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EDEXCEL A LEVEL MATHEMATICS: QUESTIONS BY TOPIC

Trigonometry (5)

A Level Mathematics Topic 5

50 questions · 345 marks · Mark schemes included

Pearson Edexcel A Level Mathematics (9MA0), June 2018 - June 2025

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HOW TO USE THIS BOOKLET

Questions are grouped by specification sub-topic and run from the lowest mark tariff to the highest within each one.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme.

Diagrams, graphs, tables and data are printed exactly as they appeared in the original papers.

QUESTION	SOURCE	MARKS	PAGE
5.1 TRIGONOMETRIC RATIOS, RADIANs, ARCS AND SECTORS		6 Qs · 42 marks	4
1. Q3 <i>also Topic 3</i>	Paper 1: Pure Mathematics 1 June2018	4	5
2. Q6 <i>also Topic 10</i>	Paper 1: Pure Mathematics 1 November2021	5	7
3. Q6 <i>also Topic 3</i>	Paper 2: Pure Mathematics 2 November2021	5	11
4. Q11	Paper 1: Pure Mathematics 1 October2020	8	15
5. Q8 <i>also Topic 3</i>	Paper 1: Pure Mathematics 1 June2023	10	19
6. Q15 <i>also Topic 1 & 3 & 7</i>	Paper 1: Pure Mathematics 1 June2022	10	23
5.2 SMALL ANGLE APPROXIMATIONS		5 Qs · 20 marks	27
7. Q1	Paper 1: Pure Mathematics 1 June2018	3	28
8. Q4	Paper 2: Pure Mathematics 2 November2021	3	30
9. Q5	Paper 2: Pure Mathematics 2 June2024	3	32
10. Q4 <i>also Topic 7</i>	Paper 1: Pure Mathematics 1 June2023	5	34
11. Q6	Paper 2: Pure Mathematics 2 June2025	6	36
5.3 TRIGONOMETRIC FUNCTIONS, GRAPHS AND EXACT VALUES		5 Qs · 24 marks	38
12. Q9	Paper 2: Pure Mathematics 2 November2021	3	39
13. Q2	Paper 1: Pure Mathematics 1 June2019	5	41
14. Q2	Paper 2: Pure Mathematics 2 June2023	6	43
15. Q6 <i>also Topic 9</i>	Paper 2: Pure Mathematics 2 June2022	7	45
16. Q13	Paper 2: Pure Mathematics 2 June2025	3	49
5.4 SEC, COSEC, COT AND INVERSE TRIG FUNCTIONS		4 Qs · 33 marks	51
17. Q13 <i>also Topic 3 & 7</i>	Paper 2: Pure Mathematics 2 November2021	6	52
18. Q15 <i>also Topic 1 & 7</i>	Paper 1: Pure Mathematics 1 October2020	7	54
19. Q12	Paper 1: Pure Mathematics 1 October2020	8	58
20. Q16 <i>also Topic 3 & 7</i>	Paper 2: Pure Mathematics 2 June2022	12	62
5.5 TRIGONOMETRIC IDENTITIES		5 Qs · 37 marks	67
21. Q5	Paper 1: Pure Mathematics 1 June2018	5	68
22. Q4 <i>also Topic 3</i>	Paper 2: Pure Mathematics 2 June2019	6	70
23. Q8 <i>also Topic 1</i>	Paper 2: Pure Mathematics 2 June2024	7	72
24. Q16 <i>also Topic 3 & 8</i>	Paper 1: Pure Mathematics 1 June2022	9	76
25. Q14 <i>also Topic 3</i>	Paper 1: Pure Mathematics 1 June2018	10	79
5.6 DOUBLE ANGLE, COMPOUND ANGLE AND R FORMULAE		9 Qs · 72 marks	82
26. Q8 <i>also Topic 4</i>	Paper 2: Pure Mathematics 2 June2023	6	83
27. Q12 <i>also Topic 1</i>	Paper 2: Pure Mathematics 2 June2019	7	87



QUESTION		SOURCE	MARKS	PAGE
28. Q10	<i>also Topic 1</i>	Paper 1: Pure Mathematics 1 November2021	8	91
29. Q13	<i>also Topic 8</i>	Paper 1: Pure Mathematics 1 June2024	8	95
30. Q14		Paper 1: Pure Mathematics 1 June2022	8	99
31. Q10		Paper 2: Pure Mathematics 2 October2020	8	103
32. Q12	<i>also Topic 1</i>	Paper 2: Pure Mathematics 2 June2018	9	107
33. Q12	<i>also Topic 3 & 8</i>	Paper 2: Pure Mathematics 2 October2020	11	111
34. Q14		Paper 1: Pure Mathematics 1 June2025	7	115
5.7 TRIGONOMETRIC EQUATIONS			6 Qs · 50 marks	119
35. Q6	<i>also Topic 1</i>	Paper 2: Pure Mathematics 2 June2018	6	120
36. Q14		Paper 2: Pure Mathematics 2 June2023	7	124
37. Q6		Paper 1: Pure Mathematics 1 June2019	8	128
38. Q7		Paper 2: Pure Mathematics 2 June2018	9	132
39. Q12	<i>also Topic 7</i>	Paper 1: Pure Mathematics 1 June2019	10	136
40. Q15	<i>also Topic 4</i>	Paper 2: Pure Mathematics 2 June2022	10	140
5.8 TRIGONOMETRIC PROOFS			3 Qs · 13 marks	144
41. Q15	<i>also Topic 1</i>	Paper 2: Pure Mathematics 2 June2023	3	145
42. Q12	<i>also Topic 1 & 7</i>	Paper 1: Pure Mathematics 1 June2023	5	148
43. Q9	<i>also Topic 1</i>	Paper 2: Pure Mathematics 2 June2018	5	150
5.9 TRIGONOMETRY IN CONTEXT			7 Qs · 54 marks	152
44. Q8		Paper 1: Pure Mathematics 1 June2018	5	153
45. Q9	<i>also Topic 6</i>	Paper 2: Pure Mathematics 2 June2022	5	155
46. Q6	<i>also Topic 6</i>	Paper 1: Pure Mathematics 1 October2020	7	157
47. Q13	<i>also Topic 6</i>	Paper 1: Pure Mathematics 1 June2023	7	161
48. Q10		Paper 1: Pure Mathematics 1 June2018	8	165
49. Q12	<i>also Topic 6</i>	Paper 1: Pure Mathematics 1 June2024	11	169
50. Q15	<i>also Topic 6</i>	Paper 2: Pure Mathematics 2 November2021	11	173



QUESTIONS

50 questions · 345 marks · 9 sub-topics



SUB-TOPIC 5.1

TRIGONOMETRIC RATIOS, RADIANS, ARCS AND SECTORS

Sub-topic 5.1 · 6 questions · 42 marks · lowest tariff first



Question 3 continued

(Total for Question 3 is 4 marks)



6.

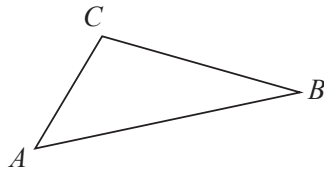


Figure 1

Figure 1 shows a sketch of triangle ABC .

Given that

- $\vec{AB} = -3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$

- $\vec{BC} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$

(a) find $\vec{AC}$

(2)

(b) show that $\cos ABC = \frac{9}{10}$

(3)

Question 6 continued





Question 6 continued

(Total for Question 6 is 5 marks)



Question 6 continued



Question 6 continued



(Total for Question 6 is 5 marks)



Question 11 continued



Question 11 continued



Question 11 continued

(Total for Question 11 is 8 marks)



Question 8 continued



Question 8 continued



Question 8 continued

(Total for Question 8 is 10 marks)



15.

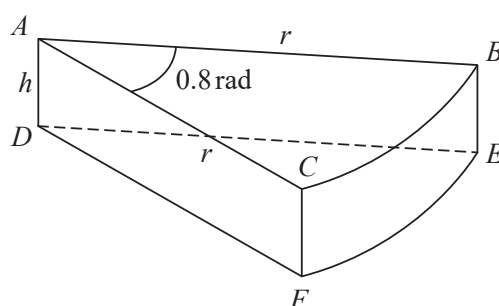


Figure 5

A company makes toys for children.

Figure 5 shows the design for a solid toy that looks like a piece of cheese.

The toy is modelled so that

- face ABC is a sector of a circle with radius r cm and centre A
- angle $BAC = 0.8$ radians
- faces ABC and DEF are congruent
- edges AD , CF and BE are perpendicular to faces ABC and DEF
- edges AD , CF and BE have length h cm

Given that the volume of the toy is 240 cm^3

(a) show that the surface area of the toy, $S \text{ cm}^2$, is given by

$$S = 0.8r^2 + \frac{1680}{r}$$

making your method clear.

(4)

Using algebraic differentiation,

(b) find the value of r for which S has a stationary point.

(4)

(c) Prove, by further differentiation, that this value of r gives the minimum surface area of the toy.

(2)



Question 15 continued



Question 15 continued

(Total for Question 15 is 10 marks)



SUB-TOPIC 5.2

SMALL ANGLE APPROXIMATIONS

Sub-topic 5.2 · 5 questions · 20 marks · lowest tariff first



1. Given that θ is small and is measured in radians, use the small angle approximations to find an approximate value of

$$\frac{1 - \cos 4\theta}{2\theta \sin 3\theta}$$

(3)

4. Given that θ is small and measured in radians, use the small angle approximations to show that

$$4 \sin \frac{\theta}{2} + 3 \cos^2 \theta \approx a + b\theta + c\theta^2$$

where a , b and c are integers to be found.

(3)



(Total for Question 4 is 3 marks)

- $$\frac{\theta \tan 2\theta}{1 - \cos 3\theta}$$

(3)





(Total for Question 4 is 5 marks)



(Total for Question 6 is 6 marks)



SUB-TOPIC 5.3

TRIGONOMETRIC FUNCTIONS, GRAPHS AND EXACT VALUES

Sub-topic 5.3 · 5 questions · 24 marks · lowest tariff first



9. Show that

$$\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n \cos(180n)^{\circ} = \frac{9}{28}$$

(3)



(Total for Question 9 is 3 marks)



Question 2 continued

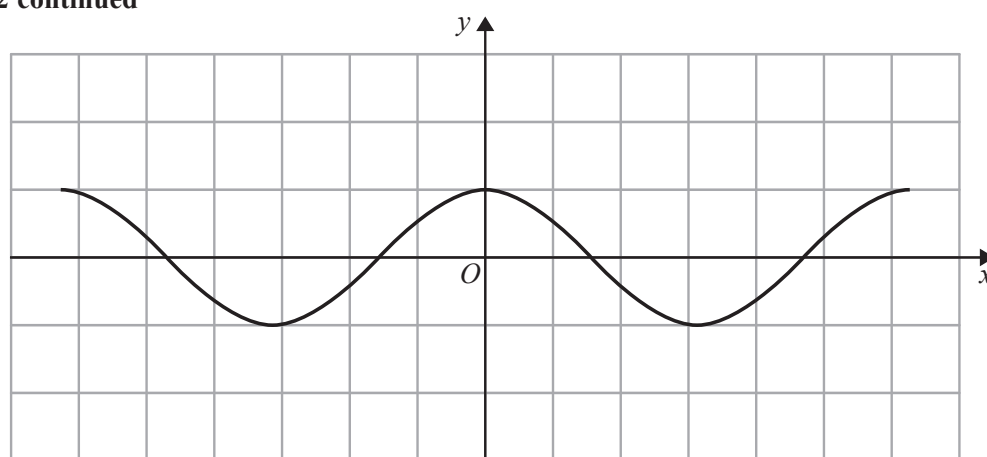


Diagram 1

(Total for Question 2 is 5 marks)

2. A sequence $u_1, u_2, u_3 \dots$ is defined by

$$u_1 = 35$$

$$u_{n+1} = u_n + 7 \cos\left(\frac{n\pi}{2}\right) - 5(-1)^n$$

(a) (i) Show that $u_2 = 40$

(ii) Find the value of u_3 and the value of u_4

(3)

Given that the sequence is periodic with order 4

(b) (i) write down the value of u_5

(ii) find the value of $\sum_{r=1}^{25} u_r$

(3)

Question 2 continued

(Total for Question 2 is 6 marks)



6.

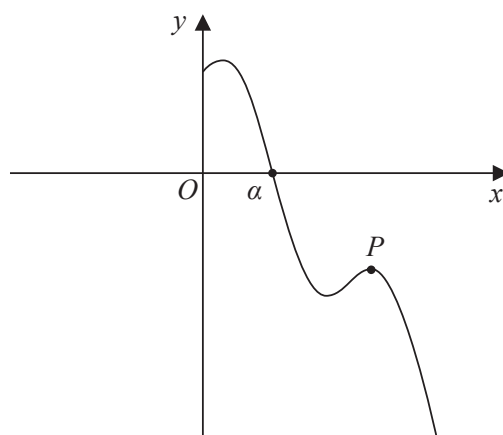


Figure 2

Figure 2 shows a sketch of part of the curve with equation $y = f(x)$ where

$$f(x) = 8 \sin\left(\frac{1}{2}x\right) - 3x + 9 \quad x > 0$$

and x is measured in radians.

The point P , shown in Figure 2, is a local maximum point on the curve.

Using calculus and the sketch in Figure 2,

- (a) find the x coordinate of P , giving your answer to 3 significant figures.

(4)

The curve crosses the x -axis at $x = \alpha$, as shown in Figure 2.

Given that, to 3 decimal places, $f(4) = 4.274$ and $f(5) = -1.212$

- (b) explain why α must lie in the interval $[4, 5]$

(1)

- (c) Taking $x_0 = 5$ as a first approximation to α , apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation to α .

Show your method and give your answer to 3 significant figures.

(2)

Question 6 continued



Question 6 continued



(Total for Question 6 is / marks)



Question 13 continued

(Total for Question 13 is 3 marks)



SUB-TOPIC 5.4

SEC, COSEC, COT AND INVERSE TRIG FUNCTIONS

Sub-topic 5.4 · 4 questions · 33 marks · lowest tariff first



13. The curve C has parametric equations

$$x = \sin 2\theta \quad y = \operatorname{cosec}^3 \theta \quad 0 < \theta < \frac{\pi}{2}$$

- (a) Find an expression for $\frac{dy}{dx}$ in terms of θ

(3)

- (b) Hence find the exact value of the gradient of the tangent to C at the point where $y = 8$

(3)



Question 13 continued

(Total for Question 13 is 6 marks)



15. The curve C has equation

$$x^2 \tan y = 9 \qquad 0 < y < \frac{\pi}{2}$$

(a) Show that

$$\frac{dy}{dx} = \frac{-18x}{x^4 + 81} \quad (4)$$

(b) Prove that C has a point of inflection at $x = \sqrt[4]{27}$

(3)

Question 15 continued



Question 15 continued

Question 15 continued

(Total for Question 15 is 7 marks)



12.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\operatorname{cosec} \theta - \sin \theta \equiv \cos \theta \cot \theta \quad \theta \neq (180n)^\circ \quad n \in \mathbb{Z} \quad (3)$$

(b) Hence, or otherwise, solve for $0 < x < 180^\circ$

$$\operatorname{cosec} x - \sin x = \cos x \cot(3x - 50^\circ) \quad (5)$$

Question 12 continued



Question 12 continued



Question 12 continued

(Total for Question 12 is 8 marks)



16.

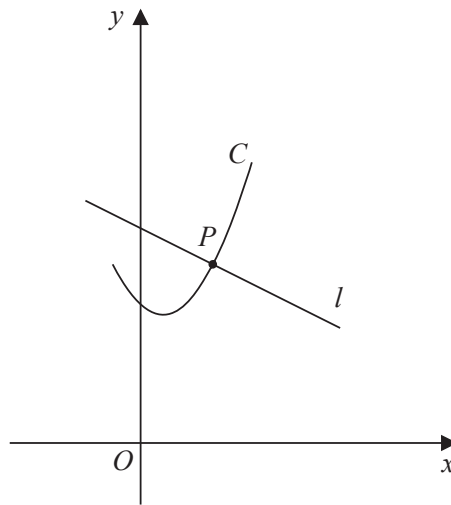


Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 2 \tan t + 1 \quad y = 2 \sec^2 t + 3 \quad -\frac{\pi}{4} \leq t \leq \frac{\pi}{3}$$

The line l is the normal to C at the point P where $t = \frac{\pi}{4}$

(a) Using parametric differentiation, show that an equation for l is

$$y = -\frac{1}{2}x + \frac{17}{2} \quad (5)$$

(b) Show that all points on C satisfy the equation

$$y = \frac{1}{2}(x-1)^2 + 5 \quad (2)$$

The straight line with equation

$$y = -\frac{1}{2}x + k \quad \text{where } k \text{ is a constant}$$

intersects C at two distinct points.

(c) Find the range of possible values for k .

(5)

Question 16 continued



Question 16 continued





Question 16 continued

(Total for Question 16 is 12 marks)



SUB-TOPIC 5.5

TRIGONOMETRIC IDENTITIES

Sub-topic 5.5 · 5 questions · 37 marks · lowest tariff first



5. Given that

$$y = \frac{3\sin\theta}{2\sin\theta + 2\cos\theta} \quad -\frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

show that

$$\frac{dy}{d\theta} = \frac{A}{1 + \sin 2\theta} \quad -\frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

where A is a rational constant to be found.

(5)

The diagram shows a Cartesian coordinate system with a horizontal x-axis and a vertical y-axis intersecting at the origin, labeled O . Two ellipses, C_1 and C_2 , are centered at O . Ellipse C_1 is elongated along the x-axis, while ellipse C_2 is more circular. They intersect at two points, with one intersection point labeled S in the first quadrant.

(6)



8.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(a) Prove that

$$\frac{1}{\operatorname{cosec} \theta - 1} + \frac{1}{\operatorname{cosec} \theta + 1} \equiv 2 \tan \theta \sec \theta \quad \theta \neq (90n)^\circ, n \in \mathbb{Z} \quad (3)$$

(b) Hence solve, for $0 < x < 90^\circ$, the equation

$$\frac{1}{\operatorname{cosec} 2x - 1} + \frac{1}{\operatorname{cosec} 2x + 1} = \cot 2x \sec 2x$$

Give each answer, in degrees, to one decimal place.

(4)







(Total for Question 8 is 7 marks)







14. A curve C has parametric equations

$$x = 3 + 2 \sin t, \quad y = 4 + 2 \cos 2t, \quad 0 \leq t < 2\pi$$

- (a) Show that all points on C satisfy $y = 6 - (x - 3)^2$ (2)

- (b) (i) Sketch the curve C .

- (ii) Explain briefly why C does not include all points of $y = 6 - (x - 3)^2$, $x \in \mathbb{R}$ (3)

The line with equation $x + y = k$, where k is a constant, intersects C at two distinct points.

- (c) State the range of values of k , writing your answer in set notation. (5)

Question 14 continued

(Total for Question 14 is 10 marks)



SUB-TOPIC 5.6

DOUBLE ANGLE, COMPOUND ANGLE AND R FORMULAE

Sub-topic 5.6 · 9 questions · 72 marks · lowest tariff first



Question 8 continued



Question 8 continued



Question 8 continued

(Total for Question 8 is 6 marks)



Question 12 continued



Question 12 continued



Question 12 continued

(Total for Question 12 is 7 marks)



10.

In this question you should show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(a) Given that $1 + \cos 2\theta + \sin 2\theta \neq 0$ prove that

$$\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} \equiv \tan \theta \quad (4)$$

(b) Hence solve, for $0 < x < 180^\circ$

$$\frac{1 - \cos 4x + \sin 4x}{1 + \cos 4x + \sin 4x} = 3 \sin 2x$$

giving your answers to one decimal place where appropriate.

(4)



Question 10 continued





(Total for Question 10 is 8 marks)



13. (a) Given that a is a positive constant, use the substitution $x = a \sin^2 \theta$ to show that

$$\int_0^a x^{\frac{1}{2}} \sqrt{a-x} \, dx = \frac{1}{2} a^2 \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta \quad (4)$$

(b) Hence use algebraic integration to show that

$$\int_0^a x^{\frac{1}{2}} \sqrt{a-x} \, dx = k\pi a^2$$

where k is a constant to be found.

(4)

Question 13 continued



Question 13 continued



Question 13 continued

(Total for Question 13 is 8 marks)



Question 14 continued





Question 14 continued

(Total for Question 14 is 8 marks)



10.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\cos 3A \equiv 4 \cos^3 A - 3 \cos A \quad (4)$$

(b) Hence solve, for $-90^\circ \leq x \leq 180^\circ$, the equation

$$1 - \cos 3x = \sin^2 x \quad (4)$$



Question 10 continued



Question 10 continued



Question 10 continued

(Total for Question 10 is 8 marks)



Question 12 continued



Question 12 continued

(Total for Question 12 is 9 marks)



12.

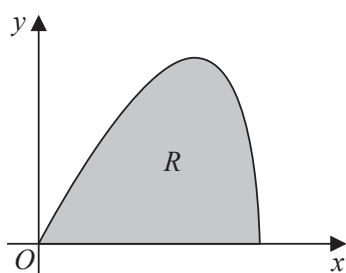


Figure 3

The curve shown in Figure 3 has parametric equations

$$x = 6 \sin t \quad y = 5 \sin 2t \quad 0 \leq t \leq \frac{\pi}{2}$$

The region R , shown shaded in Figure 3, is bounded by the curve and the x -axis.

(a) (i) Show that the area of R is given by $\int_0^{\frac{\pi}{2}} 60 \sin t \cos^2 t \, dt$

(3)

(ii) Hence show, by algebraic integration, that the area of R is exactly 20

(3)

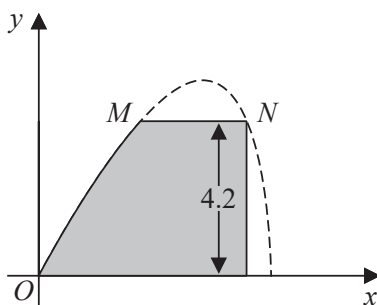


Figure 4

Part of the curve is used to model the profile of a small dam, shown shaded in Figure 4.

Using the model and given that

- x and y are in metres
- the vertical wall of the dam is 4.2 metres high
- there is a horizontal walkway of width MN along the top of the dam

(b) calculate the width of the walkway.

