2 x - 1) \, dx$ Uses $\tan^2 x = \sec^2 x - 1$		B1 (M1 on EPEN)
	$\int \sec^n x \, dx = \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$ $= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2} \Rightarrow (n-1) I_n = \dots$ Depends on all previous M and B marks Introduces I_n and I_{n-2} and makes progress to the given result.		ddM1
	$(n-1) I_n = \tan x \sec^{n-2} x + (n-2) I_{n-2}^*$	Fully correct proof.	A1cso
	(6)		
ALT	$\int \sec^n x \, dx = \int \sec^{n-2} x \sec^2 x \, dx$	Splits $\sec^n x$ into $\sec^{n-2} x \sec^2 x$	M1
	$\int \sec^{n-2} x \sec^2 x \, dx = \int \sec^{n-2} x (1 + \tan^2 x) \, dx$ $= \int \sec^{n-2} x \, dx + \int \tan^2 x \sec^{n-2} x \, dx$	Uses $\sec^2 x = 1 + \tan^2 x$ and splits into 2 integrals.	B1 (4 th mark M1 on EPEN)
	$\int \tan^2 x \sec^{n-2} x \, dx = \frac{1}{(n-2)} \tan x \sec^{n-2} x - \frac{1}{(n-2)} \int \sec^n x \, dx$ Uses integration by parts on $\int \tan^2 x \sec^{n-2} x \, dx$ to obtain $A \tan x \sec^{n-2} x - B \int \sec^n x \, dx$ Note this is the 2nd M on EPEN.		dM1
	$\int \sec^n x \, dx = \int \sec^{n-2} x \, dx + \frac{1}{(n-2)} \tan x \sec^{n-2} x - \frac{1}{(n-2)} \int \sec^n x \, dx$ Fully correct integration		A1
	$\int \sec^n x \, dx = I_{n-2} + \frac{1}{(n-2)} \tan x \sec^{n-2} x - \frac{1}{(n-2)} I_n \Rightarrow (n-1) I_n = \dots$ Depends on previous M and B marks Introduces I_n and I_{n-2} and makes progress to the given result.		ddM1
	$(n-1) I_n = \tan x \sec^{n-2} x + (n-2) I_{n-2}^*$	Fully correct proof.	A1cso

5(b)	$I_2 = 1$	Correct value for I_2 seen or implied.	B1
	$I_6 = \frac{1}{5} \tan x \sec^4 x + \frac{4}{5} I_4$ or e.g.	Applies the given reduction formula once.	M1

	$I_6 = \frac{1}{5} \tan \frac{\pi}{4} \sec^4 \frac{\pi}{4} + \frac{4}{5} I_4$ or e.g. $I_6 = \frac{1}{5} (1) (\sqrt{2})^4 + \frac{4}{5} I_4$		
	$= \frac{1}{5} \tan x \sec^4 x + \frac{4}{5} \left(\frac{1}{3} \tan x \sec^2 x + \frac{2}{3} I_2 \right) = \frac{1}{5} (1) (\sqrt{2})^4 + \frac{4}{15} (1) (\sqrt{2})^2 + \frac{8}{15} (1)$ Applies the given reduction formula again and uses the limits to reach a numerical expression for I_6	M1	
	$= \frac{28}{15}$	Correct value	A1
			(4)
ALT	$I_2 = 1$ Correct value for I_2 seen or implied.	B1	
	$I_4 = \frac{1}{3} \tan x \sec^2 x + \frac{2}{3} I_2$ or e.g. $I_4 = \frac{1}{3} \tan \frac{\pi}{4} \sec^2 \frac{\pi}{4} + \frac{2}{3} I_2$ or e.g. $I_4 = \frac{1}{3} (1) (\sqrt{2})^2 + \frac{2}{3} I_2$	Applies the given reduction formula once.	M1
	$I_6 = \frac{1}{5} \tan x \sec^4 x + \frac{4}{5} \left(\frac{1}{3} \tan x \sec^2 x + \frac{2}{3} I_2 \right) = \frac{1}{5} (1) (\sqrt{2})^4 + \frac{4}{15} (1) (\sqrt{2})^2 + \frac{8}{15}$ Applies the given reduction formula again and uses the limits to reach a numerical expression for I_6	M1	
	$= \frac{28}{15}$	Correct value	A1
			Total 10

In part (b), condone confusion with the coefficients provided the intention is clear.

For either method in part (b), all working must be shown and the given reduction formula must be used at least once. So do not allow e.g. I_4 to be evaluated with a calculator but I_4 can be evaluated directly without using the given reduction formula using an alternative method e.g. by parts or by substitution – see below:

Parts:

$$\begin{aligned}
 I_4 &= \int \sec^4 x \, dx = \int \sec^2 x \sec^2 x \, dx = \sec^2 x \tan x - 2 \int \sec^2 x \tan^2 x \, dx \\
 &= \sec^2 x \tan x - 2 \int \sec^2 x (\sec^2 x - 1) \, dx = \sec^2 x \tan x - 2 \int \sec^4 x \, dx + 2 \int \sec^2 x \, dx \\
 &= \sec^2 x \tan x - 2I_4 + 2 \int \sec^2 x \, dx \Rightarrow 3I_4 = \sec^2 x \tan x + 2 \tan x \Rightarrow I_4 = \frac{1}{3} \sec^2 x \tan x + \frac{2}{3} \tan x
 \end{aligned}$$