(5)

Question 12 continued



Question 12 continued



(Total for Question 12 is 11 marks)



Question 14 continued

Question 14 continued



Question 14 continued

(Total for Question 14 is 7 marks)



SUB-TOPIC 5.7

TRIGONOMETRIC EQUATIONS

Sub-topic 5.7 · 6 questions · 50 marks · lowest tariff first







(Total for Question 6 is 6 marks)







(Total for Question 14 is 7 marks)









7. (i) Solve, for $0 \leq x < \frac{\pi}{2}$, the equation

$$4 \sin x = \sec x$$

(4)

(ii) Solve, for $0 \leq \theta < 360^\circ$, the equation

$$5 \sin \theta - 5 \cos \theta = 2$$

giving your answers to one decimal place.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)





(Total for Question 7 is 10 marks)



12.

$$f(x) = 10e^{-0.25x} \sin x, \quad x \geq 0$$

- (a) Show that the x coordinates of the turning points of the curve with equation $y = f(x)$ satisfy the equation $\tan x = 4$

(4)

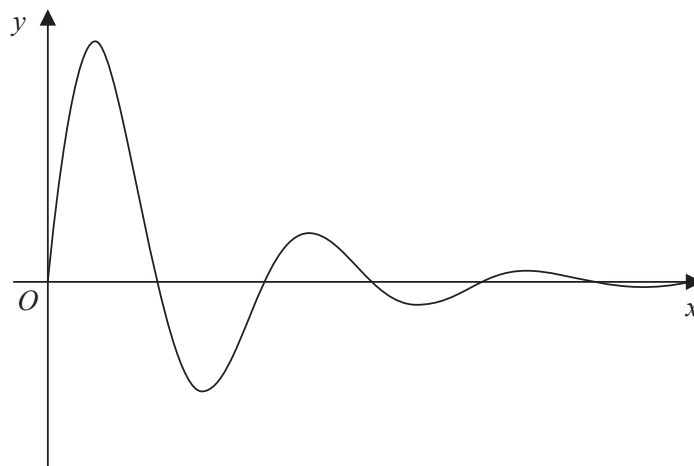


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = f(x)$.

- (b) Sketch the graph of H against t where

$$H(t) = |10e^{-0.25t} \sin t| \quad t \geq 0$$

showing the long-term behaviour of this curve.

(2)

The function $H(t)$ is used to model the height, in metres, of a ball above the ground t seconds after it has been kicked.

Using this model, find

- (c) the maximum height of the ball above the ground between the first and second bounce.

(3)

- (d) Explain why this model should not be used to predict the time of each bounce.

(1)



(Total for Question 12 is 10 marks)





(Total for Question 15 is 10 marks)

SUB-TOPIC 5.8

TRIGONOMETRIC PROOFS

Sub-topic 5.8 · 3 questions · 13 marks · lowest tariff first







(Total for Question 12 is 5 marks)



9. Given that θ is measured in radians, prove, from first principles, that

$$\frac{d}{d\theta}(\cos \theta) = -\sin \theta$$

You may assume the formula for $\cos(A \pm B)$ and that as $h \rightarrow 0$, $\frac{\sin h}{h} \rightarrow 1$ and $\frac{\cosh h - 1}{h} \rightarrow 0$

(5)

Question 9 continued

(Total for Question 9 is 5 marks)



SUB-TOPIC 5.9

TRIGONOMETRY IN CONTEXT

Sub-topic 5.9 · 7 questions · 54 marks · lowest tariff first



(Total for Question 8 is 5 marks)

9.

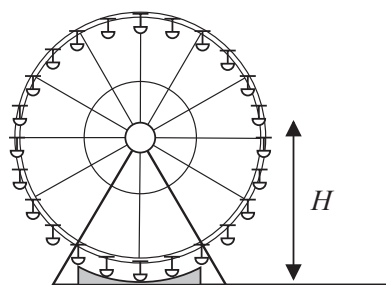


Figure 4

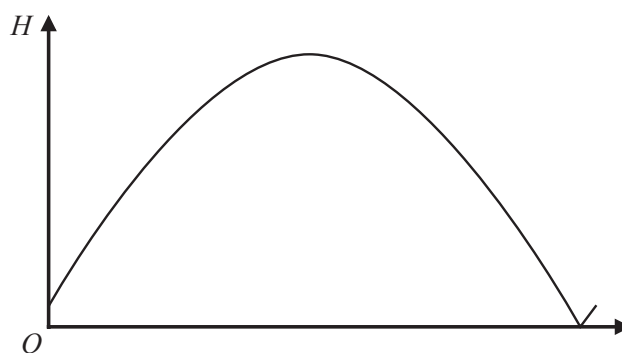


Figure 5

Figure 4 shows a sketch of a Ferris wheel.

The height above the ground, H m, of a passenger on the Ferris wheel, t seconds after the wheel starts turning, is modelled by the equation

$$H = |A \sin(bt + \alpha)^\circ|$$

where A , b and α are constants.

Figure 5 shows a sketch of the graph of H against t , for one revolution of the wheel.

Given that

- the maximum height of the passenger above the ground is 50 m
- the passenger is 1 m above the ground when the wheel starts turning
- the wheel takes 720 seconds to complete one revolution

(a) find a complete equation for the model, giving the exact value of A , the exact value of b and the value of α to 3 significant figures.

(4)

(b) Explain why an equation of the form

$$H = |A \sin(bt + \alpha)^\circ| + d$$

where d is a positive constant, would be a more appropriate model.

(1)

(Total for Question 3 is 5 marks)







(Total for Question 6 is 7 marks)

13. On a roller coaster ride, passengers travel in carriages around a track.

On the ride, carriages complete multiple circuits of the track such that

- the maximum vertical height of a carriage above the ground is 60 m
- a carriage starts a circuit at a vertical height of 2 m above the ground
- the ground is horizontal

The vertical height, H m, of a carriage above the ground, t seconds after the carriage starts the first circuit, is modelled by the equation

$$H = a - b(t - 20)^2$$

where a and b are positive constants.

- (a) Find a complete equation for the model. (3)
- (b) Use the model to determine the height of the carriage above the ground when $t = 40$ (1)

In an alternative model, the vertical height, H m, of a carriage above the ground, t seconds after the carriage starts the first circuit, is given by

$$H = 29 \cos(9t + \alpha)^\circ + \beta \quad 0 \leq \alpha < 360^\circ$$

where α and β are constants.

- (c) Find a complete equation for the alternative model. (2)

Given that the carriage moves continuously for 2 minutes,

- (d) give a reason why the alternative model would be more appropriate. (1)



Question 13 continued



(Total for Question 13 is 7 marks)







(Total for Question 10 is 6 marks)







(Total for Question 12 is 11 marks)



15. (a) Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and the value of α in radians to 3 decimal places.

(3)

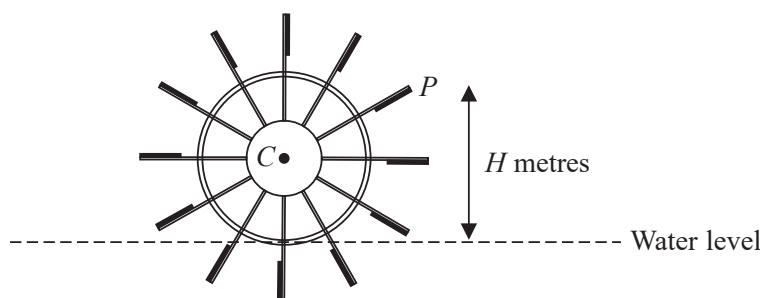


Figure 6

Figure 6 shows the cross-section of a water wheel.

The wheel is free to rotate about a fixed axis through the point C .

The point P is at the end of one of the paddles of the wheel, as shown in Figure 6.

The water level is assumed to be horizontal and of constant height.

The vertical height, H metres, of P above the water level is modelled by the equation

$$H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$$

where t is the time in seconds after the wheel starts rotating.

Using the model, find

- (b) (i) the maximum height of P above the water level,
 (ii) the value of t when this maximum height first occurs, giving your answer to one decimal place.

(3)

In a single revolution of the wheel, P is below the water level for a total of T seconds.

According to the model,

- (c) find the value of T giving your answer to 3 significant figures.

(Solutions based entirely on calculator technology are not acceptable.)

(4)

In reality, the water level may not be of constant height.

- (d) Explain how the equation of the model should be refined to take this into account.

(1)







(Total for Question 15 is 11 marks)

MARK SCHEMES

official mark schemes



Question	Scheme	Marks	AOs
3	States or uses $\frac{1}{2}r^2\theta = 11$	B1	1.1b
	States or uses $2r + r\theta = 4r\theta$	B1	1.1b
	Attempts to solve, full method $r = \dots$	M1	3.1a
	$r = \sqrt{33}$	A1	1.1b
			[4]

(4 marks)

Notes:

B1: States or uses $\frac{1}{2}r^2\theta = 11$ This may be implied with an embedded found value for θ

B1: States or uses $2r + r\theta = 4r\theta$ or equivalent

M1: Full method to find $r = \dots$ This involves combining the equations to eliminate θ or find θ
The initial equations must be of the same "form" (see **) but condone slips when attempting to solve.

It cannot be scored from impossible values for θ Hence only score if $0 < \theta < 2\pi$ FYI $\theta = \frac{2}{3}$ radians

Allow this to be scored from equations such as $\dots r^2\theta = 11$ and ones that simplify to $\dots r = \dots r\theta$ **

Allow their $2r + r\theta = 4r\theta \Rightarrow \theta = \dots$ then substitute this into their $\frac{1}{2}r^2\theta = 11$

Allow their $2r + r\theta = 4r\theta \Rightarrow r\theta = \dots$ then substitute this into their $\frac{1}{2}r^2\theta = 11$

Allow their $\frac{1}{2}r^2\theta = 11 \Rightarrow \theta = \frac{\dots}{r^2}$ then substitute into their $2r + r\theta = 4r\theta \Rightarrow r = \dots$

A1: $r = \sqrt{33}$ only but isw after a correct answer.

.....
The whole question can be attempted using θ in degrees.

B1: States or uses $\frac{\theta}{360} \times \pi r^2 = 11$

B1: States or uses $2r + \frac{\theta}{360} \times 2\pi r = 4 \times \frac{\theta}{360} \times 2\pi r$



Question	Scheme	Marks	AOs
6(a)	$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = -3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k} + \mathbf{i} + \mathbf{j} + 4\mathbf{k} = \dots$	M1	1.1b
	$= -2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$	A1	1.1b
		(2)	
(b)	At least 2 of $(AC^2) = 2^2 + 3^2 + 1^2, (AB^2) = 3^2 + 4^2 + 5^2, (BC^2) = 1^2 + 1^2 + 4^2$	M1	1.1b
	$2^2 + 3^2 + 1^2 = 3^2 + 4^2 + 5^2 + 1^2 + 1^2 + 4^2 - 2\sqrt{3^2 + 4^2 + 5^2}\sqrt{1^2 + 1^2 + 4^2} \cos ABC$	M1	3.1a
	$14 = 50 + 18 - 2\sqrt{50}\sqrt{18} \cos ABC$ $\Rightarrow \cos ABC = \frac{50 + 18 - 14}{2\sqrt{50}\sqrt{18}} = \frac{9}{10}^*$	A1*	2.1
		(3)	
	(b) Alternative		
	$AB^2 = 3^2 + 4^2 + 5^2, BC^2 = 1^2 + 1^2 + 4^2$	M1	1.1b
	$\overrightarrow{BA} \cdot \overrightarrow{BC} = (3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = 27 = \sqrt{3^2 + 4^2 + 5^2}\sqrt{1^2 + 1^2 + 4^2} \cos ABC$	M1	3.1a
	$27 = \sqrt{50}\sqrt{18} \cos ABC \Rightarrow \cos ABC = \frac{27}{\sqrt{50}\sqrt{18}} = \frac{9}{10}^*$	A1*	2.1
(5 marks)			
Notes			

(a)

 M1: Attempts $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

There must be attempt to add not subtract.

 If no method shown it may be implied by **two** correct components

 A1: Correct vector. Allow $-2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$ but not $\begin{pmatrix} -2\mathbf{i} \\ -3\mathbf{j} \\ -1\mathbf{k} \end{pmatrix}$

(b)

 M1: Attempts to "square and add" for at least 2 of the 3 sides. Follow through on their $\overrightarrow{AC}$

 Look for an attempt at either $a^2 + b^2 + c^2$ or $\sqrt{a^2 + b^2 + c^2}$

 M1: A correct attempt to apply a correct cosine rule to the given problem; Condone **slips** on the lengths of the sides but the sides must be in the correct position to find angle ABC

A1*: Correct completion with sufficient intermediate work to establish the printed result.

 Condone different labelling, e.g. $ABC \leftrightarrow \theta$ as long as it is clear what is meant

 It is OK to move from a correct cosine rule $14 = 50 + 18 - 2\sqrt{50}\sqrt{18} \cos ABC$

$$\text{via } \cos ABC = \frac{54}{2\sqrt{50}\sqrt{18}} \text{ o.e. such as } \cos ABC = \frac{(5\sqrt{2})^2 + (3\sqrt{2})^2 - (\sqrt{14})^2}{2 \times 5\sqrt{2} \times 3\sqrt{2}} \text{ to } \cos ABC = \frac{9}{10}$$

Alternative:

 M1: Correct application of Pythagoras for sides AB and BC or their squares

M1: Recognises the requirement for and applies the scalar product

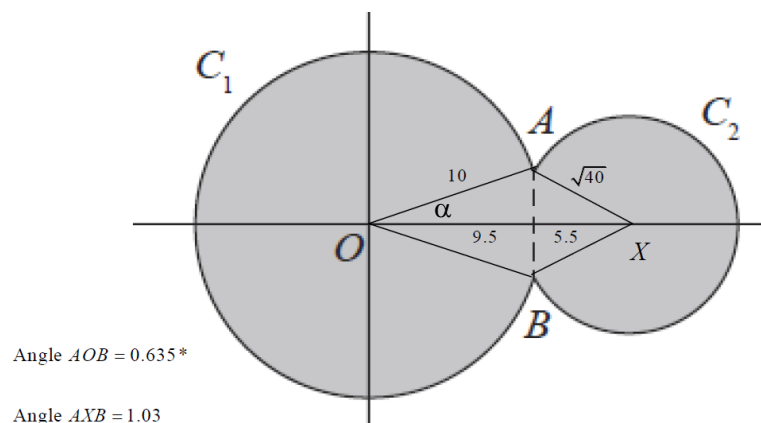
A1*: Correct completion with sufficient intermediate work to establish the printed result



Question	Scheme	Marks	AOs
6(a)	Angle $AOB = \frac{\pi - \theta}{2}$	B1	2.2a
		(1)	
(b)	Area = $2 \times \frac{1}{2} r^2 \left(\frac{\pi - \theta}{2} \right) + \frac{1}{2} (2r)^2 \theta$	M1	2.1
	$= \frac{1}{2} r^2 \pi - \frac{1}{2} r^2 \theta + 2r^2 \theta = \frac{3}{2} r^2 \theta + \frac{1}{2} r^2 \pi = \frac{1}{2} r^2 (3\theta + \pi)^*$	A1*	1.1b
		(2)	
(c)	Perimeter = $4r + 2r \left(\frac{\pi - \theta}{2} \right) + 2r\theta$	M1	3.1a
	$= 4r + r\pi + r\theta$ or e.g. $r(4 + \pi + \theta)$	A1	1.1b
		(2)	
(5 marks)			
Notes			
(a) B1: Deduces the correct expression for angle AOB Note that $\frac{180 - \theta}{2}$ scores B0 (b) M1: Fully correct strategy for the area using their angle from (a) appropriately. Need to see $2 \times \frac{1}{2} r^2 \alpha$ or just $r^2 \alpha$ where α is their angle in terms of θ from part (a) + $\frac{1}{2} (2r)^2 \theta$ with or without the brackets. A1*: Correct proof. For this mark you can condone the omission of the brackets in $\frac{1}{2} (2r)^2 \theta$ as long as they are recovered in subsequent work e.g. when this term becomes $2r^2 \theta$ The first term must be seen expanded as e.g. $\frac{1}{2} r^2 \pi - \frac{1}{2} r^2 \theta$ or equivalent (c) M1: Fully correct strategy for the perimeter using their angle from (a) appropriately Need to see $4r + 2r\alpha + 2r\theta$ where α is their angle from part (a) in terms of θ A1: Correct simplified expression Note that some candidates may change the angle to degrees at the start and all marks are available e.g. (a) $\frac{180 - \frac{180\theta}{\pi}}{2}$ (b) $2 \left(\frac{180 - \frac{180\theta}{\pi}}{2} \right) \times \frac{1}{360} \times \pi r^2 + \frac{\theta}{360} \times \frac{180}{\pi} \times \pi (2r)^2 = \frac{1}{2} \pi r^2 - \frac{1}{2} r^2 \theta + 2r^2 \theta = \frac{1}{2} r^2 (3\theta + \pi)$ (c) $4r + 2 \left(\frac{180 - \frac{180\theta}{\pi}}{2} \right) \times \frac{1}{360} \times 2\pi r + \frac{180\theta}{\pi} \times \frac{1}{360} \times 2\pi (2r) = 4r + \pi r + r\theta$			



Question	Scheme	Marks	AOs
11 (a)	Solves $x^2 + y^2 = 100$ and $(x - 15)^2 + y^2 = 40$ simultaneously to find x or y E.g. $(x - 15)^2 + 100 - x^2 = 40 \Rightarrow x = \dots$	M1	3.1a
	Either $\Rightarrow -30x + 325 = 40 \Rightarrow x = 9.5$ Or $y = \frac{\sqrt{39}}{2} = \text{awrt } \pm 3.12$	A1	1.1b
	Attempts to find the angle AOB in circle C_1 Eg Attempts $\cos \alpha = \frac{9.5}{10}$ to find α then $\times 2$	M1	3.1a
	Angle $AOB = 2 \times \arccos\left(\frac{9.5}{10}\right) = 0.635 \text{ rads (3sf)}$ *	A1*	2.1
		(4)	
(b)	Attempts $10 \times (2\pi - 0.635) = 56.48$	M1	1.1b
	Attempts to find angle AXB or AXO in circle C_2 (see diagram) E.g. $\cos \beta = \frac{15 - 9.5}{\sqrt{40}} \Rightarrow \beta = \dots$ (Note $AXB = 1.03 \text{ rads}$)	M1	3.1a
	Attempts $10 \times (2\pi - 0.635) + \sqrt{40} \times (2\pi - 2\beta)$	dM1	2.1
	$= 89.7$	A1	1.1b
		(4)	
(8 marks)			
Notes:			



(a)

M1: For the key step in an attempt to find either coordinate for where the two circles meet.

 Look for an attempt to set up an equation in a single variable leading to a value for x or y .

A1: $x = 9.5$ (or $y = \frac{\sqrt{39}}{2} = \text{awrt } \pm 3.12$)

M1: Uses the radius of the circle and correct trigonometry in an attempt to find angle AOB in circle C_1

E.g. Attempts $\cos \alpha = \frac{9.5}{10}$ to find α then $\times 2$

Alternatives include $\tan \alpha = \frac{\sqrt{100 - 9.5^2}}{9.5} = (0.3286...)$ to find α then $\times 2$

$$\text{And } \cos AOB = \frac{10^2 + 10^2 - (\sqrt{39})^2}{2 \times 10 \times 10} = \frac{161}{200}$$

A1*: Correct and careful work in proceeding to the given answer. Condone an answer with greater accuracy. Condone a solution where the intermediate value has been truncated, provided the trig equation is correct

$$\text{E.g. } \sin \alpha = \frac{\sqrt{39}}{20} \Rightarrow \alpha = 0.317 \Rightarrow AOB = 2\alpha = 0.635$$

Condone a solution written down from awrt 36.4° (without the need to shown any calculation.)

E

(b)

M1: Attempts to use the formula $s = r\theta$ with $r = 10$ and $\theta = 2\pi - 0.635$

The formula may be embedded. You may see $\underline{2\pi \times 10} - \underline{10 \times 0.635...}$ which is fine for this M1

M1: Attempts to use a correct method in order to find angle AXB or AXO in circle C_2

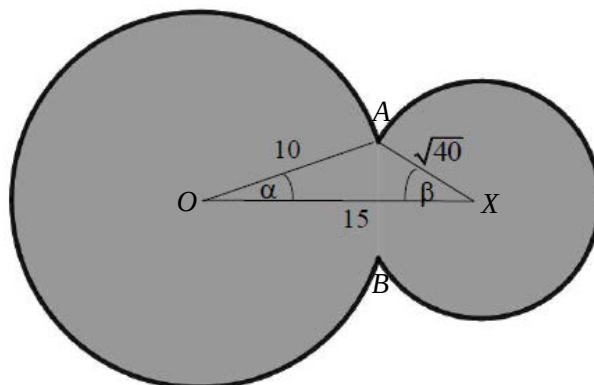
$$\text{Amongst many other methods are } \tan \beta = \frac{3.12}{15 - 9.5} \text{ and } \cos AXB = \frac{40 + 40 - (\sqrt{39})^2}{2 \times \sqrt{40} \times \sqrt{40}} = \frac{41}{80}$$

Note that many candidates believe this to be 0.635. This scores M0 dM0 A0

dM1: A full and complete attempt to find the perimeter of the region.

It is dependent upon having scored both M's.

A1: awrt 89.7



(a)

M1: For the key step in attempting to find all lengths in triangle OAX , condoning slips

A1: All three lengths correct

M1: Attempts cosine rule to find α then $\times 2$

A1*: Correct and careful work in proceeding to the given answer

Question	Scheme	Marks	AOs
8(a)	$OC \times 2.3 = 27.6$	M1	1.1b
	e.g. $OC = \frac{27.6}{2.3} = 12 \text{ m}^*$	A1*	2.1
		(2)	
(b)	e.g. $(2AOB =) \pi - 2.3$	M1	1.1b
	$\frac{\pi - 2.3}{2} \Rightarrow 0.421 \text{ rad}^*$	A1*	2.1
		(2)	
(c)	Area $OCDE = \frac{1}{2} \times 12^2 \times 2.3$	M1	1.1b
	$= 165.6 \text{ (m}^2\text{)} \text{ (accept awrt 166)}$	A1	1.1b
	$(OB =) \frac{35 - 27.6}{2} + 12 = 15.7 \text{ m}$	B1	2.1
	Area of OAB (or OFG) $= \frac{1}{2} \times "15.7" \times 7.5 \times \sin 0.421 \text{ (= 24.0...m}^2\text{)}$	M1	1.1b
	Total area $= "165.6" + 2 \times "24.1"$	dM1	3.1a
	$= \text{awrt } 214 \text{ (m}^2\text{)}$	A1	1.1b
		(6)	
(10 marks)			
Notes			
(a)			
M1: Uses $l = r\theta$ with $l = 27.6$ and $\theta = 2.3$ correctly substituted in (may be labelled differently in their equation). Values just need to be embedded in an equation or accept an expression for OC e.g. $\frac{27.6}{2.3}$. May work in degrees which is acceptable. Condone an alternative letter being used to denote OC such as r Alternatively, they use $l = r\theta$ with $r = 12$ and $\theta = 2.3$ and verify that $l = 27.6 \text{ m}$			
A1*: Achieves an expression for OC before proceeding to $OC = 12 \text{ (m)}$ with no errors seen (condone lack of units) They must show at least $\frac{27.6}{2.3} \Rightarrow OC = 12 \text{ (m)}$ which can score M1A1* $r = \frac{27.6}{2.3} = 12$ is M1A1* (condone alternative letters for OC) BUT e.g. $\frac{27.6}{2.3} = 12 \text{ (m)}$ on its own is M1A0* e.g. $OC \times 2.3 = 27.6 \Rightarrow OC = 12 \text{ (m)}$ is M1A0* In the alternative method they verify $l = 27.6$ and conclude that $OC = 12 \text{ m}$ We must see the calculation $12 \times 2.3 = 27.6$ and conclude that $OC = 12 \text{ (m)}$ e.g. $\text{arc} = 12 \times 2.3 = 27.6$ so $OC = 12 \text{ (m)}$ is M1A1* whereas $12 \times 2.3 = 27.6$ is M1A0* Also allow e.g. if $OC = 12 \text{ (m)}$ then $12 \times 2.3 = 27.6 \checkmark$ is M1A1* If they work in degrees and use rounded values this scores A0* (If they work with e.g. $\frac{414}{\pi}$ to keep the angle exact then A1* can still be scored)			

(b)

M1: Attempts to subtract 2.3 from π (which may be implied by an expression for AOB which is not the given answer)

e.g. $\frac{1}{2}(\pi - 2.3)$ or $\frac{\pi}{2} - 1.15$ score M1

May work in degrees e.g. $180 - \text{awrt } 132$ is M1

Condone invisible brackets e.g. $\pi - 2.3 \div 2$ can still score M1.

A1*: Achieves 0.421 (rad) with no errors seen (ignore any side working which is not part of their main solution). Look for a correct expression which is awrt 0.421 before proceeding to the answer.

Alternatively, they may write

e.g. $2AOB = \pi - 2.3 (= 0.8415...) \Rightarrow AOB = 0.421$

Condone if they do not round their answer at the end to 0.421.

Condone lack of units. Condone poor labelling of other angles and it does not require $AOB =$ to score this mark, but do not accept e.g. $ABO =$

If they work in degrees then withhold this mark if they do not show the conversion back to radians.

e.g. $\frac{\pi - 2.3}{2} = 0.421 \text{ (rad)}$ is M1A1*

e.g. $\frac{180 - \text{awrt } 131.8}{2} \div \frac{180}{\pi} = 0.421 \text{ (rad)}$ is M1A1* (conversion from degrees to radians seen)

e.g. $\pi - 2.3 \div 2 = 0.421 \text{ (rad)}$ is M1A0* (invisible/lack of brackets)

e.g. $\pi - 2.3 = \frac{0.842...}{2} = 0.421$ M1A0* (incorrect joined statement)

(c)

M1: Attempts to use $A = \frac{1}{2}r^2\theta$ with $r = 12$ and $\theta = 2.3$ The values embedded in the formula is sufficient for this mark. May be implied by a correct answer or further work. Look out for alternative more complex ways to find the area of the sector. e.g. area of semicircle – area of two sectors with $r = 12$ and $\theta = 0.421$

A1: awrt 166 (may be implied by later work)

B1: A correct expression or value for the length OB or OF which may be a part of a calculation (may see 15.7 in the equation to find the area of AOB). May be implied by a correct value for the area of a congruent triangle (or both)

M1: Attempts to find the area of at least one of the two congruent triangles using their OB found from $\frac{35 - 27.6}{2} + 12 (= 15.7)$, $OA = 7.5$ and $\theta = 0.421$ in $\frac{1}{2} \times OA \times OB \times \sin C$ (may work in degrees)

Be aware that omitting sine in the formula may give a value close to the area of the triangle which would be M0. **Condone use of $\theta = 0.4$ or $\theta = 0.42$** if they have rounded angle AOB .

The values embedded in the expression is sufficient to score the mark or may be implied by the value.

Look out for more complex methods to find the area of one or both of the two congruent triangles

e.g. they may split the congruent triangle into two right angled triangles and add the separate areas.



Other alternatives

 e.g. finding the area of the trapezium $ABFG$:

$$BF = 2 \times 15.7 \cos 0.421$$

$$\text{Area of } AOB = \frac{1}{2} \left(\left(\frac{15 + 2 \times 15.7 \cos 0.421}{2} \right) \times 15.7 \sin 0.421 - \frac{1}{2} \times 15.7^2 \times \sin 2.3 \right) \text{ o.e.}$$

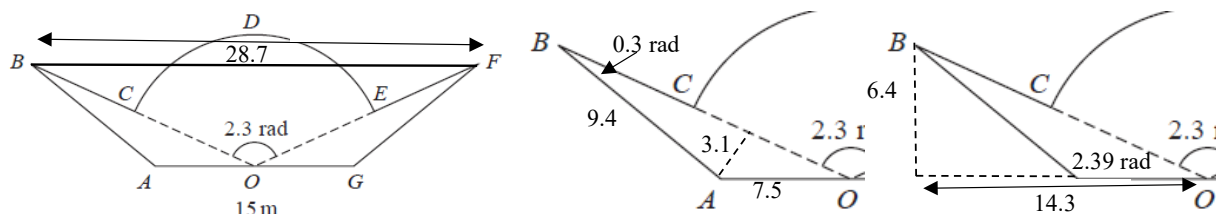
 e.g. finding the length AB and either angle OAB or angle ABO :

$$AB^2 = 15.7^2 + 7.5^2 - 2 \times 15.7 \times 7.5 \times \cos 0.421 \Rightarrow AB = 9.37...$$

$$\frac{\sin ABO}{7.5} = \frac{\sin 0.421}{9.37...} \Rightarrow ABO = 0.333... \quad \text{or} \quad \frac{\sin OAB}{15.7} = \frac{\sin 0.421}{9.37...} \Rightarrow OAB = 2.4$$

$$\text{Area of } ABO = \frac{1}{2} \times 15.7 \times 9.37... \times \sin 0.333... \quad \text{or} \quad \frac{1}{2} \times 7.5 \times 9.37... \times \sin 2.4$$

Approximate values are shown below for some of the lengths you may see in calculations:



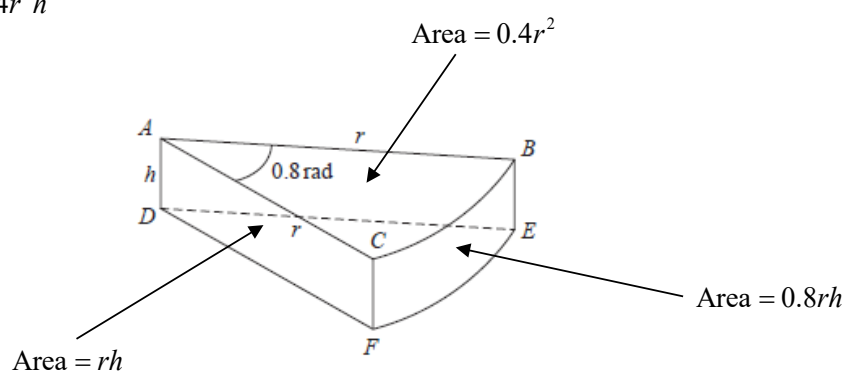
dM1: Solves the problem by combining appropriate areas together which result in the total area of the concert stage (usually the sum of the areas of the **two** congruent triangles and the area of the sector).

It is dependent on the previous method marks and the B mark.

A1: awrt 214 (m^2) (condone lack of units). Must follow from a correct method. Isw if they round incorrectly.

Question	Scheme	Marks	AOs
15 (a)	Sets up an allowable equation using volume = 240 E.g. $\frac{1}{2}r^2 \times 0.8h = 240 \Rightarrow h = \frac{600}{r^2}$ o.e.	M1 A1	3.4 1.1b
	Attempts to substitute their $h = \frac{600}{r^2}$ into $(S =) \frac{1}{2}r^2 \times 0.8 + \frac{1}{2}r^2 \times 0.8 + 2rh + 0.8rh$	dM1	3.4
	$S = 0.8r^2 + 2.8rh = 0.8r^2 + 2.8 \times \frac{600}{r} = 0.8r^2 + \frac{1680}{r}$ *	A1*	2.1
		(4)	
(b)	$\left(\frac{dS}{dr}\right) = 1.6r - \frac{1680}{r^2}$	M1 A1	3.1a 1.1b
	Sets $\frac{dS}{dr} = 0 \Rightarrow r^3 = 1050$ $r = \text{awrt } 10.2$	dM1 A1	2.1 1.1b
		(4)	
(c)	Attempts to substitute their positive r into $\left(\frac{d^2S}{dr^2}\right) = 1.6 + \frac{3360}{r^3}$ and considers its value or sign	M1	1.1b
	E.g. Correct $\frac{d^2S}{dr^2} = 1.6 + \frac{3360}{r^3}$ with $\frac{d^2S}{dr^2} \bigg _{r=10.2} = 5 > 0$ proving a minimum value of S	A1	1.1b
		(2)	
(10 marks)			
Notes:			

$$\text{Volume} = 0.4r^2h$$



$$\text{Total surface area} = 2rh + 0.8r^2 + 0.8rh$$

(a)



M1: Attempts to use the fact that the volume of the toy is 240 cm^3

Sight of $\frac{1}{2}r^2 \times 0.8 \times h = 240$ leading to $h = \dots$ or $rh = \dots$ scores this mark

But condone an equation of the correct form so allow for $kr^2h = 240 \Rightarrow h = \dots$ or $rh = \dots$

A1: A correct expression for $h = \frac{600}{r^2}$ or $rh = \frac{600}{r}$ which may be left unsimplified.

This may be implied when you see an expression for S or part of S E.g. $2rh = 2r \times \frac{600}{r^2}$

dM1: Attempts to substitute their $h = \frac{a}{r^2}$ o.e. such as $hr = \frac{a}{r}$ into a **correct** expression for S

Sight of $\frac{1}{2}r^2 \times 0.8 + \frac{1}{2}r^2 \times 0.8 + rh + rh + 0.8rh$ with an appropriate substitution

Simplified versions such as $0.8r^2 + 2rh + 0.8rh$ used with an appropriate substitution is fine.

A1*: Correct work leading to the given result.

$S =$, $SA =$ or surface area = must be seen at least once in the correct place

The method must be made clear so expect to see evidence. For example

$S = 0.8r^2 + 2rh + 0.8rh \Rightarrow S = 0.8r^2 + 2r \times \frac{600}{r^2} + 0.8r \times \frac{600}{r^2} \Rightarrow S = 0.8r^2 + \frac{1680}{r}$ would be fine.

(b) There is no requirement to see $\frac{dS}{dr}$ in part (b). It may even be called $\frac{dy}{dx}$.

M1: Achieves a derivative of the form $pr \pm \frac{q}{r^2}$ where p and q are non-zero constants

A1: Achieves $\left(\frac{dS}{dr}\right) = 1.6r - \frac{1680}{r^2}$

dM1: Sets or implies that their $\frac{dS}{dr} = 0$ and proceeds to $mr^3 = n$, $m \times n > 0$. It is dependent upon a

correct attempt at differentiation. This mark may be implied by a correct answer to their $pr - \frac{q}{r^2} = 0$

A1: $r = \text{awrt } 10.2$ or $\sqrt[3]{1050}$

(c)

M1: Attempts to substitute their positive r (found in (b)) into $\left(\frac{d^2S}{dr^2} = \right) e \pm \frac{f}{r^3}$ where e and f are non zero

and finds its value or sign.

Alternatively considers the sign of $\left(\frac{d^2S}{dr^2} = \right) e \pm \frac{f}{r^3}$ (at their positive r found in (b))

Condone the $\frac{d^2S}{dr^2}$ to be $\frac{d^2y}{dx^2}$ or being absent, but only for this mark.

A1: States that $\frac{d^2S}{dr^2}$ or $S'' = 1.6 + \frac{3360}{r^3} = \text{awrt } 5 > 0$ proving a minimum value of S

This is dependent upon having achieved $r = \text{awrt } 10$ and a correct $\frac{d^2S}{dr^2} = 1.6 + \frac{3360}{r^3}$

It can be argued without finding the value of $\frac{d^2S}{dr^2}$. E.g. $\frac{d^2S}{dr^2} = 1.6 + \frac{3360}{r^3} > 0$ as $r > 0$, so

minimum value of S. For consistency it is also dependent upon having achieved $r = \text{awrt } 10$



Question	Scheme	Marks	AOs
1	Attempts either $\sin 3\theta \approx 3\theta$ or $\cos 4\theta \approx 1 - \frac{(4\theta)^2}{2}$ in $\frac{1 - \cos 4\theta}{2\theta \sin 3\theta}$	M1	1.1b
	Attempts both $\sin 3\theta \approx 3\theta$ and $\cos 4\theta \approx 1 - \frac{(4\theta)^2}{2} \rightarrow \frac{1 - \left(1 - \frac{(4\theta)^2}{2}\right)}{2\theta \times 3\theta}$ and attempts to simplify	M1	2.1
	$= \frac{4}{3}$ oe	A1	1.1b
		(3)	
(3 marks)			
M1: Attempts either $\sin 3\theta \approx 3\theta$ or $\cos 4\theta \approx 1 - \frac{(4\theta)^2}{2}$ in the given expression. See below for description of marking of $\cos 4\theta$ M1: Attempts to substitute both $\sin 3\theta \approx 3\theta$ and $\cos 4\theta \approx 1 - \frac{(4\theta)^2}{2}$ $\rightarrow \frac{1 - \left(1 - \frac{(4\theta)^2}{2}\right)}{2\theta \times 3\theta}$ and attempts to simplify. Condone missing bracket on the 4θ so $\cos 4\theta \approx 1 - \frac{4\theta^2}{2}$ would score the method Expect to see it simplified to a single term which could be in terms of θ Look for an answer of k but condone $k\theta$ following a slip A1: Uses both identities and simplifies to $\frac{4}{3}$ or exact equivalent with no incorrect lines BUT allow recovery on missing bracket for $\cos 4\theta \approx 1 - \frac{4\theta^2}{2}$. Eg. $\frac{1 - \left(1 - \frac{(4\theta)^2}{2}\right)}{2\theta \times 3\theta} = \frac{8\theta^2}{6\theta} = \frac{4}{3}$ is M1 M1 A0 Condone awrt 1.33. Alt: $\frac{1 - \cos 4\theta}{2\theta \sin 3\theta} = \frac{1 - (1 - 2\sin^2 2\theta)}{2\theta \sin 3\theta} = \frac{2\sin^2 2\theta}{2\theta \sin 3\theta} = \frac{2 \times (2\theta)^2}{2\theta \times 3\theta} = \frac{4}{3}$ M1 For an attempt at $\sin 3\theta \approx 3\theta$ or the identity $\cos 4\theta = 1 - 2\sin^2 2\theta$ with $\sin 2\theta \approx 2\theta$ M1 For both of the above and attempts to simplify to a single term. A1 $\frac{4}{3}$ oe			

Question	Scheme	Marks	AOs
4	Examples: $4\sin\frac{\theta}{2}\approx 4\left(\frac{\theta}{2}\right), \quad 3\cos^2\theta\approx 3\left(1-\frac{\theta^2}{2}\right)^2$ $3\cos^2\theta=3(1-\sin^2\theta)\approx 3(1-\theta^2)$ $3\cos^2\theta=3\frac{(\cos 2\theta+1)}{2}\approx \frac{3}{2}\left(1-\frac{4\theta^2}{2}+1\right)$	M1	1.1a
	Examples: $4\sin\frac{\theta}{2}+3\cos^2\theta\approx 4\left(\frac{\theta}{2}\right)+3\left(1-\frac{\theta^2}{2}\right)^2$ $4\sin\frac{\theta}{2}+3\cos^2\theta=4\left(\frac{\theta}{2}\right)+3(1-\sin^2\theta)\approx 2\theta+3(1-\theta^2)$ $4\sin\frac{\theta}{2}+3\cos^2\theta=4\sin\frac{\theta}{2}+3\frac{(\cos 2\theta+1)}{2}\approx 4\left(\frac{\theta}{2}\right)+\frac{3}{2}\left(1-\frac{4\theta^2}{2}+1\right)$	dM1	1.1b
	$=2\theta+3(1-\theta^2+\dots)=3+2\theta-3\theta^2$	A1	2.1
		(3)	
	(3 marks)		
Notes			
M1: Attempts to use at least one correct approximation within the given expression. Either $\sin\frac{\theta}{2}\approx\frac{\theta}{2}$ or $\cos\theta\approx 1-\frac{\theta^2}{2}$ or e.g. $\sin\theta\approx\theta$ if they write $\cos^2\theta$ as $1-\sin^2\theta$ or e.g. $\cos 2\theta\approx 1-\frac{(2\theta)^2}{2}$ (condone missing brackets) if they write $\cos^2\theta$ as $\frac{1+\cos 2\theta}{2}$. Allow sign slips only with any identities used but the appropriate approximations must be applied. dM1: Attempts to use correct approximations with the given expression to obtain an expression in terms of θ only. Depends on the first method mark. A1: Correct terms following correct work. Allow the terms in any order and ignore any extra terms if given correct or incorrect.			



Question	Scheme	Marks	AOs												
5	One of $\theta \tan 2\theta = \theta \times 2\theta$ or $1 - \cos 3\theta = 1 - \left(1 - \frac{(3\theta)^2}{2}\right)$ or equivalents.	B1	1.1a												
	$\frac{\theta \tan 2\theta}{1 - \cos 3\theta} = \frac{\theta \times 2\theta}{1 - \left(1 - \frac{(3\theta)^2}{2}\right)}$	M1	2.1												
	$= \frac{4}{9} \text{ or exact equivalent.}$	A1	1.1b												
		(3)													
(3 marks)															
Notes															
B1: Award this mark for $\theta \tan 2\theta = \theta \times 2\theta$ or $1 - \cos 3\theta = 1 - \left(1 - \frac{(3\theta)^2}{2}\right)$ or equivalents. May be seen when working on numerator or denominator separately or within the fraction. This is a B mark so if awarding for $\cos 3\theta$ do not condone missing brackets e.g. $1 - \frac{3\theta^2}{2}$ unless they are recovered or are implied by subsequent work. M1: Attempts to use both correct small angle approximations in the given expression. For this mark they must have attempted to use $\tan 2\theta = 2\theta$ and $\cos 3\theta = 1 - \frac{(3\theta)^2}{2}$ in the given expression but condone poor bracketing e.g. $\frac{\theta \times 2\theta}{1 - \left(1 - \frac{3\theta^2}{2}\right)}$ or e.g. $\frac{\theta \times 2\theta}{1 - 1 - \frac{3\theta^2}{2}}$ Do not allow e.g. $\frac{\theta \times 2\theta}{1 - \frac{(3\theta)^2}{2}}$ as this suggests they are approximating $\frac{\theta \tan 2\theta}{\cos 3\theta}$ A1: Correct value. Do not allow rounded decimals e.g. 0.444 but allow if recurring decimals are clearly indicated e.g. $0.\dot{4}$ Do not allow e.g. $\frac{2}{4.5}$. Ignore any units if given. Isw once a correct answer is seen. Examples: <table><tr><td>$\frac{\theta \times 2\theta}{1 - \left(1 - \frac{3\theta^2}{2}\right)} = \frac{4}{3} \left(\text{or } -\frac{4}{3} \right)$</td><td>scores B1M1A0 (Missing brackets not recovered)</td></tr><tr><td>$\frac{\theta \times 2\theta}{1 - \frac{(3\theta)^2}{2}}$</td><td>scores B1M0A0 (Missing “1 –” in the denominator so M0)</td></tr><tr><td>$\frac{\theta \times 2\theta}{1 + \left(1 - \frac{(3\theta)^2}{2}\right)}$</td><td>scores B1M0A0 (Has “1 +” in the denominator so M0)</td></tr><tr><td>$\frac{\theta \times \theta}{1 - \left(1 - \frac{3\theta^2}{2}\right)} = \dots$</td><td>scores B0M0A0 (The B mark could be recovered but M0 because of the incorrect numerator)</td></tr><tr><td>$\frac{\theta \times 2\theta}{1 - \left(1 - \frac{3\theta^2}{2}\right)} = \frac{2\theta^2}{\frac{9\theta^2}{2}} = \frac{2}{9}$</td><td>scores B1M1A0 (Missing brackets recovered)</td></tr><tr><td>$\frac{\theta \times 2\theta}{1 - \left(1 - \left(\frac{3\theta}{2}\right)^2\right)}$</td><td>Scores B1M0A0 (The denominator suggests an incorrect expansion – unless it was recovered.)</td></tr></table>				$\frac{\theta \times 2\theta}{1 - \left(1 - \frac{3\theta^2}{2}\right)} = \frac{4}{3} \left(\text{or } -\frac{4}{3} \right)$	scores B1M1A0 (Missing brackets not recovered)	$\frac{\theta \times 2\theta}{1 - \frac{(3\theta)^2}{2}}$	scores B1M0A0 (Missing “1 –” in the denominator so M0)	$\frac{\theta \times 2\theta}{1 + \left(1 - \frac{(3\theta)^2}{2}\right)}$	scores B1M0A0 (Has “1 +” in the denominator so M0)	$\frac{\theta \times \theta}{1 - \left(1 - \frac{3\theta^2}{2}\right)} = \dots$	scores B0M0A0 (The B mark could be recovered but M0 because of the incorrect numerator)	$\frac{\theta \times 2\theta}{1 - \left(1 - \frac{3\theta^2}{2}\right)} = \frac{2\theta^2}{\frac{9\theta^2}{2}} = \frac{2}{9}$	scores B1M1A0 (Missing brackets recovered)	$\frac{\theta \times 2\theta}{1 - \left(1 - \left(\frac{3\theta}{2}\right)^2\right)}$	Scores B1M0A0 (The denominator suggests an incorrect expansion – unless it was recovered.)
$\frac{\theta \times 2\theta}{1 - \left(1 - \frac{3\theta^2}{2}\right)} = \frac{4}{3} \left(\text{or } -\frac{4}{3} \right)$	scores B1M1A0 (Missing brackets not recovered)														
$\frac{\theta \times 2\theta}{1 - \frac{(3\theta)^2}{2}}$	scores B1M0A0 (Missing “1 –” in the denominator so M0)														
$\frac{\theta \times 2\theta}{1 + \left(1 - \frac{(3\theta)^2}{2}\right)}$	scores B1M0A0 (Has “1 +” in the denominator so M0)														
$\frac{\theta \times \theta}{1 - \left(1 - \frac{3\theta^2}{2}\right)} = \dots$	scores B0M0A0 (The B mark could be recovered but M0 because of the incorrect numerator)														
$\frac{\theta \times 2\theta}{1 - \left(1 - \frac{3\theta^2}{2}\right)} = \frac{2\theta^2}{\frac{9\theta^2}{2}} = \frac{2}{9}$	scores B1M1A0 (Missing brackets recovered)														
$\frac{\theta \times 2\theta}{1 - \left(1 - \left(\frac{3\theta}{2}\right)^2\right)}$	Scores B1M0A0 (The denominator suggests an incorrect expansion – unless it was recovered.)														

$\frac{\theta \times 2\theta}{\frac{9\theta^2}{2}} = \frac{2}{18}$	B1M1A0 (The B1 is awarded for the numerator but can be implied by the denominator. The M1 is implied)
$\frac{\theta \times 2\theta}{1 - \left(1 - \frac{3\theta^2}{2}\right)} = \frac{4}{9}$	B1M1A1 (The correct value implies correct recovery of missing brackets.)

Note that other approaches are possible using identities.

In such cases we will allow correct work leading to an expression that if terms in θ^8 and higher can be ignored will lead to $\frac{4}{9}$

But to score the M mark they must be using correct identities and correct approximations but condone bracketing errors as in the main scheme.