Substitution:

$$\begin{aligned}
 I_4 &= \int \sec^4 x \, dx = \int \sec^2 x \sec^2 x \, dx = \int \sec^2 x (1 + \tan^2 x) \, dx \\
 u = \tan x &\Rightarrow \int \sec^2 x (1 + \tan^2 x) \, dx = \int \sec^2 x (1 + u^2) \frac{du}{\sec^2 x} = \frac{u^3}{3} + u = \frac{\tan^3 x}{3} + \tan x
 \end{aligned}$$

Question Number	Scheme	Notes	Marks
-----------------	--------	-------	-------

6(a)	Normal to plane given by $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = \dots$	Attempt cross product of direction vectors. If the method is unclear, look for at least 2 correct components.	M1
	$= 6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$	Or any multiple of this vector.	A1
	Substitute appropriate point into $6x + 2y - 2z = d$ e.g. (1, 1, 1) or (2, 1, 4) to find "d"	Use a valid point and use scalar product with normal or substitute into Cartesian equation.	M1
	$6x + 2y - 2z = 6$ $3x + y - z = 3^*$	Given answer. No errors seen	A1* cso
			(4)
6(a) ALT	$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ $\Rightarrow x = 1 + \lambda + \mu, y = 1 - 2\mu, z = 1 + 3\lambda + \mu$ M1: Forms equation of plane using (1, 1, 1) and direction vectors and extracts 3 equations for x, y and z in terms of λ and μ A1: Correct equations		M1A1
	$x = 1 + \frac{1}{2} - \frac{1}{2}y + \frac{1}{3}z - \frac{1}{2} + \frac{1}{6}y$	Eliminates λ and μ and achieves an equation in x, y and z only.	M1
	$3x + y - z = 3^*$	Given answer. No errors seen.	A1
6(b)	$s = -3$	cao	B1
			(1)
6(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 3 & 1 & -1 \end{vmatrix} = \mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$	Attempts cross product of normal vectors. If the method is unclear, look for at least 2 correct components.	M1
	e.g. $x = 0, 2y - 2z = 6, y - 2z = 3$ $\Rightarrow y = 3, z = 0$	Any valid attempt to find a point on the line.	M1
	e.g. (0,3,0)	Any valid point on the line	A1
	$\mathbf{r} = 3\mathbf{j} + \lambda(\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$	Correct equation including "r =" or equivalent e.g. $x = \frac{y-3}{-5} = \frac{z}{-2}$	A1
			(4)
6(c) ALT 1	$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k}), \mathbf{r} \cdot (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 3$ $\Rightarrow 1 + \lambda + \mu + 1 - 2\mu - 2 - 6\lambda - 2\mu = 3$ Forms equation of first plane using (1, 1, 1) and direction vectors and substitutes into the second plane to form an equation in λ and μ		M1
	$\Rightarrow \mu = \frac{1}{3}(-5\lambda - 3)$	Solves to obtain μ in terms of λ or λ in terms of μ	M1
	E.g. $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \frac{1}{3}(-5\lambda - 3)(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ Correct equation including "r ="	Correct equation	A1
6(c) ALT 2	$3x + y - z = 3, x + y - 2z = 3 \Rightarrow 2x + z = 0$	Uses the Cartesian equations of both planes and eliminates one variable	M1
	$z = \lambda \Rightarrow x = -\frac{1}{2}\lambda, y = 3 + 2z - x = 3 + \frac{5}{2}\lambda$	Introduces parameter and expresses other 2 variables in terms of the parameter	M1
	$\mathbf{r} = 3\mathbf{j} + \lambda(\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$	Correct equations	A1
		Correct equation including "r =" or equivalent e.g. $x = \frac{y-3}{-5} = \frac{z}{-2}$	A1
6(c) ALT 3	$3x + y - z = 3, x + y - 2z = 3 \Rightarrow 2x + z = 0$	Uses the Cartesian equations of both planes and eliminates one variable	M1

	$3x + y - z = 3, \quad x + y - 2z = 3 \Rightarrow 5x + y = 3$	Uses the Cartesian equations of both planes and eliminates another variable	M1
	$\Rightarrow x = -\frac{z}{2}, \quad x = \frac{3-y}{5}$	Correct equations for one variable in terms of the other 2	A1
	$x = \frac{y-3}{-5} = \frac{z}{-2}$	Correct equation or equivalent e.g. $x = \frac{3-y}{5} = \frac{z}{-2}$	A1
6(d)	$(3\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 6$	Correct value for scalar product	B1
	$\cos \theta = \frac{(3\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} - 2\mathbf{k})}{\sqrt{9+1+1}\sqrt{1+1+4}} = \frac{\sqrt{6}}{\sqrt{11}}$	Full scalar product attempt to reach a value for $\cos \theta$	M1
		For $\cos \theta = \frac{\sqrt{6}}{\sqrt{11}}$	A1
	$\theta = 42.4^\circ$	Correct value. Mark their final answer.	A1
			(4)
6(d) ALT	$ (3\mathbf{i} + \mathbf{j} - \mathbf{k}) \times (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \sqrt{30}$	Correct value for magnitude of cross product	B1
	$\sin \theta = \frac{ (3\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) }{\sqrt{9+1+1}\sqrt{1+1+4}} = \frac{\sqrt{55}}{11}$	Full attempt to reach a value for $\sin \theta$	M1
		For $\sin \theta = \frac{\sqrt{55}}{11}$	A1
	$\theta = 42.4^\circ$	Correct value. Mark their final answer.	A1
			Total 13