Examples:

$$\frac{\theta \tan 2\theta}{1 - \cos 3\theta} = \frac{\theta \times \frac{\sin 2\theta}{\cos 2\theta}}{1 - \cos 3\theta} = \frac{\theta \times \frac{2\theta}{1 - \frac{(2\theta)^2}{2}}}{1 - \left(1 - \frac{(3\theta)^2}{2}\right)} = \frac{2\theta^2}{1 - 2\theta^2} \times \frac{2}{9\theta^2} = \frac{4\theta^2}{9\theta^2 - 18\theta^4}$$

$$= \frac{4\theta^2}{9\theta^2} = \frac{4}{9}$$

Scores B1M1A1

Similarly: $\frac{\theta \tan 2\theta}{1 - \cos 3\theta} = \frac{\theta \times \sin 2\theta}{\cos 2\theta (1 - \cos 3\theta)} = \frac{\theta \times 2\theta}{\left(1 - \frac{(2\theta)^2}{2}\right) \left(1 - \left(1 - \frac{(3\theta)^2}{2}\right)\right)} = \frac{2\theta^2}{1 - 2\theta^2} \times \frac{2}{9\theta^2}$ etc.

$$\frac{4\theta^2}{9\theta^2 - 18\theta^4} = \frac{4\theta^2}{9\theta^2} = \frac{4}{9}$$

Scores B1M1A1

$$\frac{\theta \tan 2\theta}{1 - \cos 3\theta} = \frac{\theta \times \sin 2\theta}{\cos 2\theta (1 - \cos 3\theta)} = \frac{\theta \times 2\theta}{\left(1 - \frac{(2\theta)^2}{2}\right) \left(1 - \left(1 - \frac{(3\theta)^2}{2}\right)\right)} = \frac{2\theta^2}{1 - 2\theta^2} \times \frac{2}{9\theta^2}$$

$$= \frac{4\theta^2}{9\theta^2 - 18\theta^4} = \frac{4}{9 - 18\theta^2} = \frac{4}{9}$$

Scores B1M1A0

(They cannot just assume the term in θ^2 is 0 unless they provide a convincing limiting

argument e.g. $\lim_{\theta \rightarrow 0} \frac{4}{9 - 18\theta^2} = \frac{4}{9}$ or equivalent)



Scores B1M1A0

$$\frac{\theta \tan 2\theta}{1 - \cos 3\theta} = \frac{\theta \times \frac{2 \tan \theta}{1 - \tan^2 \theta}}{1 - (4 \cos^3 \theta - 3 \cos \theta)} = \frac{\theta \times \frac{2\theta}{1 - \theta^2}}{1 - \left(4 \left(1 - \frac{\theta^2}{2} \right)^3 - 3 \left(1 - \frac{\theta^2}{2} \right) \right)}$$

Scores B1M1A0

Note that attempts to use expansions in higher powers of θ should be sent to review.



A1: $\frac{4}{15}x^{\frac{5}{2}} - \frac{10}{9}x^{\frac{3}{2}} + c$ and including the constant or simplified exact equivalent such as

$\frac{4}{15}\sqrt{x^5} - \frac{10}{9}\sqrt{x^3} + c$ or $\frac{4}{15}x^{2.5} - \frac{10}{9}x^{1.5} + c$ or $\frac{1}{45}\left(12x^{\frac{5}{2}} - 50x^{\frac{3}{2}}\right) + c$ or $\frac{x^{\frac{3}{2}}}{45}(12x - 50) + c$. Fractions must be in their lowest terms and indices processed.

Do not accept e.g. $0.267x^{\frac{5}{2}} - 1.11x^{\frac{3}{2}} + c$ but allow $0.2\dot{6}x^{\frac{5}{2}} - 1.\dot{1}x^{\frac{3}{2}} + c$

Is w once a correct answer is seen but withhold this mark if there is spurious notation around their final answer e.g. $\int \frac{4}{15}x^{\frac{5}{2}} - \frac{10}{9}x^{\frac{3}{2}} + c \, dx$ is M1A1dM1A0

Alternative method using integration by parts example

M1: e.g. $\int x^{\frac{1}{2}}(2x-5) \, dx = \dots x^{\frac{3}{2}}(2x-5) - \int \dots x^{\frac{3}{2}} \, dx$ (applies integration by parts correctly to typically achieve this form – the $(2x-5)$ may also be split up as well – send to review if unsure how to mark)

This may also be done the other way round e.g. $\int x^{\frac{1}{2}}(2x-5) \, dx = \dots x^{\frac{1}{2}}(x^2-5x) - \int \dots x^{\frac{3}{2}} \pm \dots x^{\frac{1}{2}} \, dx$

The $\frac{1}{3}$ does not need to be considered for this mark.

A1: A correct intermediate stage applying integration by parts with correct coefficients.

e.g. $\int \frac{x^{\frac{1}{2}}(2x-5)}{3} \, dx = \frac{2}{3}x^{\frac{3}{2}}\left(\frac{2x-5}{3}\right) - \int \frac{4}{9}x^{\frac{3}{2}} \, dx$ (or unsimplified equivalent).

Coefficients must be exact. (See main scheme notes above) The other way round this could appear as

e.g. $\int \frac{x^{\frac{1}{2}}(2x-5)}{3} \, dx = \frac{1}{3}x^{\frac{1}{2}}(x^2-5x) - \frac{1}{6}\int x^{\frac{3}{2}} - 5x^{\frac{1}{2}} \, dx$. Condone a missing dx . May be implied.

dM1: Increases the power by one on an x^n term where n is a fraction e.g. $\int \dots x^{\frac{3}{2}} \, dx \rightarrow \dots x^{\frac{5}{2}}$ The index does not need to be processed. It is dependent on the previous method mark.

A1: $\frac{4}{15}x^{\frac{5}{2}} - \frac{10}{9}x^{\frac{3}{2}} + c$ or exact simplified equivalent. (See main scheme notes above)

Alternative method using the substitution method

M1: e.g. let $u = x^{\frac{1}{2}} \Rightarrow \int \dots u^4 + \dots u^2 \, du$ (uses a substitution to express the integral in terms of another variable. Allow slips with the coefficients, but the indices should be correct for their substitution)

The $\frac{1}{3}$ does not need to be considered for this mark.

A1: e.g. $\int \frac{4u^4}{3} - \frac{10u^2}{3} \, du$ or unsimplified equivalent. Coefficients must be exact. See main scheme notes above). May be implied by further work. Condone a missing dx .

dM1: $\int \dots u^4 + \dots u^2 \, du \rightarrow \dots u^5 + \dots u^3$ (increases the power by one on at least one of their indices – does not need to be processed. It is dependent on the previous method mark.

A1: $\frac{4}{15}x^{\frac{5}{2}} - \frac{10}{9}x^{\frac{3}{2}} + c$ or exact simplified equivalent. (See main scheme notes above)

There may be alternative substitutions, but the same marking principles apply.

Question	Scheme	Marks	AOs
2(a)	$(f(a) =) 4a^3 + 5a^2 - 10a + 4a = 0 \Rightarrow a(\dots) = 0$	M1	3.1a
	$a(4a^2 + 5a - 6) = 0$ *	A1*	1.1b
		(2)	
(b)(i)	$a = \frac{3}{4}$	B1	2.2a
(ii)	$4x^3 + 5x^2 - 10x + 4 \times \frac{3}{4} = 3 \Rightarrow 4x^3 + 5x^2 - 10x (= 0)$	M1	1.1b
	$x = 0$	B1	1.1b
	$x = \frac{-5 \pm \sqrt{185}}{8}$	A1	1.1b
		(4)	
(6 marks)			
Notes			
(a) M1: Attempts $f(a) = 0$ leading to an equation in a only and attempts to take a factor of a out. Condone slips. The $= 0$ may appear on a later line which is fine, but must be seen at some point in their solution in part (a). A1*: Achieves the given answer with no errors including brackets. Minimum acceptable is $4a^3 + 5a^2 - 10a + 4a = 0 \Rightarrow a(4a^2 + 5a - 6) = 0$ If the $= 0$ is absent at the start of their solution, it must appear before achieving the given answer. Do not allow attempts to find the value of a and substitute that into $f(x)$ More difficult alternative methods may be seen: Alt 1: You may see attempts via division / inspection $ \begin{array}{r} 4x^2 \quad + (5+4a)x \quad + (-10+5a+4a^2) \\ x-a \overline{) 4x^3 \quad + 5x^2 \quad - 10x \quad + 4a} \\ \underline{4x^3 \quad - 4ax^2} \\ (5+4a)x^2 \quad - 10x \\ \underline{(5+4a)x^2 \quad - a(5+4a)x} \\ (-10+5a+4a^2)x \quad + 4a \\ \underline{(-10+5a+4a^2)x - a(-10+5a+4a^2)} \\ -6a+5a^2+4a^3 \end{array} $ Then sets remainder $-6a + 5a^2 + 4a^3 = 0$ M1: For dividing the cubic by $(x - a)$ leading to a quadratic quotient in x and a cubic remainder in a which is then set $= 0$ and attempts to take a factor of a out. A1*: Completely correct with $a(4a^2 + 5a - 6) = 0$			

Alt 2: You may also see a grid or an attempt at factorisation via inspection

	$4x^2$	$+(5+4a)x$	$+(-10+5a+4a^2)$
x	$4x^3$	$+(5+4a)x^2$	$(-10+5a+4a^2)x$
$-a$	$-4ax^2$	$-a(5+4a)x$	$-a(-10+5a+4a^2)$

OR $4x^3 + 5x^2 - 10x + 4a \equiv (x-a)(4x^2 + px - 4)$ which should be followed by equating the x terms and x^2 terms to form two equations which can be solved simultaneously.

$$-10 = -ap - 4 \quad \text{and} \quad 5 = -4a + p \Rightarrow p = 5 + 4a$$

$$\Rightarrow -10 = -a(5+4a) - 4 \Rightarrow 4a^2 + 5a - 6 = 0 \Rightarrow a(4a^2 + 5a - 6) = 0$$

The above are examples. There may be other correct attempts so look at what is done.

M1: For an attempt to set up two simultaneous equations by equating coefficients for x and equating coefficients for x^2 . Condone slips.

A1*: $4a^2 + 5a - 6 = 0 \Rightarrow a(4a^2 + 5a - 6) = 0$ Completely correct with no errors.

(b) Mark (i) and (ii) together

(i)

B1: Deduces that $a = \frac{3}{4}$ only. May be implied by their resultant cubic. If they do (b)(ii) multiple times using other roots for which $a \neq \frac{3}{4}$, then the solutions arising from using the other roots $a \neq \frac{3}{4}$ must be rejected

(ii)

M1: Attempts to substitute their $a = \frac{3}{4}$ (which must be positive) into $f(x)$, sets their $f(x) = 3$ and collects terms on one side ($= 0$ may be implied). Condone arithmetical and sign slips. Condone if they repeat this step using their other root(s).

B1: $(x =) 0$

A1: $(x =) \frac{-5 \pm \sqrt{185}}{8}$ (**and these values only**) or exact equivalent (ignore 0 for this mark). Withhold this mark if the fraction line was clearly not intended to be under both terms. This mark cannot be scored if they proceed directly to the roots from $4x^3 + 5x^2 - 10x$ without taking a factor of or dividing by x first to **see the quadratic factor**. Isw once the correct answers are seen if they proceed to provide rounded answers after.

e.g. $4x^3 + 5x^2 - 10x + 4 \times \frac{3}{4} = 3 \Rightarrow 0, \frac{-5 \pm \sqrt{185}}{8}$ is M0B1A0

e.g. $4x^3 + 5x^2 - 10x = 0 \Rightarrow 0, \frac{-5 \pm \sqrt{185}}{8}$ is M1B1A0

e.g. $4x^3 + 5x^2 - 10x = 0 \Rightarrow 4x^2 + 5x - 10 = 0 \Rightarrow \frac{-5 \pm \sqrt{185}}{8}$ is M1B0A1

e.g. $4x^3 + 5x^2 - 10x = 0 \Rightarrow x(4x^2 + 5x - 10) = 0 \Rightarrow 0, \frac{-5 \pm \sqrt{185}}{8}$ is M1B1A1

Question	Scheme	Marks	AOs
3(a)	$(\overline{OA} =) \sqrt{5^2 + 3^2 + 2^2} = \sqrt{38} *$	B1*	1.1b
(b)		(1)	
	$ \overline{OB} = \sqrt{2^2 + 4^2 + a^2} = \sqrt{20 + a^2}$ so when $a = 5$ $ \overline{OB} = \sqrt{20 + 25} = \sqrt{45}$	M1	1.1b
	$= 5$	A1 cso	2.3
		(2)	
(3 marks)			
Notes			
(a)			
B1*: Shows the magnitude of $\overline{OA}$ is $\sqrt{38}$. Must see $\sqrt{5^2 + 3^2 + 2^2}$ or e.g. $\sqrt{25 + 9 + 4}$. We need to see how the value 38 or $\sqrt{38}$ is formed using the three components. Withhold this mark for incorrect working such as $\overline{OA} = 5^2 + 3^2 + 2^2 = 38 \Rightarrow \overline{OA} = \sqrt{38}$ but do not penalise poor notation to denote vectors or the magnitude as long as the intention is clear as to what they are finding (OA instead of $\overline{OA}$ is fine). Do not penalise if their square root does not go fully over all three terms as long as the intention is clear. May find $\overline{AO}$ instead which is acceptable. $(\overline{OA} ^2 =) 25 + 9 + 4 = 38 \Rightarrow (\overline{OA} =) \sqrt{38}$ scores B1 (we see how 38 is found) $(\overline{OA} ^2 =) 25 + 9 + 4 \Rightarrow \sqrt{38}$ scores B1 (we see how 38 is found) $25 + 9 + 4 = \sqrt{38}$ scores B0 (they are not equal) $(\overline{OA} ^2 =) 38 \Rightarrow (\overline{OA} =) \sqrt{38}$ scores B0 (no method seen to show how 38 is found)			
(b)			
M1: Attempts to find $\overline{OB}$ (or $\overline{OB} ^2$) in terms of a and substitutes in a positive integer for a to find a value for $\overline{OB}$ (or $\overline{OB} ^2$). e.g. $\overline{OB} = \sqrt{20 + a^2} \Rightarrow$ when $a = 2 \Rightarrow \sqrt{24}$. Also accept e.g. $\overline{BO}$ (or $\overline{BO} ^2$). Alternatively sets up an equation or an inequality e.g. $\sqrt{20 + a^2} > \sqrt{38}$ and proceeds to $a^2 > \dots$ (or $a^2 = \dots$). Condone sign slips in their rearrangement only. Allow the use of $=$, $<$ or $>$ for this mark. "$20 + a^2 \dots 38 \Rightarrow a^2 \dots 18$" (may be implied by sight of $\sqrt{18} = 4.24\dots$) A1: 5 cso (answer on its own with no incorrect working seen scores M1A1). Withhold this mark if $\overline{OB}$ (or $\overline{OB} ^2$) is incorrect (or $\overline{BO}$ or $\overline{BO} ^2$). Do not be concerned with the notation as long as the intention is clear or implied as to what they are trying to calculate (the calculations must be correct) Withhold this mark if at any point they set $\overline{OB} < \overline{OA}$ but accept an argument leading to an answer of 5 from either $\overline{OB} > \overline{OA}$ or $\overline{OB} = \overline{OA}$			

Question	Scheme	Marks	AOs
4a	$2\alpha + \frac{1}{2}\left(1 - \frac{\alpha^2}{2}\right)$	M1	1.2
	$2\alpha + \frac{1}{2}\left(1 - \frac{\alpha^2}{2}\right) = 0 \Rightarrow 2\alpha + \frac{1}{2} - \frac{\alpha^2}{4} = 0 \Rightarrow \alpha = \dots$	dM1	1.1b
	$\alpha = -0.243$ (3dp) only	A1	2.3
		(3)	
b	$f'(0) = \frac{1}{2} \cos 0 \Rightarrow \dots \Rightarrow y = \dots x + 3$	M1	1.1b
	$y = \frac{1}{2}x + 3$	A1	1.1b
		(2)	
(5 marks)			
Notes			
(a) Accept to be in terms of α or another variable e.g. x Note: -0.243 with no working is 0 marks M1: Fully substitutes $\cos x = 1 - \frac{x^2}{2}$ into the derivative. dM1: Attempts to multiply out to achieve a 3TQ (= 0) and attempts to find a value for α . Condone slips. Allow solving the quadratic via any method (usual rules apply). If they use a calculator then you may need to check this. A1: ($\alpha =$) -0.243 only cao Can only be scored provided a correct 3TQ is seen. If both roots found then the other one must be rejected (or a choice made of -0.243 e.g. underlining it or a tick) Condone $x = -0.243$			
(b) M1: Attempts to find the gradient of the curve when $x = 0$ and achieves an equation of the form $y = "f'(0)"x + 3$. $x = 0$ must be fully substituted in and a value must be found for the gradient. Do not allow this mark if they attempt to use a changed gradient e.g. the gradient of the normal. Also allow attempts using the small angle approximation: $f'(x) \approx 2x + \frac{1}{2}\left(1 - \frac{x^2}{2}\right)$ when $x = 0$, $f'(0) = \frac{1}{2} \Rightarrow y = "f'(0)"x + 3$ A1: $y = \frac{1}{2}x + 3$ or equivalent in the form $y = mx + c$ isw Stating just the values $m = 0.5$, $c = 3$ without the correct equation is A0			

Question	Scheme	Marks	AOs
6(a)	Attempts $\cos^2(2x) \approx \left(1 - \frac{(2x)^2}{2}\right)^2 = \dots(1 - 4x^2 + 4x^4)$	M1	1.1b
	$1 - \cos^2(2x) \approx 1 - (1 - 4x^2 + 4x^4) = 4x^2 - 4x^4$ *	A1*	1.1b
		(2)	
(b)	$\frac{1 - \cos^2(2x)}{\sin\left(\frac{x}{3}\right)\tan\left(\frac{x}{2}\right)} \approx \frac{4x^2 - 4x^4}{\left(\frac{x}{3}\right)\left(\frac{x}{2}\right)}$	M1	1.1b
	$= \frac{6}{x^2}(4x^2 - 4x^4) = 24 - 24x^2$	A1	2.1
		(2)	
(c)	24	B1ft	2.2a
	If x is (very) small, then any terms in x^2 are negligible.	dB1ft	2.4
		(2)	
(6 marks)			
Notes			
(a)			
M1: Attempts to use $\cos \theta \approx 1 - \frac{\theta^2}{2}$ with θ , 2θ , x or $2x$ and squares the resulting expression.			
Condone poor squaring, e.g., $(1 - 2x^2)^2 = 1 - 4x^4$ or missing brackets $\left(1 - \frac{2x^2}{2}\right)^2 = (1 - x^2)^2 = \dots$			
but not e.g. $\cos 2x \approx 2\left(1 - \frac{x^2}{2}\right)$			
May be implied by e.g. $1 - (1 - 2x^2)^2 = 2x^2(2 - 2x^2)$ from difference of two squares.			
A1*: Correct proof with an intermediate line such as $1 - (1 - 4x^2 + 4x^4)$			
Do not be concerned if the LHS does not appear and ignore any spurious $= 0$ in their work.			
There should be no obvious incorrect statements in the proof e.g. $1 - (1 + 4x^2 - 4x^4)$			
Do not condone incorrect work including invisible brackets such as $1 - 1 - 4x^2 + 4x^4$ or $\left(1 - \frac{2x^2}{2}\right)$ but condone a missing trailing bracket e.g. $1 - (1 - 4x^2 + 4x^4$			
Condone an attempt that starts in stages, which may not deal with the full expression e.g., $1 - \left(1 - \frac{4x^2}{2}\right) \rightarrow 1 - (1 - 2x^2)^2 = \dots$ has a missing square on the first bracket but is recovered in the next stage/step.			
The final line should be in terms of x but condone a slip to e.g. θ in the workings of the proof.			
If they complete the proof using θ they must revert to x to score this mark.			
Special Cases: In each of these cases below, please send to review.			
You may see attempts that use e.g.			
$\cos 2x = \pm 2 \cos^2 x \pm 1$ or $\cos^2 2x = \pm \frac{1}{2} \pm \frac{1}{2} \cos 4x$ $\cos 2x = \pm 1 \pm 2 \sin^2 x$ followed by the small angle approximation for $\sin x$ - Maclaurin expansions			

(b)

M1: Uses the **given answer** to part (a) and writes $\sin\left(\frac{x}{3}\right)$ **and** $\tan\left(\frac{x}{2}\right)$ using correct small angle approximations. Only allow misreads that are clearly misreads e.g. they cannot replace $\tan\left(\frac{x}{2}\right)$ with $\frac{x}{3}$ unless $\tan\left(\frac{x}{3}\right)$ is seen first. In such cases they will lose the A mark in (b) but

both of the marks in (c) are available. Condone mixed variables for this mark.

A1: $24 - 24x^2$ but condone e.g. $24 + -24x^2$ or e.g. $a = 24$ and $b = -24$
Condone recovery of missing/invisible brackets but the work must otherwise be correct.
Do not condone mixed variables being recovered unless explicitly replaced with x before cancelling x^2 .

No marks are scored in (b) for using the Maclaurin expansions for $\sin\left(\frac{x}{3}\right)$ and/or $\tan\left(\frac{x}{2}\right)$

(c)

B1ft: 24 but this must follow from the non-zero constant term in their answer to (b).

Do not allow e.g. $x = 24$

Allow follow through on their non-zero constant term from a polynomial in x .

dB1ft: Suitable reason given but it should refer to their x^2 term in some way (and any additional terms if they have any). Ignore spurious remarks e.g. “if $x < 1$ ” unless contradictory.
Dependent on the previous B1ft mark.

Some acceptable examples:

- “ $24x^2 \rightarrow 0$ or $x^2 \rightarrow 0$ ”
- “We can ignore the x^2 term”
- “ x^2 is much smaller than 24”
- As $x \rightarrow 0$ their $a + bx^2 \rightarrow a$ (condone as $x \rightarrow 0$, bx^2 “becomes” 0)
- $\lim_{x \rightarrow 0} a + bx^2 = (a + b(0)^2) = a$
- Since x is very small, $24 - 24(0)^2 = 24$ (and condone if the squared is missing).

They cannot *just* substitute in 0 or a very small value for x to score the mark for the reason.

A reason such as “the answer rounds to 24” is not acceptable.

There must be some justification, either “as $x \rightarrow 0$ ” or “since x is very small”

so e.g. “ $24 - 24(0)^2 = 24$ on its own scores B1ft dB0ft

This mark may be scored from any polynomial in x that includes at least a constant term and an x^2 term.

They cannot go back to e.g. $4x^4$ or $\frac{4x^4}{\frac{1}{6}x^2}$ and say this is negligible or $\rightarrow 0$ as they haven’t

dealt with the x in the denominator.



Question	Scheme	Marks	AOs
9	$a = \left(\frac{3}{4}\right)^2 \text{ or } a = \frac{9}{16}$ or $r = -\frac{3}{4}$	B1	2.2a
	$\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n \cos(180n)^\circ = \frac{\frac{9}{16}}{1 - \left(-\frac{3}{4}\right)} = \dots$	M1	3.1a
	$= \frac{9}{28}^*$	A1*	1.1b
		(3)	
Alternative 1:			
	$\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \cos(180n)^\circ = \frac{-\frac{3}{4}}{1 - \left(-\frac{3}{4}\right)} = \dots \text{ or } r = -\frac{3}{4}$	B1	2.2a
	$\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n \cos(180n)^\circ = -\frac{3}{7} - \left(-\frac{3}{4}\right)$	M1	3.1a
	$= \frac{9}{28}^*$	A1*	1.1b
Alternative 2:			
	$\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n \cos(180n)^\circ = \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 - \dots$	B1	2.2a
	$= \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^4 + \dots - \left(\frac{3}{4}\right)^3 - \left(\frac{3}{4}\right)^5 - \dots$ $\left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^4 + \dots = \left(\frac{3}{4}\right)^2 \left(\frac{1}{1 - \left(\frac{3}{4}\right)^2}\right) \text{ or } -\left(\frac{3}{4}\right)^3 - \left(\frac{3}{4}\right)^5 - \dots = -\left(\frac{3}{4}\right)^3 \left(\frac{1}{1 - \left(\frac{3}{4}\right)^2}\right)$ $\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n \cos(180n)^\circ = \left(\frac{3}{4}\right)^2 \left(\frac{1}{1 - \left(\frac{3}{4}\right)^2}\right) - \left(\frac{3}{4}\right)^3 \left(\frac{1}{1 - \left(\frac{3}{4}\right)^2}\right)$	M1	3.1a
	$= \frac{9}{28}^*$	A1*	1.1b
Alternative 3:			
	$\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n \cos(180n)^\circ = S = \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 - \dots$	B1	2.2a
	$= \left(\frac{3}{4}\right)^2 \left(1 - \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 - \dots\right) = \left(\frac{3}{4}\right)^2 \left(\frac{1}{4} + S\right) \Rightarrow \frac{7}{16}S = \frac{9}{64} \Rightarrow S = \dots$	M1	3.1a
	$= \frac{9}{28}^*$	A1*	1.1b
(3 marks)			
Notes			



B1: Deduces the correct value of the **first** term or the common ratio. The correct first term can be seen as part of them writing down the sequence but must be the **first** term.

M1: Recognises that the series is infinite geometric and applies the sum to infinity GP formula

with $a = \frac{9}{16}$ and $r = \pm \frac{3}{4}$

A1*: Correct proof

Alternative 1:

B1: Deduces the correct value for the sum to infinity (starting at $n = 1$) or the common ratio

M1: Calculates the required value by subtracting the first term from their sum to infinity

A1*: Correct proof

Alternative 2:

B1: Deduces the correct value of the **first** term or the common ratio.

M1: Splits the series into “odds” and “evens”, attempts the sum of both parts and calculates the required value by adding both sums

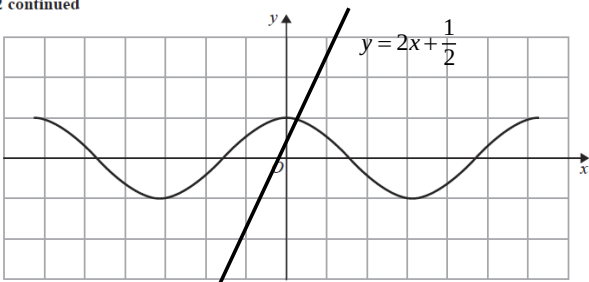
A1*: Correct proof

Alternative 3:

B1: Deduces the correct value of the **first** term

M1: A complete method by taking out the first term, expresses the rhs in terms of the original sum and rearranges for “S”

A1*: Correct proof

Question	Scheme	Marks	AOs
2(a)	2 continued  Diagram 1 For an allowable linear graph and explaining that there is only one intersection	B1	3.1a
		B1	2.4
		(2)	
(b)	$\cos x - 2x - \frac{1}{2} = 0 \Rightarrow 1 - \frac{x^2}{2} - 2x - \frac{1}{2} = 0$	M1	1.1b
	Solves their $x^2 + 4x - 1 = 0$	dM1	1.1b
	Allow awrt 0.236 but accept $-2 + \sqrt{5}$	A1	1.1b
		(3)	
(5 marks)			

(a)

B1: Draws $y = 2x + \frac{1}{2}$ on Figure 1 or Diagram 1 with an attempt at the correct gradient and the correct

intercept. Look for a straight line with an intercept at $\approx \frac{1}{2}$ and a further point at $\approx \left(\frac{1}{2}, 1\frac{1}{2}\right)$ Allow a tolerance of

0.25 of a square in either direction on these two points. It must appear in quadrants 1, 2 and 3.

B1: There must be an allowable linear graph on Figure 1 or Diagram 1 for this to be awarded

Explains that as there is only one intersection so there is just one root.

This requires a reason and a minimal conclusion.

The question asks candidates to explain but as a bare minimum allow one "intersection"

Note: An allowable linear graph is one with intercept of $\pm \frac{1}{2}$ with one intersection with $\cos x$ **OR** gradient of

± 2 with one intersection with $\cos x$

(b)

M1: Attempts to use the small angle approximation $\cos x = 1 - \frac{x^2}{2}$ in the given equation.

The equation must be in a single variable but may be recovered later in the question.

dM1: Proceeds to a 3TQ in a single variable and attempts to solve. See General Principles

The previous M must have been scored. Allow completion of square or formula or calculator. Do not allow attempts via factorisation unless their equation does factorise. You may have to use your calculator to check if a calculator is used.

A1: Allow $-2 + \sqrt{5}$ or awrt 0.236.

Do not allow this where there is another root given and it is not obvious that 0.236 has been chosen.

Question	Scheme	Marks	AOs
2(a)(i)	e.g. $(u_2 =)35 + 7 \cos\left(\frac{\pi}{2}\right) - 5(-1)^1 = 40$ *	B1*	2.1
(ii)	$u_3 = 40 + 7 \cos\left(\frac{2\pi}{2}\right) - 5(-1)^2 (= 28)$ or $u_4 = "28" + 7 \cos\left(\frac{3\pi}{2}\right) - 5(-1)^3 (= 33)$	M1	1.1b
	$u_3 = 28$ and $u_4 = 33$	A1	1.1b
		(3)	
(b)(i)	$(u_5 =)35$	B1	2.2a
(ii)	e.g. $\sum_{r=1}^{25} u_r = 6(35 + 40 + "28" + "33") + 35$	M1	3.1a
	$= 851$	A1	1.1b
		(3)	
(6 marks)			
Notes			
(a) (i) B1*: Correct application of the formula with $n = 1$ and proceeds correctly to achieve an answer of 40 with no errors. Note that e.g., $(u_2 =)35 + 7 \cos\left(\frac{35\pi}{2}\right) - 5(-1)^{35} = 35 + 0 + 5 = 40$ scores B0 As a minimum need to see e.g. $(u_2 =)35 + 7 \cos\left(\frac{\pi}{2}\right) - 5(-1)^1 = 40$, $35 + 0 + 5 = 40$, $35 + 5 = 40$, $35 - 5(-1)^1 = 40$ (ii) M1: A correct attempt to use the formula to find a value for u_3 or u_4 Look for $n = 2$ substituted correctly into the given formula with $u_2 = 40$. May be implied by $u_3 = 28$ Or their calculated value of u_3 used with $n = 3$ substituted correctly into the given formula to find u_4 Condone use of calculator in degree mode which gives $u_3 = 41.989...$ which may imply this mark if no working is shown. If there is no working and u_3 is incorrect and u_4 is correct score M0A0 A1: Both correct $u_3 = 28$ and $u_4 = 33$ If 28, 33 are listed then allow M1A1. For both correct values only score M1A1			
(b)(i) B1: $(u_5 =)35$ (ii) M1: Attempts a correct method to find $\sum_{r=1}^{25} u_r$ There are various ways e.g. attempts to add 35 to $6 \times$ the sum of their four values. Some other examples are: $\sum_{r=1}^{25} u_r = 7 \times 35 + 6 \times 40 + 6 \times "28" + 6 \times "33", \quad \sum_{r=1}^{25} u_r = 7(35 + 40 + "28" + "33") - (40 + "28" + "33"),$ $\sum_{r=1}^{25} u_r = \frac{25}{4}(35 + 40 + "28" + "33") + 1, \quad 2(35 + 40 + "28" + "33") = 272, \quad 272 \times 3 = 816, \quad 816 + 35$ There may be other methods seen but the calculation must be correct for their values. If there is no working, with incorrect u_3 and/or u_4 you will need to check if their answer implies a correct method using $6(35 + 40 + "28" + "33") + 35$ Attempts to use an AP/GP formula score M0 A1: 851 (Correct answer with no working scores both marks)			

Question	Scheme	Marks	AOs
6(a)	$(f'(x) =) 4 \cos\left(\frac{1}{2}x\right) - 3$	M1 A1	1.1b 1.1b
	Sets $f'(x) = 4 \cos\left(\frac{1}{2}x\right) - 3 = 0 \Rightarrow x =$	dM1	3.1a
	$x = 14.0$ Cao	A1	3.2a
		(4)	
(b)	Explains that $f(4) > 0$, $f(5) < 0$ and the function is continuous	B1	2.4
		(1)	
(c)	Attempts $x_1 = 5 - \frac{8 \sin 2.5 - 15 + 9}{4 \cos 2.5 - 3}$ (NB $f(5) = -1.212\dots$ and $f'(5) = -6.204\dots$)	M1	1.1b
	$x_1 = \text{awrt } 4.80$	A1	1.1b
		(2)	
(7 marks)			
Notes:			

(a)

M1: Differentiates to obtain $k \cos\left(\frac{1}{2}x\right) \pm \alpha$ where α is a constant which may be zero and no other terms. The brackets are not required.

A1: Correct derivative $f'(x) = 4 \cos\left(\frac{1}{2}x\right) - 3$. Allow unsimplified e.g. $f'(x) = \frac{1}{2} \times 8 \cos\left(\frac{1}{2}x\right) - 3x^0$

There is no need for $f'(x) = \dots$ or $\frac{dy}{dx} = \dots$ just look for the expression and the brackets are not required.

dM1: For the complete strategy of proceeding to a value for x .

Look for

- $f'(x) = a \cos\left(\frac{1}{2}x\right) + b = 0$, $a, b \neq 0$
- Correct method of finding a valid solution to $a \cos\left(\frac{1}{2}x\right) + b = 0$

Allow for $a \cos\left(\frac{1}{2}x\right) + b = 0 \Rightarrow \cos\left(\frac{1}{2}x\right) = \pm k \Rightarrow x = 2 \cos^{-1}(\pm k)$ where $|k| < 1$

If this working is not shown then you may need to check their value(s).

For example $4 \cos\left(\frac{1}{2}x\right) - 3 = 0 \Rightarrow x = 1.4\dots$ or $11.1\dots$ (or $82.8\dots$ or $637\dots$ or 803 in degrees) would indicate this method.

A1: Selects the correct turning point $x = 14.0$ and not just 14 or unrounded e.g. $14.011\dots$

Must be this value only and no other values unless they are clearly rejected or 14.0 clearly selected. Ignore any attempts to find the y coordinate.

Correct answer with no working scores no marks.

(b)

B1: See scheme. Must be a full reason, (e.g. change of sign and continuous)

Accept equivalent statements for $f(4) > 0$, $f(5) < 0$ e.g. $f(4) \times f(5) < 0$, "there is a change of sign", "one negative one positive". A minimum is "change of sign and continuous" but do **not** allow this mark if the comment about continuity is clearly incorrect e.g. "because x is

(c)

M1: Attempts $x_1 = 5 - \frac{f(5)}{f'(5)}$ to obtain a value following through on their $f'(x)$ as long as it is a “changed” function.

Must be a correct N-R formula used – may need to check their values.

Allow if attempted in degrees. For reference in degrees $f(5) = -5.65\dots$ and $f'(5) = 0.996\dots$ and gives $x_1 = 10.67\dots$

There must be clear evidence that $5 - \frac{f(5)}{f'(5)}$ is being attempted.

$$\text{so e.g. } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow x_1 = 4.80 \text{ scores M0 as does e.g. } x_1 = x - \frac{8\sin\left(\frac{1}{2}x\right) - 3x + 9}{4\cos\left(\frac{1}{2}x\right) - 3} = 4.80$$

BUT evidence may be provided by the accuracy of their answer. Note that the full N-R accuracy is 4.804624337 so e.g. 4.805 or 4.804 (truncated) with no evidence of incorrect work may imply the method.

A1: $x_1 = \text{awrt } 4.80$ not awrt 4.8 but isw if awrt 4.80 is seen. Ignore any subsequent iterations.

Note that work for part (a) cannot be recovered in part (c)

Note also:

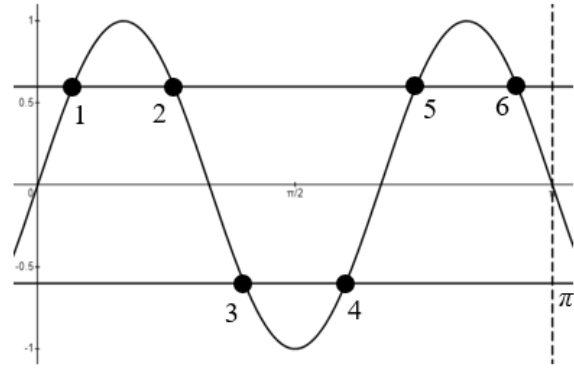
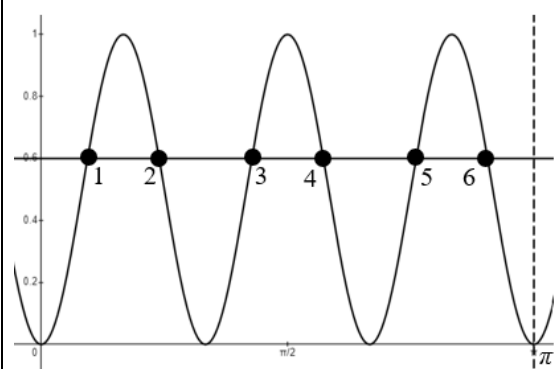
$$5 - \frac{f(5)}{f'(5)} = \text{awrt } 4.80 \text{ following a correct derivative scores M1A1}$$

$$5 - \frac{f(5)}{f'(5)} \neq \text{awrt } 4.80 \text{ with no evidence that } 5 - \frac{f(5)}{f'(5)} \text{ was attempted scores M0}$$



Question	Scheme	Marks	AOs
13(i)	$(n =) 3$	B1	2.2a
		(1)	
(ii)	30 solutions or e.g. each interval of length π has 6 solutions	B1	2.2a
	e.g., Each interval of length π has 6 solutions, and there are 5 intervals of length π , so $5 \times 6 = 30$ solutions.	dB1	2.4
		(2)	
(3 marks)			
Notes			
(i)			
B1: cao $(n =) 3$			
(ii)			
B1: Either:			
- Deduces 30 solutions or deduces the correct total number of solutions for $\sin(nx) = k$ and $\sin(nx) = -k$ (or $\sin^2(nx) = k^2$) in any relevant interval (0 to π, 2π, 4π or 5π) or deduces the correct number of solutions in 0 to 5π for either $\sin(nx) = k$ or $\sin(nx) = -k$ or deduces the correct total number of solutions for $\sin(x) = k$ and $\sin(x) = -k$ in any relevant interval (0 to π, 2π, 4π or 5π)			
dB1: Requires 30 (solutions) and correct justification comprising all the elements in one of the bullet points below. See the tables on the next page for the correct values.			
- Deduces the correct total number of solutions for $\sin(nx) = k$ and $\sin(nx) = -k$ (or $\sin^2(nx) = k^2$) in any relevant interval (0 to π, 2π or 4π) and scales the interval to 0 to 5π or deduces the correct number of solutions in 0 to 5π for $\sin(nx) = k$ and for $\sin(nx) = -k$ and then adds or finds the total correct number of solutions for $\sin(x) = k$ and $\sin(x) = -k$ in any relevant interval (0 to π, 2π, 4π or 5π) and then scales the interval to 0 to 5π and multiplies by 3			
Just stating e.g. $2 \times 6 \times 2.5$ is insufficient for the dB1 mark without further justification. There are many acceptable variations, and some examples are below and on the next page.			
Some examples:			
6 solutions in each interval of length π (B1) and 5 intervals of length π so 30 (dB1) 12 solutions in each interval of length 2π (B1) and 2.5 intervals of length 2π so 30 (dB1) 2 solutions for $\sin x = k$ up to 2π, so for $\sin^2 x$ there are 4 per 2π (B1)			
and since $0 \leq x < 5\pi$ then $0 \leq 3x < 15\pi$ so $\frac{15}{2} \times 4 = 30$ (dB1)			
- Double the 6 solutions since $\sin(nx) = \pm k$ (or $\sin^2(nx) = k^2$) so 12 (B1) and so 24 in 4π, hence 30 in 5π (dB1) 16 solutions for $\sin(nx) = k$ (B1) and 14 (solutions) for $\sin(nx) = -k \rightarrow 30$ (dB1) $\sin^2(3x) = \frac{1 - \cos(6x)}{2}$ and as $\cos x = k$ has 5 solutions in 0 to 5π (B1) and so $\cos 6x$ has $5 \times 6 = 30$ (dB1)			
Graphical approaches may be used but these must be convincing, clearly showing the correct number of solutions in one of the relevant intervals. Be lenient with the shape of the curve.			

e.g. 30 solutions in $0 \leq x < 5\pi$ [with 6 solutions on, for example, either sketch below]
 $y = \sin^2(3x) \quad 0 \leq x < \pi$ $y = \sin(3x) \quad 0 \leq x < \pi$



Note that a suitable graph of $\sin x$ showing, e.g. 10 solutions up to 5π , followed by 30 (calculation of $\times 3$ clearly implied) would also be eligible to score the dB1.

The following are examples of an incorrect justification but scores B1dB0 for reaching 30:

- $\sin(nx) = k$ has 3×6 solutions in the interval 0 to 5π , $\sin(nx) = -k$ has 3×4 solutions (in the interval 0 to 5π) so $18 + 12 = 30$ solutions in total.
- $\sin(nx) = k$ has 15 solutions in the interval 0 to 5π , so for $\sin^2(nx)$ has $15 \times 2 = 30$ solutions in total.

For reference, the tables below show the total number of solutions for each branch of $\sin^2(nx) = k^2$ with $n = 3$ and the bold values score the first B1 **provided they are in the correct interval**.

	$0 \leq x < \pi$	$0 \leq x < 2\pi$	$0 \leq x < 4\pi$	$0 \leq x < 5\pi$
$\sin(nx) = k$	4	6	12	16
$\sin(nx) = -k$	2	6	12	14
Total solutions	6	12	24	30

and $n = 1$ for those that deal with nx ($3x$) last.

	$0 \leq x < \pi$	$0 \leq x < 2\pi$	$0 \leq x < 4\pi$	$0 \leq x < 5\pi$
$\sin(x) = k$	2	2	4	6
$\sin(x) = -k$	0	2	4	4
Total solutions	2	4	8	10

Question	Scheme	Marks	AOs
13(a)	$y = \operatorname{cosec}^3 \theta \Rightarrow \frac{dy}{d\theta} = -3 \operatorname{cosec}^2 \theta \operatorname{cosec} \theta \cot \theta$	B1	1.1b
	$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$	M1	1.1b
	$\frac{dy}{dx} = \frac{-3 \operatorname{cosec}^3 \theta \cot \theta}{2 \cos 2\theta}$	A1	1.1b
		(3)	
(b)	$y = 8 \Rightarrow \operatorname{cosec}^3 \theta = 8 \Rightarrow \sin^3 \theta = \frac{1}{8} \Rightarrow \sin \theta = \frac{1}{2}$	M1	3.1a
	$\theta = \frac{\pi}{6} \Rightarrow \frac{dy}{dx} = \frac{-3 \operatorname{cosec}^3 \left(\frac{\pi}{6} \right) \cot \left(\frac{\pi}{6} \right)}{2 \cos \left(\frac{2\pi}{6} \right)} = \dots$ or $\sin \theta = \frac{1}{2} \Rightarrow \frac{dy}{dx} = \frac{\frac{-3}{\sin^3 \theta} \times \frac{\cos \theta}{\sin \theta}}{2(1 - 2 \sin^2 \theta)} = \frac{-3 \times 8 \times \frac{\sqrt{3}/2}{1/2}}{2 \left(1 - 2 \times \frac{1}{4} \right)}$	M1	2.1
	$= -24\sqrt{3}$	A1	2.2a
		(3)	
(6 marks)			
Notes			
(a) B1: Correct expression for $\frac{dy}{d\theta}$ seen or implied in any form e.g. $\frac{-3 \cos \theta}{\sin^4 \theta}$ M1: Obtains $\frac{dx}{d\theta} = k \cos 2\theta$ or $\alpha \cos^2 \theta + \beta \sin^2 \theta$ (from product rule on $\sin \theta \cos \theta$) and attempts $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$ A1: Correct expression in any form. May see e.g. $\frac{-3 \cos \theta}{2 \sin^4 \theta \cos 2\theta}, -\frac{3}{4 \sin^4 \theta \cos \theta - 2 \sin^3 \theta \tan \theta}$ (b) M1: Recognises the need to find the value of $\sin \theta$ or θ when $y = 8$ and uses the y parameter to establish its value. This should be correct work leading to $\sin \theta = \frac{1}{2}$ or e.g. $\theta = \frac{\pi}{6}$ or 30°. M1: Uses their value of $\sin \theta$ or θ in their $\frac{dy}{dx}$ from part (a) (working in exact form) in an attempt to obtain an exact value for $\frac{dy}{dx}$. May be implied by a correct exact answer. If no working is shown but an exact answer is given you may need to check that this follows their $\frac{dy}{dx}$. A1: Deduces the correct gradient			

Question	Scheme	Marks	AOs
15 (a)	$x^2 \tan y = 9 \Rightarrow 2x \tan y + x^2 \sec^2 y \frac{dy}{dx} = 0$	M1 A1	3.1a 1.1b
	Full method to get $\frac{dy}{dx}$ in terms of x using $\sec^2 y = 1 + \tan^2 y = 1 + f(x)$	M1	1.1b
	$\frac{dy}{dx} = \frac{-2x \times \frac{9}{x^2}}{x^2 \left(1 + \frac{81}{x^4}\right)} = \frac{-18x}{x^4 + 81} \quad *$	A1*	2.1
		(4)	
(b)	$\frac{dy}{dx} = \frac{-18x}{x^4 + 81}$ $\Rightarrow \frac{d^2y}{dx^2} = \frac{-18 \times (x^4 + 81) - (-18x)(4x^3)}{(x^4 + 81)^2} = \frac{54(x^4 - 27)}{(x^4 + 81)^2} \text{ o.e.}$	M1 A1	1.1b 1.1b
	States that when $x < \sqrt[4]{27} \Rightarrow \frac{d^2y}{dx^2} < 0$ when $x = \sqrt[4]{27} \Rightarrow \frac{d^2y}{dx^2} = 0$ AND when $x > \sqrt[4]{27} \Rightarrow \frac{d^2y}{dx^2} > 0$ giving a point of inflection when $x = \sqrt[4]{27}$	A1	2.4
		(3)	
(7 marks)			
Notes:			

(a)

M1: Attempts to differentiate $\tan y$ implicitly. Eg. $\tan y \rightarrow \sec^2 y \frac{dy}{dx}$ or $\cot y \rightarrow -\operatorname{cosec}^2 y \frac{dy}{dx}$

You may well see an attempt $\tan y = \frac{9}{x^2} \Rightarrow \sec^2 y \frac{dy}{dx} = \dots$

When a candidate writes $x^2 \tan y = 9 \Rightarrow x = 3 \tan^{\frac{1}{2}} y$ the mark is scored for $\tan^{\frac{1}{2}} y \rightarrow \dots \tan^{\frac{3}{2}} y \sec^2 y$

A1: Correct differentiation $2x \tan y + x^2 \sec^2 y \frac{dy}{dx} = 0$

Allow also $\sec^2 y \frac{dy}{dx} = -\frac{18}{x^3}$ or $2x = -9 \operatorname{cosec}^2 y \frac{dy}{dx}$ amongst others

M1: Full method to get $\frac{dy}{dx}$ in terms of x using $\sec^2 y = 1 + \tan^2 y = 1 + f(x)$

A1*: Proceeds correctly to the given answer of $\frac{dy}{dx} = \frac{-18x}{x^4 + 81}$

(b)

M1: Attempts to differentiate the given expression using the product or quotient rule.For example look for a correct attempt at $\frac{vu' - uv'}{v^2}$ with $u = -18x, v = x^4 + 81, u' = \pm 18, v' = \dots x^3$ If no method is seen or implied award for $\frac{\pm 18 \times (x^4 + 81) \pm 18x(ax^3)}{(x^4 + 81)^2}$ Using the product rule award for $\pm 18(x^4 + 81)^{-1} \pm 18x(x^4 + 81)^{-2} \times cx^3$ **A1:** Correct **simplified** $\frac{d^2y}{dx^2} = \frac{54(x^4 - 27)}{(x^4 + 81)^2}$ o.e. such as $\frac{d^2y}{dx^2} = \frac{54x^4 - 1458}{(x^4 + 81)^2}$ Alternatively score for showing that when a correct (unsimplified) $\frac{d^2y}{dx^2} = 0 \Rightarrow x^4 = 27 \Rightarrow x = \sqrt[4]{27}$ Or for substituting $x = \sqrt[4]{27}$ into an unsimplified but correct $\frac{d^2y}{dx^2}$ and showing that it is 0**A1:** Correct explanation with a minimal conclusion and correct second derivative.

See scheme.

It can be also be argued from $x^4 < 27$, $x^4 = 27$ and $x^4 > 27$ provided the conclusion states that the point of inflection is at $x = \sqrt[4]{27}$ Alternatively substitutes values of x either side of $\sqrt[4]{27}$ and at $\sqrt[4]{27}$, into $\frac{d^2y}{dx^2}$, finds all three values and makes a minimal conclusion.A different method involves finding $\frac{d^3y}{dx^3}$ and showing that $\frac{d^3y}{dx^3} \neq 0$ and $\frac{d^2y}{dx^2} = 0$ when $x = \sqrt[4]{27}$

FYI $\frac{d^3y}{dx^3} = \frac{23328x^3}{(x^4 + 81)^3} = 0.219$ when $x = \sqrt[4]{27}$

Alternative part (a) using arctan

M1: Sets $y = \arctan \frac{9}{x^2} \rightarrow \frac{dy}{dx} = \frac{1}{1 + \left(\frac{9}{x^2}\right)^2} \times \dots$ where $\dots$ could be 1

A2: $y = \arctan \frac{9}{x^2} \rightarrow \frac{dy}{dx} = \frac{1}{1 + \left(\frac{9}{x^2}\right)^2} \times -\frac{18}{x^3}$

A1*: $\frac{dy}{dx} = \frac{1}{1 + \frac{81}{x^4}} \times -\frac{18}{x^3} = \frac{-18x}{x^4 + 1}$ showing correct intermediate step and no errors.

Question	Scheme	Marks	AOs
12 (a)	States or uses $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$	B1	1.2
	$\operatorname{cosec} \theta - \sin \theta = \frac{1}{\sin \theta} - \sin \theta = \frac{1 - \sin^2 \theta}{\sin \theta}$	M1	2.1
	$= \frac{\cos^2 \theta}{\sin \theta} = \cos \theta \times \frac{\cos \theta}{\sin \theta} = \cos \theta \cot \theta$ *	A1*	2.1
		(3)	
(b)	$\operatorname{cosec} x - \sin x = \cos x \cot(3x - 50^\circ)$ $\Rightarrow \cos x \cot x = \cos x \cot(3x - 50^\circ)$		
	$\cot x = \cot(3x - 50^\circ) \Rightarrow x = 3x - 50^\circ$	M1	3.1a
	$x = 25^\circ$	A1	1.1b
	Also $\cot x = \cot(3x - 50^\circ) \Rightarrow x + 180^\circ = 3x - 50^\circ$	M1	2.1
	$x = 115^\circ$	A1	1.1b
	Deduces $x = 90^\circ$	B1	2.2a
		(5)	
(8 marks)			
Notes:			

(a) Condone a full proof in x (or other variable) instead of θ 's here

B1: States or uses $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$ Do not accept $\operatorname{cosec} \theta = \frac{1}{\sin}$ with the θ missing

M1: For the key step in forming a single fraction/common denominator

E.g. $\operatorname{cosec} \theta - \sin \theta = \frac{1}{\sin \theta} - \sin \theta = \frac{1 - \sin^2 \theta}{\sin \theta}$. Allow if written separately $\frac{1}{\sin \theta} - \sin \theta = \frac{1}{\sin \theta} - \frac{\sin^2 \theta}{\sin \theta}$

Condone missing variables for this M mark

A1*: Shows careful work with all necessary steps shown leading to given answer. See scheme for necessary steps. There should not be any notational or bracketing errors.

(b) Condone θ 's instead of x 's here

M1: Uses part (a), cancels or factorises out the $\cos x$ term, to establish that one solution is found when $x = 3x - 50^\circ$.

You may see solutions where $\cot A - \cot B = 0 \Rightarrow \cot(A - B) = 0$ or $\tan A - \tan B = 0 \Rightarrow \tan(A - B) = 0$.

As long as they don't state $\cot A - \cot B = \cot(A - B)$ or $\tan A - \tan B = \tan(A - B)$ this is acceptable

A1: $x = 25^\circ$

M1: For the key step in realising that $\cot x$ has a period of 180° and a second solution can be found by solving $x + 180^\circ = 3x - 50^\circ$. The sight of $x = 115^\circ$ can imply this mark provided the step $x = 3x - 50^\circ$ has been seen. Using reciprocal functions it is for realising that $\tan x$ has a period of 180°

A1: $x = 115^\circ$ Withhold this mark if there are additional values in the range (0,180) but ignore values outside.

B1: Deduces that a solution can be found from $\cos x = 0 \Rightarrow x = 90^\circ$. Ignore additional values here.

.....
Solutions with limited working. The question demands that candidates show all stages of working.

SC: $\cos x \cot x = \cos x \cot(3x - 50^\circ) \Rightarrow \cot x = \cot(3x - 50^\circ) \Rightarrow x = 25^\circ, 115^\circ$

They have shown some working so can score B1, B1 marked on open as 11000

Question	Scheme	Marks	AOs
12 (a)	States or uses $\cot \theta = \frac{\cos \theta}{\sin \theta}$	B1	1.2
	$\cos \theta \cot \theta = \frac{\cos^2 \theta}{\sin \theta} = \frac{1 - \sin^2 \theta}{\sin \theta}$	M1	2.1
	$= \frac{1}{\sin \theta} - \sin \theta = \operatorname{cosec} \theta - \sin \theta$ *	A1*	2.1
		(3)	

Alt 2- Works on both sides

Question	Scheme	Marks	AOs
12 (a)	States or uses $\cot \theta = \frac{\cos \theta}{\sin \theta}$ or $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$	B1	1.2
	$LHS = \frac{1}{\sin \theta} - \sin \theta = \frac{1 - \sin^2 \theta}{\sin \theta} = \frac{\cos^2 \theta}{\sin \theta}$ $RHS = \cos \theta \cot \theta = \frac{\cos^2 \theta}{\sin \theta}$	M1	2.1
	States a conclusion E.g. "HENCE TRUE", "QED" or $\operatorname{cosec} \theta - \sin \theta \equiv \cos \theta \cot \theta$ o.e. (condone = for $\equiv$)	A1*	2.1
		(3)	

Alt (b)

Question	Scheme	Marks	AOs
	$\cot x = \cot(3x - 50^\circ) \Rightarrow \frac{\cos x}{\sin x} = \frac{\cos(3x - 50^\circ)}{\sin(3x - 50^\circ)}$ $\sin(3x - 50^\circ) \cos x - \cos(3x - 50^\circ) \sin x = 0$ $\sin((3x - 50^\circ) - x) = 0$ $2x - 50^\circ = 0$	M1	3.1a
	$x = 25^\circ$	A1	1.1b
	Also $2x - 50^\circ = 180^\circ$	M1	2.1
	$x = 115^\circ$	A1	1.1b
	Deduces $\cos x = 0 \Rightarrow x = 90^\circ$	B1	2.2a
		(5)	

Question	Scheme	Marks	AOs
16(a)	Attempts $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\sec^2 t \tan t}{2\sec^2 t} (= 2 \tan t)$	M1 A1	1.1b 1.1b
	At $t = \frac{\pi}{4}$, $\frac{dy}{dx} = 2, x = 3, y = 7$	M1	2.1
	Attempts equation of normal $y - 7 = -\frac{1}{2}(x - 3)$	M1	1.1b
	$y = -\frac{1}{2}x + \frac{17}{2}$ *	A1*	2.1
		(5)	
(b)	Attempts to use $\sec^2 t = 1 + \tan^2 t \Rightarrow \frac{y-3}{2} = 1 + \left(\frac{x-1}{2}\right)^2$	M1	3.1a
	$\Rightarrow y - 3 = 2 + \frac{(x-1)^2}{2} \Rightarrow y = \frac{1}{2}(x-1)^2 + 5$ *	A1*	2.1
		(2)	
(b) Alternative 1:			
	$y = \frac{1}{2}(x-1)^2 + 5 = \frac{1}{2}(2 \tan t + 1 - 1)^2 + 5$ $= \frac{1}{2}4 \tan^2 t + 5 = 2(\sec^2 t - 1) + 5$	M1	3.1a
	$= 2\sec^2 t + 3 = y$ *	A1	2.1
(b) Alternative 2:			
	$x = 2 \tan t + 1 \Rightarrow t = \tan^{-1}\left(\frac{x-1}{2}\right) \Rightarrow y = 2\sec^2\left(\tan^{-1}\left(\frac{x-1}{2}\right)\right) + 3$ $\Rightarrow y = 2\left(1 + \tan^2\left(\tan^{-1}\left(\frac{x-1}{2}\right)\right)\right) + 3$	M1	3.1a
	$\Rightarrow y = 2\left(1 + \left(\frac{x-1}{2}\right)^2\right) + 3 = \frac{1}{2}(x-1)^2 + 5$ *	A1	2.1
(b) Alternative 3:			
	$\frac{dy}{dx} = 2 \tan t = x - 1 \Rightarrow y = \int (x-1) dx = \frac{x^2}{2} - x + c$ $(3, 7) \rightarrow 7 = \frac{3^2}{2} - 3 + c \Rightarrow c = \frac{11}{2}$	M1	3.1a
	$\frac{x^2}{2} - x + \frac{11}{2} = \frac{1}{2}(x^2 - 2x) + \frac{11}{2} = \frac{1}{2}(x-1)^2 - \frac{1}{2} + \frac{11}{2} = \frac{1}{2}(x-1)^2 + 5$ *	A1	2.1

(c)	Attempts the lower limit for k: $\frac{1}{2}(x-1)^2 + 5 = -\frac{1}{2}x + k \Rightarrow x^2 - x + (11 - 2k) = 0$ $b^2 - 4ac = 1 - 4(11 - 2k) = 0 \Rightarrow k = \dots$	M1	2.1
	$(k =) \frac{43}{8}$	A1	1.1b
	Attempts the upper limit for k: $(x, y)_{t=-\frac{\pi}{4}} : t = -\frac{\pi}{4} \Rightarrow x = 2 \tan\left(-\frac{\pi}{4}\right) + 1 = -1, y = 2 \sec^2\left(-\frac{\pi}{4}\right) + 3 = 7$ $(-1, 7), y = -\frac{1}{2}x + k \Rightarrow 7 = \frac{1}{2} + k \Rightarrow k = \dots$	M1	2.1
	$(k =) \frac{13}{2}$	A1	1.1b
	$\frac{43}{8} < k \leq \frac{13}{2}$	A1	2.2a
		(5)	
(12 marks)			
Notes:			

(a) **Must use parametric differentiation to score any marks in this part and not e.g. Cartesian form**

M1: For the key step of attempting $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$. There must be some attempt to differentiate both

parameters however poor and divide the right way round so using $\frac{dy}{dx} = \frac{y}{x}$ scores M0.

This may be implied by e.g. $\frac{dx}{dt} = 2 \sec^2 t$, $\frac{dy}{dt} = 4 \sec^2 t \tan t$, $t = \frac{\pi}{4} \Rightarrow \frac{dx}{dt} = 4$, $\frac{dy}{dt} = 8 \Rightarrow \frac{dy}{dx} = 2$

A1: $\frac{dy}{dx} = \frac{4 \sec^2 t \tan t}{2 \sec^2 t}$. Correct expression in any form. May be implied as above.

Condone the confusion with variables as long as the intention is clear e.g.

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4 \sec^2 x \tan x}{2 \sec^2 x} (= 2 \tan x)$ and allow subsequent marks if this is interpreted correctly

M1: For attempting to find the values of x, y and the gradient at $t = \frac{\pi}{4}$ **AND** getting at least two correct.

Follow through on their $\frac{dy}{dx}$ so allow for any two of $x = 3$, $y = 7$, $\frac{dy}{dx} = 2$ (or their $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$)

Note that the $x = 3$, $y = 7$ may be seen as e.g. (3, 7) on the diagram. There must be a non-trivial $\frac{dy}{dx}$ for this mark e.g. they must have a $\frac{dy}{dx}$ to substitute into.

M1: For a correct attempt at the normal equation using their x and y at $t = \frac{\pi}{4}$ with the negative

reciprocal of their $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ having made some attempt at $\frac{dy}{dx}$ and all correctly placed.

For attempts using $y = mx + c$ they must reach as far as a value for c using their x and y at $t = \frac{\pi}{4}$

with the negative reciprocal of their $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ all correctly placed.

(b)

M1: Attempts to use $\sec^2 t = 1 + \tan^2 t$ oe to obtain an equation involving y and $(x-1)^2$ E.g. as above or e.g. $y = 2\sec^2 t + 3 = 2(1 + \tan^2 t) + 3 = 2\left(1 + \left(\frac{x-1}{2}\right)^2\right) + 3$ for M1 and then

$$y = \frac{1}{2}(x-1)^2 + 5^* \text{ for A1}$$

A1*: Proceeds with a clear argument to the given answer with no errors**Alternative 1:****M1:** Uses the given result, substitutes for x and attempts to use $\sec^2 t = 1 + \tan^2 t$ oe**A1:** Proceeds with a clear argument to the y parameter and makes a (minimal) conclusion e.g. “ $= y$ ”
QED, hence proven etc.**Alternative 2:****M1:** Uses the x parameter to obtain t in terms of arctan, substitutes into y and attempts to use
 $\sec^2 t = 1 + \tan^2 t$ oe**A1:** Proceeds with a clear argument to the given answer with no errors**Alternative 3:****M1:** Uses $\frac{dy}{dx}$ from part (a) to express $\frac{dy}{dx}$ in terms of x , integrates and uses (3, 7) to find “ c ” to reach
a Cartesian equation.**A1:** Proceeds with a clear argument to the given answer with no errors**Allow the marks for (b) to score anywhere in their solution e.g. if they find the Cartesian equation in part (a)**

(c)

M1: A full attempt to find the **lower** limit for k .

$$\frac{1}{2}(x-1)^2 + 5 = -\frac{1}{2}x + k \Rightarrow x^2 - x + (11 - 2k) = 0 \Rightarrow b^2 - 4ac = 1 - 4(11 - 2k) = 0 \Rightarrow k = \dots$$

Score **M1** for setting $\frac{1}{2}(x-1)^2 + 5 = -\frac{1}{2}x + k$, rearranging to 3TQ form and attempts $b^2 - 4ac \dots 0$ e.g. $b^2 - 4ac > 0$ or e.g. $b^2 - 4ac < 0$ correctly to find a value for k .**A1:** $k = \frac{43}{8}$ oe. Look for this **value** e.g. may appear in an inequality e.g. $k > \frac{43}{8}$, $k < \frac{43}{8}$ **An alternative method using calculus for lower limit:**

$$y = \frac{1}{2}(x-1)^2 + 5 \Rightarrow \frac{dy}{dx} = x-1, x-1 = -\frac{1}{2} \Rightarrow x = \frac{1}{2}$$

$$x = \frac{1}{2} \Rightarrow y = \frac{1}{2}\left(\frac{1}{2}-1\right)^2 + 5 = \frac{41}{8}$$

$$y = -\frac{1}{2}x + k \Rightarrow \frac{41}{8} = -\frac{1}{4} + k \Rightarrow k = \dots$$

Score **M1** for $\frac{dy}{dx} =$ “a linear expression in x ”, sets $= -\frac{1}{2}$, solves a linear equation to find x andthen substitutes into the given result in (b) to find y and then uses $y = -\frac{1}{2}x + k$ to find a valuefor k . **A1:** $k = \frac{43}{8}$ oe. Look for this **value** e.g. may appear in an inequality e.g. $k > \frac{43}{8}$, $k < \frac{43}{8}$

An alternative method using parameters for lower limit:

$$\begin{aligned}
 y &= -\frac{1}{2}x + k \Rightarrow 2\sec^2 t + 3 = -\frac{1}{2}(2\tan t + 1) + k \\
 \Rightarrow 2(1 + \tan^2 t) + 3 &= -\frac{1}{2}(2\tan t + 1) + k \Rightarrow 2\tan^2 t + \tan t + 5.5 - k = 0 \\
 b^2 - 4ac &= 0 \Rightarrow 1 - 4 \times 2(5.5 - k) = 0 \Rightarrow k = \frac{43}{8}
 \end{aligned}$$

Score **M1** for substituting parametric form of x and y into $y = -\frac{1}{2}x + k$, uses $\sec^2 t = 1 + \tan^2 t$ rearranges to 3TQ form and attempts $b^2 - 4ac \dots 0$ or e.g. $b^2 - 4ac > 0$ or $b^2 - 4ac < 0$ correctly to find a value for k .

A1: $k = \frac{43}{8}$ oe. Look for this **value** e.g. may appear in an inequality e.g. $k > \frac{43}{8}$, $k < \frac{43}{8}$

M1: A full attempt to find the **upper** limit for k . This requires an attempt to find the value of x and the value of y using $t = -\frac{\pi}{4}$, the substitution of these values into $y = -\frac{1}{2}x + k$ and solves for k .

A1: $k = \frac{13}{2}$. Look for this value e.g. may appear in an inequality.

A1: Deduces the correct range for k : $\frac{43}{8} < k \leq \frac{13}{2}$

Allow equivalent notation e.g. $\left(k \leq \frac{13}{2} \text{ and } k > \frac{43}{8}\right)$, $\left(k \leq \frac{13}{2} \cap k > \frac{43}{8}\right)$, $\left(\frac{43}{8}, \frac{13}{2}\right]$

But not e.g. $\left(k \leq \frac{13}{2}, k > \frac{43}{8}\right)$, $\left(k \leq \frac{13}{2} \cup k > \frac{43}{8}\right)$, $\left(k \leq \frac{13}{2} \text{ or } k > \frac{43}{8}\right)$ and do not allow if in terms of x .

Allow equivalent exact values for $\frac{43}{8}$, $\frac{13}{2}$

There may be other methods for finding the upper limit which are valid. If you are in any doubt if a method deserves credit then use Review.

Question	Scheme	Marks	AOs
5	$\frac{dy}{d\theta} = \frac{(2\sin\theta + 2\cos\theta)3\cos\theta - 3\sin\theta(2\cos\theta - 2\sin\theta)}{(2\sin\theta + 2\cos\theta)^2}$	M1 A1	1.1b 1.1b
	Expands and uses $\sin^2\theta + \cos^2\theta = 1$ at least once in the numerator or the denominator or uses $2\sin\theta\cos\theta = \sin 2\theta$ in $\Rightarrow \frac{dy}{d\theta} = \frac{\dots}{\dots\dots C \sin\theta \cos\theta}$	M1	3.1a
	Expands and uses $\sin^2\theta + \cos^2\theta = 1$ the numerator and the denominator AND uses $2\sin\theta\cos\theta = \sin 2\theta$ in $\Rightarrow \frac{dy}{d\theta} = \frac{P}{Q + R \sin 2\theta}$	M1	2.1
	$\Rightarrow \frac{dy}{d\theta} = \frac{3}{2 + 2\sin 2\theta} = \frac{\frac{3}{2}}{1 + \sin 2\theta}$	A1	1.1b
(5 marks)			
Notes: M1: For choosing either the quotient, product rule or implicit differentiation and applying it to the given function. Look for the correct form of $\frac{dy}{d\theta}$ (condone it being stated as $\frac{dy}{dx}$) but tolerate slips on the coefficients and also condone $\frac{d(\sin\theta)}{d\theta} = \pm \cos\theta$ and $\frac{d(\cos\theta)}{d\theta} = \pm \sin\theta$ For quotient rule look for $\frac{dy}{d\theta} = \frac{(2\sin\theta + 2\cos\theta) \times \pm \dots \cos\theta - 3\sin\theta(\pm \dots \cos\theta \pm \dots \sin\theta)}{(2\sin\theta + 2\cos\theta)^2}$ For product rule look for $\frac{dy}{d\theta} = (2\sin\theta + 2\cos\theta)^{-1} \times \pm \dots \cos\theta \pm 3\sin\theta \times (2\sin\theta + 2\cos\theta)^{-2} \times (\pm \dots \cos\theta \pm \dots \sin\theta)$ Implicit differentiation look for $(\dots \cos\theta \pm \dots \sin\theta)y + (2\sin\theta + 2\cos\theta)\frac{dy}{d\theta} = \dots \cos\theta$ A1: A correct expression involving $\frac{dy}{d\theta}$ condoning it appearing as $\frac{dy}{dx}$ M1: Expands and uses $\sin^2\theta + \cos^2\theta = 1$ at least once in the numerator or the denominator OR uses $2\sin\theta\cos\theta = \sin 2\theta$ in $\Rightarrow \frac{dy}{d\theta} = \frac{\dots}{\dots\dots C \sin\theta \cos\theta}$ M1: Expands and uses $\sin^2\theta + \cos^2\theta = 1$ in the numerator and the denominator AND uses $2\sin\theta\cos\theta = \sin 2\theta$ in the denominator to reach an expression of the form $\frac{dy}{d\theta} = \frac{P}{Q + R \sin 2\theta}$. A1: Fully correct proof with $A = \frac{3}{2}$ stated but allow for example $\frac{\frac{3}{2}}{1 + \sin 2\theta}$ Allow recovery from missing brackets. Condone notation slips. This is not a given answer			

Question	Scheme	Marks	AOs
4	$C_1: x=10\cos t, y=4\sqrt{2}\sin t, 0 \leq t < 2\pi; C_2: x^2 + y^2 = 66$		
Way 1	$(10\cos t)^2 + (4\sqrt{2}\sin t)^2 = 66$	M1	3.1a
	$100(1 - \sin^2 t) + 32\sin^2 t = 66$	M1	2.1
	$100\cos^2 t + 32(1 - \cos^2 t) = 66$	A1	1.1b
	$100 - 68\sin^2 t = 66 \Rightarrow \sin^2 t = \frac{1}{2}$ $\Rightarrow \sin t = \dots$	dM1	1.1b
	$68\cos^2 t + 32 = 66 \Rightarrow \cos^2 t = \frac{1}{2}$ $\Rightarrow \cos t = \dots$		
	Substitutes their solution back into the relevant original equation(s) to get the value of the x-coordinate and value of the corresponding y-coordinate. Note: These may not be in the correct quadrant	M1	1.1b
	$S = (5\sqrt{2}, -4)$ or $x = 5\sqrt{2}, y = -4$ or $S = (\text{awrt } 7.07, -4)$	A1	3.2a
		(6)	
Way 2	$\{\cos^2 t + \sin^2 t = 1 \Rightarrow \left(\frac{x}{10}\right)^2 + \left(\frac{y}{4\sqrt{2}}\right)^2 = 1 \Rightarrow 32x^2 + 100y^2 = 3200\}$	M1	3.1a
	$\frac{x^2}{100} + \frac{66 - x^2}{32} = 1$	M1	2.1
	$\frac{66 - y^2}{100} + \frac{y^2}{32} = 1$	A1	1.1b
	$32x^2 + 6600 - 100x^2 = 3200$ $x^2 = 50 \Rightarrow x = \dots$	dM1	1.1b
	$2112 - 32y^2 + 100y^2 = 3200$ $y^2 = 16 \Rightarrow y = \dots$		
	Substitutes their solution back into the relevant original equation(s) to get the value of the corresponding x-coordinate or y-coordinate. Note: These may not be in the correct quadrant	M1	1.1b
	$S = (5\sqrt{2}, -4)$ or $x = 5\sqrt{2}, y = -4$ or $S = (\text{awrt } 7.07, -4)$	A1	3.2a
		(6)	
Way 3	$\{C_2: x^2 + y^2 = 66 \Rightarrow x = \sqrt{66}\cos\alpha, y = \sqrt{66}\sin\alpha$ $\{C_1 = C_2 \Rightarrow 10\cos t = \sqrt{66}\cos\alpha, 4\sqrt{2}\sin t = \sqrt{66}\sin\alpha$ $\{\cos^2\alpha + \sin^2\alpha = 1 \Rightarrow \left(\frac{10\cos t}{\sqrt{66}}\right)^2 + \left(\frac{4\sqrt{2}\sin t}{\sqrt{66}}\right)^2 = 1$	M1	3.1a
	<i>then continue with applying the mark scheme for Way 1</i>		
Way 4	$(10\cos t)^2 + (4\sqrt{2}\sin t)^2 = 66$	M1	3.1a
	$100\left(\frac{1 + \cos 2t}{2}\right) + 32\left(\frac{1 - \cos 2t}{2}\right) = 66$	M1	2.1
	$50 + 50\cos 2t + 16 - 16\cos 2t = 66 \Rightarrow 34\cos 2t + 66 = 66$ $\Rightarrow \cos 2t = \dots$	A1	1.1b
	$50 + 50\cos 2t + 16 - 16\cos 2t = 66 \Rightarrow 34\cos 2t + 66 = 66$ $\Rightarrow \cos 2t = \dots$	dM1	1.1b
	Substitutes their solution back into the original equation(s) to get the value of the x-coordinate and value of the y-coordinate. Note: These may not be in the correct quadrant	M1	1.1b
	$S = (5\sqrt{2}, -4)$ or $x = 5\sqrt{2}, y = -4$ or $S = (\text{awrt } 7.07, -4)$	A1	3.2a
		(6)	
	Note: Give final A0 for writing $x = 5\sqrt{2}, y = -4$ followed by $S = (-4, 5\sqrt{2})$		
(6 marks)			
Notes for Question 4			

4	$C_1: x = 10 \cos t, y = 4\sqrt{2} \sin t, 0 \leq t < 2\pi; C_2: x^2 + y^2 = 66$		
Way 5	$(10 \cos t)^2 + (4\sqrt{2} \sin t)^2 = 66$	M1	3.1a
	$(10 \cos t)^2 + (4\sqrt{2} \sin t)^2 = 66(\sin^2 t + \cos^2 t)$	M1	2.1
		A1	1.1b
	$100 \cos^2 t + 32 \sin^2 t = 66 \sin^2 t + 66 \cos^2 t \Rightarrow 34 \cos^2 t = 34 \sin^2 t$ $\Rightarrow \tan t = \dots$	dM1	1.1b
	Substitutes their solution back into the relevant original equation(s) to get the value of the x-coordinate and value of the corresponding y-coordinate. Note: These may not be in the correct quadrant	M1	1.1b
	$S = (5\sqrt{2}, -4)$ or $x = 5\sqrt{2}, y = -4$ or $S = (\text{awrt } 7.07, -4)$	A1	3.2a
		(6)	
	Way 5		
M1:	Begins to solve the problem by applying an appropriate strategy. E.g. Way 5: A complete process of combining equations for C_1 and C_2 by substituting the parametric equation into the Cartesian equation to give an equation in one variable (i.e. t) only.		
M1:	Uses the identity $\sin^2 t + \cos^2 t \equiv 1$ to achieve an equation in $\sin^2 t$ only and $\cos^2 t$ only with no constant term		
A1:	A correct equation in $\sin^2 t$ and $\cos^2 t$ containing no constant term		
dM1:	dependent on both the previous M marks Rearranges to make $\tan t = \dots$		
M1:	See scheme		
A1:	Selects the correct coordinates for S Allow either $S = (5\sqrt{2}, -4)$ or $S = (\text{awrt } 7.07, -4)$ or $S = (\sqrt{50}, -4)$ or $S = \left(\frac{10}{\sqrt{2}}, -4\right)$		

Question	Scheme	Marks	AOs
8(a) Way 1	$\left(\frac{1}{\operatorname{cosec} \theta - 1} + \frac{1}{\operatorname{cosec} \theta + 1} \right) \frac{\operatorname{cosec} \theta + 1 + \operatorname{cosec} \theta - 1}{(\operatorname{cosec} \theta - 1)(\operatorname{cosec} \theta + 1)}$	B1	1.1b
	$\equiv \frac{2 \operatorname{cosec} \theta}{\operatorname{cosec}^2 \theta - 1} \equiv \frac{2 \operatorname{cosec} \theta}{\cot^2 \theta} \text{ or e.g. } \equiv \frac{2 \sin \theta}{1 - \sin^2 \theta} = \frac{2 \sin \theta}{\cos^2 \theta}$	M1	1.1b
	$\frac{2 \operatorname{cosec} \theta}{\cot^2 \theta} \equiv \frac{2}{\sin \theta} \times \frac{\sin^2 \theta}{\cos^2 \theta} \equiv 2 \tan \theta \sec \theta^*$ or $\frac{2 \sin \theta}{\cos^2 \theta} \equiv 2 \tan \theta \sec \theta^*$	A1*	2.1
		(3)	
(a) “Meets in the middle”			
Way 2	$\left(\text{LHS} = \frac{1}{\operatorname{cosec} \theta - 1} + \frac{1}{\operatorname{cosec} \theta + 1} \right) \frac{\operatorname{cosec} \theta + 1 + \operatorname{cosec} \theta - 1}{(\operatorname{cosec} \theta - 1)(\operatorname{cosec} \theta + 1)}$	B1	1.1b
	$\equiv \frac{2 \operatorname{cosec} \theta}{\operatorname{cosec}^2 \theta - 1} \equiv \frac{2 \operatorname{cosec} \theta}{\cot^2 \theta}$	M1	1.1b
	$\text{RHS} = 2 \tan \theta \sec \theta \equiv \frac{2 \sin \theta}{\cos^2 \theta} \equiv \frac{2 \sin^2 \theta}{\sin \theta \cos^2 \theta}$ $\equiv \frac{2}{\sin \theta} \times \frac{\sin^2 \theta}{\cos^2 \theta} \equiv \frac{2 \operatorname{cosec} \theta}{\cot^2 \theta} = \text{LHS or e.g. QED or e.g. Proven}$	A1*	2.1
		(3)	
Part (a) Notes			
(a) Condone a complete proof <u>entirely</u> in x (or another variable) instead of θ Condone “=” for “$\equiv$” Note that we are marking this as B1M1A1 not M1M1A1 B1: Adds the fractions to obtain a correct single fraction (not fractions over fractions) in any form. Condone missing brackets when they combine their fractions as long as they are recovered to give a correct fraction. This can be done in a variety of ways but when combined, the fraction must be correct e.g. $\frac{\operatorname{cosec} \theta + 1 + \operatorname{cosec} \theta - 1}{(\operatorname{cosec} \theta - 1)(\operatorname{cosec} \theta + 1)} \text{ or } \frac{2 \operatorname{cosec} \theta}{(\operatorname{cosec} \theta - 1)(\operatorname{cosec} \theta + 1)} \text{ or } \frac{2 \operatorname{cosec} \theta}{\operatorname{cosec}^2 \theta - 1}$ or e.g. $\left(\frac{\frac{1}{\frac{1}{\sin \theta} - 1}}{\frac{1}{\sin \theta} - 1} + \frac{\frac{1}{\frac{1}{\sin \theta} + 1}}{\frac{1}{\sin \theta} + 1} = \frac{\sin \theta}{1 - \sin \theta} + \frac{\sin \theta}{1 + \sin \theta} \right) = \frac{2 \sin \theta}{1 - \sin^2 \theta} \text{ etc.}$ M1: Uses a correct Pythagorean identity anywhere in their attempt e.g. $\operatorname{cosec}^2 \theta - 1 = \cot^2 \theta$, $\sin^2 \theta + \cos^2 \theta = 1$ etc. or equivalent A1*: Correct work with all necessary steps shown leading to the given answer. See scheme for the necessary steps. They need to proceed via sine and cosine to the given answer. There should be no notational or bracketing errors and no mixed or missing variables. E.g. we would consider $\cos^2 \theta$ written as $\cos \theta^2$ a notational error. Condone reaching $2 \sec \theta \tan \theta^*$			
Way 2 (Meet in the middle)			



B1: See Way 1

M1: See Way 1

A1*: Correct work on the RHS with all necessary steps shown leading to showing the equivalence with the LHS. See scheme for the necessary steps. There should be no notational or bracketing errors and no mixed or missing variables.
For this approach there must be a (minimal) conclusion e.g. “= LHS”, “QED”, “Hence proven” etc.

It is possible to start with the rhs e.g.:

$$\begin{aligned}
 2 \tan \theta \sec \theta &= 2 \frac{\sin \theta}{\cos \theta} \times \frac{1}{\cos \theta} \\
 &= \frac{2 \sin \theta}{\cos^2 \theta} = \frac{2 \operatorname{cosec} \theta}{\cot^2 \theta} \\
 &= \frac{2 \operatorname{cosec} \theta}{\operatorname{cosec}^2 \theta - 1} = \frac{2 \operatorname{cosec} \theta}{(\operatorname{cosec} \theta - 1)(\operatorname{cosec} \theta + 1)} \\
 &= \frac{\operatorname{cosec} \theta + 1 + \operatorname{cosec} \theta - 1}{(\operatorname{cosec} \theta - 1)(\operatorname{cosec} \theta + 1)} \\
 &= \frac{1}{\operatorname{cosec} \theta - 1} + \frac{1}{\operatorname{cosec} \theta + 1}
 \end{aligned}$$

B1: Correctly reaches $2 \tan \theta \sec \theta = \frac{2 \operatorname{cosec} \theta}{(\operatorname{cosec} \theta - 1)(\operatorname{cosec} \theta + 1)}$

M1: Uses a **correct** Pythagorean identity anywhere in their attempt e.g.
 $\operatorname{cosec}^2 \theta - 1 = \cot^2 \theta$, $\sin^2 \theta + \cos^2 \theta = 1$ etc. or equivalent

A1*: Correct work with all necessary steps shown leading to the lhs. See scheme for the necessary steps. There should be no notational or bracketing errors and no mixed or missing variables.

8(b)	$2 \tan 2x \sec 2x = \cot 2x \sec 2x$	B1	2.2a
	$2 \tan 2x \sec 2x - \cot 2x \sec 2x = 0$ $\Rightarrow \sec 2x(2 \tan 2x - \cot 2x) = 0$ $2 \tan 2x - \cot 2x = 0 \Rightarrow 2 \tan 2x = \cot 2x \Rightarrow \tan^2 2x = \frac{1}{2}$ or $2 \tan 2x - \cot 2x = 0 \Rightarrow 2 \frac{\sin 2x}{\cos 2x} = \frac{\cos 2x}{\sin 2x} \Rightarrow 2 \sin^2 2x = \cos^2 2x$ $\Rightarrow 2 \sin^2 2x = 1 - \sin^2 2x \Rightarrow \sin^2 2x = \frac{1}{3}$ or $2(1 - \cos^2 2x) = \cos^2 2x \Rightarrow \cos^2 2x = \frac{2}{3}$ or $2 \tan 2x = \cot 2x \Rightarrow \frac{4 \tan x}{1 - \tan^2 x} = \frac{1 - \tan^2 x}{2 \tan x}$ $\Rightarrow \tan^4 x - 10 \tan^2 x + 1 = 0$ or $2 \tan 2x - \cot 2x = 0 \Rightarrow 2 \frac{\sin 2x}{\cos 2x} = \frac{\cos 2x}{\sin 2x} \Rightarrow 2 \sin^2 2x = \cos^2 2x$ $2 \sin^2 2x = \cos^2 2x \Rightarrow 1 - \cos 4x = \frac{1}{2}(\cos 4x + 1) \Rightarrow \cos 4x = \frac{1}{3}$	M1	2.1
	$2x = \tan^{-1} \frac{1}{\sqrt{2}} = K \Rightarrow x = \frac{K}{2}$ or $2x = \sin^{-1} \frac{1}{\sqrt{3}} = K \Rightarrow x = \frac{K}{2}$ or $2x = \cos^{-1} \sqrt{\frac{2}{3}} = K \Rightarrow x = \frac{K}{2}$ or $\tan^2 x = 5 \pm 2\sqrt{6} \Rightarrow x = \tan^{-1}(\sqrt{5 \pm 2\sqrt{6}})$ or $\cos 4x = \frac{1}{3} \Rightarrow 4x = \cos^{-1}\left(\frac{1}{3}\right) = K \Rightarrow x = \frac{1}{4}K$	M1	1.1b
	$x = 17.6^\circ, 72.4^\circ$	A1	1.1b
		(4)	
(7 marks)			
(b) Notes			

- (b) Note that attempts solve an equation of the form:
 $2 \tan x \sec x = \cot 2x \sec 2x$ or e.g. $2 \tan \theta \sec \theta = \cot 2\theta \sec 2\theta$ or e.g. $2 \tan \theta \sec \theta = \cot 2x \sec 2x$
 Will generally score no marks in part (b)

Condone the use of θ instead of x here.

- B1:** Deduces the correct equation using the result from part (a)
M1: Factors out or cancels the $\sec 2x$ to obtain $\dots \tan 2x \pm \dots \cot 2x = 0$ or e.g.
 $\dots \tan 2x = \pm \dots \cot 2x$ leading to an equation of the form:
 $\tan^2 2x = \alpha$ or e.g. $\cot^2 2x = \frac{1}{\alpha}$ where $\alpha > 0$
Or uses $\tan 2x = \frac{\sin 2x}{\cos 2x}$ and $\cot 2x = \frac{\cos 2x}{\sin 2x}$ and then $\sin^2 2x = \pm 1 \pm \cos^2 2x$ or
 $\cos^2 2x = \pm 1 \pm \sin^2 2x$ to obtain an equation of the form $\sin^2 2x = \beta$ or $\cos^2 2x = \gamma$ or
 e.g. $\operatorname{cosec}^2 2x = \frac{1}{\beta}$ or $\sec^2 2x = \frac{1}{\gamma}$ where $0 < \beta < 1$ or $0 < \gamma < 1$
Or uses $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$ and $\cot 2x = \frac{1 - \tan^2 x}{2 \tan x}$ to obtain a 3TQ in $\tan^2 x$ (or possibly
 in $\sec^2 x$)
Or uses $\tan 2x = \frac{\sin 2x}{\cos 2x}$ and $\cot 2x = \frac{\cos 2x}{\sin 2x}$ and then $2 \sin^2 2x = \pm 1 \pm \cos 4x$ and
 $2 \cos^2 2x = \pm 1 \pm \cos 4x$ to obtain an equation of the form $\cos 4x = k$, $0 < k < 1$
M1: Correct order of operations from $\tan^2 2x = \alpha$ or $\sin^2 2x = \beta$ or $\cos^2 2x = \gamma$ or $\cos 4x = k$
 or equivalents e.g. $\operatorname{cosec}^2 2x = \frac{1}{\beta}$ where $\alpha > 0$ or $0 < \beta < 1$ or $0 < \gamma < 1$ or $0 < k < 1$
 leading to at least one value for x e.g. square roots, finds inverse tan/sin/cos/cosec and
 divides by 2 or inverse cos and divides by 4
or from $\tan^2 x = k$, $k > 0$ which follows their equation (may need to check) and then
 finds $x = \tan^{-1} \sqrt{k}$
 You may need to check their value(s) (in degrees or radians) to see if the correct order of
 operations has been used. May be implied by e.g. 17.6° or 17.7° provided no incorrect
 work is seen.
A1: Correct values. Allow awrt 17.6° and awrt 72.4° . The degrees symbol is not required.
 Ignore any values outside the range (correct or incorrect) but if there are extra angles in
 range score A0. Answers in radians score A0.

Note that some candidates may convert to $\sin x$ or $\cos x$ and then solve:

E.g.

$$\tan^2 2x = \frac{1}{2} \Rightarrow \frac{\sin^2 2x}{\cos^2 2x} = \frac{1}{2} \Rightarrow \frac{4 \sin^2 x \cos^2 x}{(2 \cos^2 x - 1)^2} \rightarrow 12 \cos^4 x - 12 \cos^2 x + 1 = 0$$

$$\text{or } \frac{4 \sin^2 x \cos^2 x}{(1 - 2 \sin^2 x)^2} \rightarrow 12 \sin^4 x - 12 \sin^2 x + 1 = 0$$

$$\Rightarrow \cos^2 x / \sin^2 x = \frac{3 \pm \sqrt{6}}{6} \Rightarrow \cos x / \sin x = \pm \sqrt{\frac{3 \pm \sqrt{6}}{6}} \Rightarrow x = 17.6^\circ, 72.4^\circ$$

These can be marked in a similar way.



Alternative not using part (a):

$$\frac{1}{\operatorname{cosec} 2x - 1} + \frac{1}{\operatorname{cosec} 2x + 1} = \cot 2x \sec 2x$$

$$\Rightarrow \frac{2 \operatorname{cosec} 2x}{\operatorname{cosec}^2 2x - 1} = \cot 2x \sec 2x$$

$$\Rightarrow \frac{2 \operatorname{cosec} 2x}{\cot^2 2x} = \frac{\cos 2x}{\sin 2x} \times \frac{1}{\cos 2x} = \operatorname{cosec} 2x$$

$$\Rightarrow 2 \tan^2 2x = 1 \Rightarrow \tan^2 2x = \frac{1}{2}$$

Score as:

M1: For correct work leading to one of the forms in the main scheme e.g.

$$\tan^2 2x = \alpha \text{ or e.g. } \cot^2 2x = \frac{1}{\alpha} \text{ where } \alpha > 0$$

or

$$\sin^2 2x = \beta \text{ or } \cos^2 2x = \gamma \text{ or e.g. } \operatorname{cosec}^2 2x = \frac{1}{\beta} \text{ or } \sec^2 2x = \frac{1}{\gamma}$$

where $0 < \beta < 1$ or $0 < \gamma < 1$

B1: Any correct equation e.g. $\tan^2 2x = \frac{1}{2}$, $\cot^2 2x = 2$, $\sin^2 2x = \frac{1}{3}$ etc.

Then **M1A1** as main scheme



Question	Scheme	Marks	AOs
16 (a)	Attempts $y \cdot \frac{dx}{dt} = (2 \sin 2t + 3 \sin t) \times 16 \sin t \cos t$ and uses $\sin 2t = 2 \sin t \cos t$	M1	2.1
	Correct expanded integrand. Usually for one of $(R) = \int \underline{48 \sin^2 t \cos t + 16 \sin^2 2t} dt$ $(R) = \int \underline{48 \sin^2 t \cos t + 64 \sin^2 t \cos^2 t} dt$ $(R) = \int \underline{24 \sin 2t \sin t + 16 \sin^2 2t} dt$	A1	1.1b
	Attempts to use $\cos 4t = 1 - 2 \sin^2 2t = (1 - 8 \sin^2 t \cos^2 t)$	M1	1.1b
	$R = \int_0^a 8 - 8 \cos 4t + 48 \sin^2 t \cos t dt$ *	A1*	2.1
	Deduces $a = \frac{\pi}{4}$	B1	2.2a
		(5)	
(b)	$\int 8 - 8 \cos 4t + 48 \sin^2 t \cos t dt = 8t - 2 \sin 4t + 16 \sin^3 t$	M1 A1	2.1 1.1b
	$\left[8t - 2 \sin 4t + 16 \sin^3 t \right]_0^{\frac{\pi}{4}} = 2\pi + 4\sqrt{2}$	M1 A1	2.1 1.1b
		(4)	
(9 marks)			
Notes:			

(a) Condone work in another variable, say $\theta \leftrightarrow t$ if used consistently for the first 3 marks

M1: For the key step in attempting $y \cdot \frac{dx}{dt} = (2 \sin 2t + 3 \sin t) \times 16 \sin t \cos t$ with an attempt to use

$\sin 2t = 2 \sin t \cos t$ Condone slips in finding $\frac{dx}{dt}$ but it must be of the form $k \sin t \cos t$

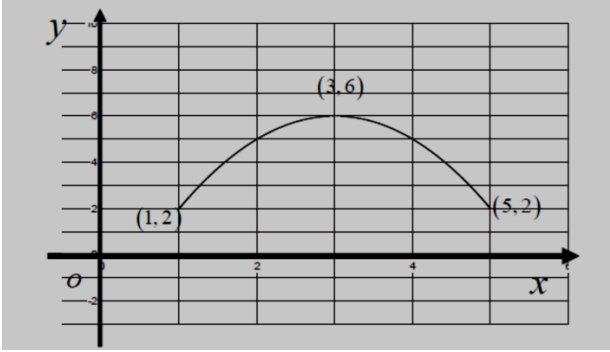
E.g. I $y \cdot \frac{dx}{dt} = (2 \sin 2t + 3 \sin t) \times k \sin t \cos t = (4 \sin t \cos t + 3 \sin t) \times k \sin t \cos t$

E.g. II $y \cdot \frac{dx}{dt} = (2 \sin 2t + 3 \sin t) \times k \sin t \cos t = (2 \sin 2t + 3 \sin t) \times \frac{k}{2} \sin 2t$

A1: A correct (expanded) integrand in t . Don't be concerned by the absence of $\int$ or dt or limits

$$(R) = \int \underline{48 \sin^2 t \cos t + 16 \sin^2 2t} dt \quad \text{or} \quad (R) = \int \underline{48 \sin^2 t \cos t + 64 \sin^2 t \cos^2 t} dt$$

but watch for other correct versions such as $(R) = \int \underline{24 \sin 2t \sin t + 16 \sin^2 2t} dt$

Question	Scheme	Marks	AOs
14(a)	Attempts to use $\cos 2t = 1 - 2\sin^2 t \Rightarrow \frac{y-4}{2} = 1 - 2\left(\frac{x-3}{2}\right)^2$	M1	2.1
	$\Rightarrow y - 4 = 2 - 4 \times \frac{(x-3)^2}{4} \Rightarrow y = 6 - (x-3)^2$ *	A1*	1.1b
		(2)	
(b)	 shaped parabola Fully correct with 'ends' at (1,2) & (5,2) Suitable reason : Eg states as $x = 3 + 2\sin t$, $1 \leq x \leq 5$	M1	1.1b
		A1	1.1b
		B1	2.4
		(3)	
(c)	Either finds the lower value for $k = 7$ or deduces that $k < \frac{37}{4}$	B1	2.2a
	Finds where $x + y = k$ meets $y = 6 - (x-3)^2$ $\Rightarrow k - x = 6 - (x-3)^2$ and proceeds to 3TQ in x or y	M1	3.1a
	Correct 3TQ in x $x^2 - 7x + (k+3) = 0$ Or y $y^2 + (7-2k)y + (k^2 - 6k + 3) = 0$	A1	1.1b
	Uses $b^2 - 4ac = 0 \Rightarrow 49 - 4 \times 1 \times (k+3) = 0 \Rightarrow k = \left(\frac{37}{4}\right)$ or $(7-2k)^2 - 4 \times 1 \times (k^2 - 6k + 3) = 0 \Rightarrow k = \left(\frac{37}{4}\right)$	M1	2.1
	Range of values for $k = \left\{k : 7 \leq k < \frac{37}{4}\right\}$	A1	2.5
		(5)	
(10 marks)			
(a)			

A1*: Proceeds to $y = 6 - (x - 3)^2$ without any errors

Allow a proof where they start with $y = 6 - (x - 3)^2$ and substitute the parametric coordinates. M1 is scored for a correct $\cos 2t = 1 - 2\sin^2 t$ but A1 is only scored when both sides are seen to be the same AND a comment is made, hence proven, or similar.

(b)

M1: For sketching a $\cap$ parabola with a maximum in quadrant one. It does not need to be symmetrical

A1: For sketching a $\cap$ parabola with a maximum in quadrant one and with end coordinates of $(1, 2)$ and $(5, 2)$

B1: Any suitable explanation as to why C does not include all points of $y = 6 - (x - 3)^2$, $x \in \mathbb{R}$

This should include a reference to **the limits on sin or cos** with a **link to a restriction on x or y**. For example

‘As $-1 \leq \sin t \leq 1$ then $1 \leq x \leq 5$ ’ Condone in words ‘x lies between 1 and 5’ and strict inequalities

‘As $\sin t \leq 1$ then $x \leq 5$ ’ Condone in words ‘x is less than 5’

‘As $-1 \leq \cos(2t) \leq 1$ then $2 \leq y \leq 6$ ’ Condone in words ‘y lies between 2 and 6’

Withhold if the statement is incorrect Eg "because the domain is $2 \leq x \leq 5$ "

Do not allow a statement on the top limit of y as this is the same for both curves

(c)

B1: Deduces either

- the correct that the lower value of $k = 7$ This can be found by substituting into $(5, 2)$
 $x + y = k \Rightarrow k = 7$ or substituting $x = 5$ into $x^2 - 7x + (k + 3) = 0 \Rightarrow 25 - 35 + k + 3 = 0$
 $\Rightarrow k = 7$
- or deduces that $k < \frac{37}{4}$ This may be awarded from later work

M1: For an attempt at the upper value for k.

Finds where $x + y = k$ meets $y = 6 - (x - 3)^2$ once by using an appropriate method.

Eg. Sets $k - x = 6 - (x - 3)^2$ and proceeds to a 3TQ

A1: Correct 3TQ $x^2 - 7x + (k + 3) = 0$ The $= 0$ may be implied by subsequent work

M1: Uses the "discriminant" condition. Accept use of $b^2 = 4ac$ or $b^2 \leq 4ac$ where ... is any inequality

leading to a critical value for k. Eg. one root $\Rightarrow 49 - 4 \times 1 \times (k + 3) = 0 \Rightarrow k = \frac{37}{4}$

A1: Range of values for $k = \left\{ k : 7 \leq k < \frac{37}{4} \right\}$ Accept $k \in \left[7, \frac{37}{4} \right)$ or exact equivalent

ALT	As above	B1	2.2a
	Finds where $x + y = k$ meets $y = 6 - (x - 3)^2$ once by using an appropriate method. Eg. Sets gradient of $y = 6 - (x - 3)^2$ equal to -1	M1	3.1a
	$-2x + 6 = -1 \Rightarrow x = 3.5$	A1	1.1b
	Finds point of intersection and uses this to find upper value of k. $y = 6 - (3.5 - 3)^2 = 5.75$ Hence using $k = 3.5 + 5.75 = 9.25$	M1	2.1
	Range of values for $k = \{k : 7 \leq k < 9.25\}$	A1	2.5

Question	Scheme	Marks	AOs
8(a)	$R = \sqrt{2^2 + 8^2} = \sqrt{68} = 2\sqrt{17}$	B1	1.1b
	$2 \cos \theta + 8 \sin \theta = R \cos \theta \cos \alpha + R \sin \theta \sin \alpha$ $2 = R \cos \alpha \quad 8 = R \sin \alpha$ $\tan \alpha = \frac{8}{2} \Rightarrow \alpha = \dots$	M1	1.1b
	$\alpha = \text{awrt } 1.326$	A1	2.2a
		(3)	
(b)(i)	$4.5 \times "2\sqrt{17}"$	M1	1.1b
	$9\sqrt{17}$	A1	2.2a
(ii)	awrt 1.33	B1ft	2.2a
		(3)	
(6 marks)			
Notes			
(a) B1: $R = 2\sqrt{17}$ or $\sqrt{68}$. $\pm 2\sqrt{17}$ or $\pm\sqrt{68}$ score B0 (Condone if this comes from e.g., $8 = R \cos \alpha \quad 2 = R \sin \alpha$) Decimal answers score B0 unless the exact value is seen then apply isw. M1: Proceeds to a value for α from $\tan \alpha = \pm \frac{8}{2}$, $\cos \alpha = \pm \frac{2}{\sqrt{68}}$, $\sin \alpha = \pm \frac{8}{\sqrt{68}}$ May be implied by awrt 1.33 radians or 76 degrees A1: awrt 1.326 for α. Apply isw if this is then subsequently rounded to e.g. 1.33			
(b)(i) M1: For a value of $\pm 4.5 \times$ their R or allow $\pm 4.5R$ (with the letter R) But not embedded in an expression e.g. $9\sqrt{17} \cos(\theta - \alpha)$ unless extracted later. Note that the sum may be found as $9 \cos x + 36 \sin x$ with the maximum then found using calculus e.g. $S = 9 \cos x + 36 \sin x \Rightarrow \frac{dS}{dx} = -9 \sin x + 36 \cos x = 0 \Rightarrow \tan x = 4 \Rightarrow \sin x = \frac{4}{\sqrt{17}}$, $\cos x = \frac{1}{\sqrt{17}}$ $\Rightarrow 9 \cos x + 36 \sin x = 9\sqrt{17}$. This will score M1 once they reach $\pm 4.5 \times$ their R May be implied by $9\sqrt{17}$ or awrt 37.1 (which may come from a graphical method) May also see e.g. $\text{Max}(9 \cos x + 36 \sin x) = \sqrt{9^2 + 36^2} = \dots$ A1: $9\sqrt{17}$ or exact equivalent e.g. $\sqrt{1377}$, $4.5\sqrt{68}$, $4.5(2\sqrt{17})$ and apply isw once a correct answer is seen			
(ii) B1ft: awrt 1.33 (or follow through on their α even if in degrees (76), no matter how accurate)			

Question	Scheme	Marks	AOs
12	$\frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} \equiv 2 \cot 2\theta$		
(a) Way 1	$\{\text{LHS} = \} \frac{\cos 3\theta \cos \theta + \sin 3\theta \sin \theta}{\sin \theta \cos \theta}$	M1	3.1a
	$= \frac{\cos(3\theta - \theta)}{\sin \theta \cos \theta} \left\{ = \frac{\cos 2\theta}{\sin \theta \cos \theta} \right\}$	A1	2.1
	$= \frac{\cos 2\theta}{\frac{1}{2} \sin 2\theta} = 2 \cot 2\theta *$	dM1	1.1b
		A1 *	2.1
		(4)	
(a) Way 2	$\{\text{LHS} = \} \frac{\cos 2\theta \cos \theta - \sin 2\theta \sin \theta}{\sin \theta} + \frac{\sin 2\theta \cos \theta + \cos 2\theta \sin \theta}{\cos \theta}$		
	$= \frac{\cos 2\theta \cos^2 \theta - \sin 2\theta \sin \theta \cos \theta + \sin 2\theta \cos \theta \sin \theta + \cos 2\theta \sin^2 \theta}{\sin \theta \cos \theta}$	M1	3.1a
	$= \frac{\cos 2\theta (\cos^2 \theta + \sin^2 \theta)}{\sin \theta \cos \theta} \left\{ = \frac{\cos 2\theta}{\sin \theta \cos \theta} \right\}$	A1	2.1
	$= \frac{\cos 2\theta}{\frac{1}{2} \sin 2\theta} = 2 \cot 2\theta *$	dM1	1.1b
		A1 *	2.1
(a) Way 3		(4)	
	$\{\text{RHS} = \} \frac{2 \cos 2\theta}{\sin 2\theta} = \frac{2 \cos(3\theta - \theta)}{\sin 2\theta} = \frac{2(\cos 3\theta \cos \theta + \sin 3\theta \sin \theta)}{\sin 2\theta}$	M1	3.1a
		A1	2.1
	$= \frac{2(\cos 3\theta \cos \theta + \sin 3\theta \sin \theta)}{2 \sin \theta \cos \theta}$	dM1	1.1b
	$= \frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} *$	A1 *	2.1
		(4)	
(b) Way 1	$\left\{ \frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} = 4 \Rightarrow \right\} 2 \cot 2\theta = 4 \Rightarrow 2 \left(\frac{1}{\tan 2\theta} \right) = 4$	M1	1.1b
	Rearranges to give $\tan 2\theta = k; k \neq 0$ and applies $\arctan k$	dM1	1.1b
	$\left\{ 90^\circ < \theta < 180^\circ, \tan 2\theta = \frac{1}{2} \Rightarrow \right\}$		
	Only one solution of $\theta = 103.3^\circ$ (1 dp) or awrt 103.3°	A1	2.2a
		(3)	
(b) Way 2	$\left\{ \frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} = 4 \Rightarrow \right\} 2 \cot 2\theta = 4 \Rightarrow \frac{2}{\tan 2\theta} = 4$	M1	1.1b
	$\frac{2}{\left(\frac{2 \tan \theta}{1 - \tan^2 \theta} \right)} = 4 \Rightarrow 2(1 - \tan^2 \theta) = 8 \tan \theta$		
	$\Rightarrow \tan^2 \theta + 4 \tan \theta - 1 = 0 \Rightarrow \tan \theta = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-1)}}{2(1)}$	dM1	1.1b
	$\{\Rightarrow \tan \theta = -2 \pm \sqrt{5}\} \Rightarrow \tan \theta = k; k \neq 0 \Rightarrow \text{applies } \arctan k$		
	$\{90^\circ < \theta < 180^\circ, \tan \theta = -2 - \sqrt{5} \Rightarrow \}$		
	Only one solution of $\theta = 103.3^\circ$ (1 dp) or awrt 103.3°	A1	2.2a
		(3)	
(7 marks)			

Notes for Question 12	
(a)	Way 1 and Way 2
M1:	Correct valid method forming a common denominator of $\sin \theta \cos \theta$ i.e. correct process of $\frac{(\dots)\cos \theta + (\dots)\sin \theta}{\sin \theta \cos \theta}$
A1:	Proceeds to show that the numerator of their resulting fraction simplifies to $\cos(3\theta - \theta)$ or $\cos 2\theta$
dM1:	dependent on the previous M mark Applies a correct $\sin 2\theta \equiv 2\sin \theta \cos \theta$ to the common denominator $\sin \theta \cos \theta$
A1*:	Correct proof
Note:	Writing $\frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} = \frac{\cos 3\theta \cos \theta}{\sin \theta \cos \theta} + \frac{\sin 3\theta \sin \theta}{\sin \theta \cos \theta}$ is considered a correct valid method of forming a common denominator of $\sin \theta \cos \theta$ for the 1 st M1 mark
Note:	Give 1 st M0 e.g. for $\frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} = \frac{\cos 4\theta + \sin 4\theta}{\sin \theta \cos \theta}$ but allow 1 st M1 for $\frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} = \frac{\cos 3\theta \cos \theta + \sin 3\theta \sin \theta}{\sin \theta \cos \theta} = \frac{\cos 4\theta + \sin 4\theta}{\sin \theta \cos \theta}$
Note:	Give 1 st M0 e.g. for $\frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} = \frac{\cos^2 3\theta + \sin^2 3\theta}{\sin \theta \cos \theta}$ but allow 1 st M1 for $\frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} = \frac{\cos 3\theta \cos \theta + \sin 3\theta \sin \theta}{\sin \theta \cos \theta} = \frac{\cos^2 3\theta + \sin^2 3\theta}{\sin \theta \cos \theta}$
Note:	Allow 2 nd M1 for stating a correct $\sin 2\theta = 2\sin \theta \cos \theta$ and for attempting to apply it to the common denominator $\sin \theta \cos \theta$
(a)	Way 3
M1:	Starts from RHS and proceeds to expand $\cos 2\theta$ in the form $\cos 3\theta \cos \theta \pm \sin 3\theta \sin \theta$
A1:	Shows, as part of their proof, that $\cos 2\theta = \cos 3\theta \cos \theta + \sin 3\theta \sin \theta$
dM1:	dependent on the previous M mark Applies $\sin 2\theta \equiv 2\sin \theta \cos \theta$ to their denominator
A1*:	Correct proof
Note:	Allow 1 st M1 1 st A1 (together) for any of $\text{LHS} \rightarrow \frac{\cos 2\theta}{\sin \theta \cos \theta}$ or $\text{LHS} \rightarrow \frac{\cos 2\theta(\cos^2 \theta + \sin^2 \theta)}{\sin \theta \cos \theta}$ or $\text{LHS} \rightarrow \cos 2\theta(\cot \theta + \tan \theta)$ or $\text{LHS} \rightarrow \cos 2\theta \left(\frac{1 + \tan^2 \theta}{\tan \theta} \right)$ (i.e. where $\cos 2\theta$ has been factorised out)
Note:	Allow 1 st M1 1 st A1 for progressing as far as $\text{LHS} = \dots = \cot x - \tan x$
Note:	The following is a correct alternative solution $\frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} = \frac{\cos 3\theta \cos \theta + \sin 3\theta \sin \theta}{\sin \theta \cos \theta} = \frac{\frac{1}{2}(\cos 4\theta + \cos 2\theta) - \frac{1}{2}(\cos 4\theta - \cos 2\theta)}{\sin \theta \cos \theta}$ $= \frac{\cos 2\theta}{\sin \theta \cos \theta} = \frac{\cos 2\theta}{\frac{1}{2}\sin 2\theta} = 2 \cot 2\theta *$
Note:	E.g. going from $\frac{\cos 2\theta \cos^2 \theta - \sin 2\theta \sin \theta \cos \theta + \sin 2\theta \cos \theta \sin \theta + \cos 2\theta \sin^2 \theta}{\sin \theta \cos \theta}$ to $\frac{\cos 2\theta}{\sin \theta \cos \theta}$ with no intermediate working is 1 st A0

Notes for Question 12 Continued	
(b)	Way 1
M1:	Evidence of applying $\cot 2\theta = \frac{1}{\tan 2\theta}$
dM1:	dependent on the previous M mark Rearranges to give $\tan 2\theta = k$, $k \neq 0$, and applies $\arctan k$
A1:	Uses $90^\circ < \theta < 180^\circ$ to deduce the only solution $\theta = \text{awrt } 103.3^\circ$
Note:	Give M0M0A0 for writing, for example, $\tan 2\theta = 2$ with no evidence of applying $\cot 2\theta = \frac{1}{\tan 2\theta}$
Note:	1 st M1 can be implied by seeing $\tan 2\theta = \frac{1}{2}$
Note:	Condone 2 nd M1 for applying $\frac{1}{2} \arctan\left(\frac{1}{2}\right) \{= 13.28...\}$
(b)	Way 2
M1:	Evidence of applying $\cot 2\theta = \frac{1}{\tan 2\theta}$
dM1:	dependent on the previous M mark Applies $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$, forms and uses a correct method for solving a 3TQ to give $\tan \theta = k$, $k \neq 0$, and applies $\arctan k$
A1:	Uses $90^\circ < \theta < 180^\circ$ to deduce the only solution $\theta = \text{awrt } 103.3^\circ$
Note:	Give M1 dM1 A1 for no working leading to $\theta = \text{awrt } 103.3^\circ$ and no other solutions
Note:	Give M1 dM1 A0 for no working leading to $\theta = \text{awrt } 103.3^\circ$ and other solutions which can be either outside or inside the range $90^\circ < \theta < 180^\circ$

Question	Scheme	Marks	AOs
10(a)	$\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} = \frac{1 - (1 - 2\sin^2 \theta) + 2\sin \theta \cos \theta}{1 + \cos 2\theta + \sin 2\theta}$ or $\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} = \frac{1 - \cos 2\theta + \sin 2\theta}{1 + (2\cos^2 \theta - 1) + 2\sin \theta \cos \theta}$	M1	2.1
	$\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} = \frac{1 - (1 - 2\sin^2 \theta) + 2\sin \theta \cos \theta}{1 + (2\cos^2 \theta - 1) + 2\sin \theta \cos \theta}$	A1	1.1b
	$= \frac{2\sin^2 \theta + 2\sin \theta \cos \theta}{2\cos^2 \theta + 2\sin \theta \cos \theta} = \frac{2\sin \theta (\sin \theta + \cos \theta)}{2\cos \theta (\cos \theta + \sin \theta)}$	dM1	2.1
	$= \frac{\sin \theta}{\cos \theta} = \tan \theta^*$	A1*	1.1b
		(4)	
(b)	$\frac{1 - \cos 4x + \sin 4x}{1 + \cos 4x + \sin 4x} = 3\sin 2x \Rightarrow \tan 2x = 3\sin 2x \quad \text{o.e.}$	M1	3.1a
	$\Rightarrow \sin 2x - 3\sin 2x \cos 2x = 0$ $\Rightarrow \sin 2x(1 - 3\cos 2x) = 0$ $\Rightarrow (\sin 2x = 0,) \cos 2x = \frac{1}{3}$	A1	1.1b
	$x = 90^\circ, \text{ awrt } 35.3^\circ, \text{ awrt } 144.7^\circ$	A1 A1	1.1b 2.1
		(4)	
(8 marks)			
Notes			

(a)

 M1: Attempts to use a correct double angle formulae for both $\sin 2\theta$ and $\cos 2\theta$ (seen once).

 The application of the formula for $\cos 2\theta$ must be the one that cancels out the "1"

 So look for $\cos 2\theta = 1 - 2\sin^2 \theta$ in the numerator or $\cos 2\theta = 2\cos^2 \theta - 1$ in the denominator

 Note that $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ may be used as well as using $\cos^2 \theta + \sin^2 \theta = 1$

$$\text{A1: } \frac{1 - (1 - 2\sin^2 \theta) + 2\sin \theta \cos \theta}{1 + (2\cos^2 \theta - 1) + 2\sin \theta \cos \theta} \text{ or } \frac{2\sin^2 \theta + 2\sin \theta \cos \theta}{2\cos^2 \theta + 2\sin \theta \cos \theta}$$

 dM1: Factorises numerator and denominator in order to demonstrate cancelling of $(\sin \theta + \cos \theta)$

A1*: Fully correct proof with no errors.

You must see an intermediate line of $\frac{2\sin \theta (\cancel{\sin \theta + \cos \theta})}{2\cos \theta (\cancel{\cos \theta + \sin \theta})}$ or $\frac{\sin \theta}{\cos \theta}$ or even $\frac{2\sin \theta}{2\cos \theta}$

Withhold this mark if you see, within the body of the proof,

- notational errors. E.g. $\cos 2\theta = 1 - 2\sin^2$ or $\cos \theta^2$ for $\cos^2 \theta$
- mixed variables. E.g. $\cos 2\theta = 2\cos^2 x - 1$

(b)

 M1: Makes the connection with part (a) and writes the lhs as $\tan 2x$. Condone $x \leftrightarrow \theta$ $\tan 2\theta = 3\sin 2\theta$

 A1: Obtains $\cos 2x = \frac{1}{3}$ o.e. with $x \leftrightarrow \theta$. You may see $\sin^2 x = \frac{1}{3}$ or $\cos^2 x = \frac{2}{3}$ after use of double angle formulae.

 A1: Two "correct" values. Condone accuracy of awrt 90° , 35° , 145°

Also condone radian values here. Look for 2 of awrt 0.62, 1.57, 2.53

 A1: All correct (allow awrt) and no other values in range. Condone $x \leftrightarrow \theta$ if used consistently

 Answers without working in (b): Just answers and no working score 0 marks.

 If the first line is written out, i.e. $\tan 2x = 3\sin 2x$ followed by all three correct answers score 1100.


Question	Scheme	Marks	AOs
13(a)	$x = a \sin^2 \theta \Rightarrow \frac{dx}{d\theta} = 2a \sin \theta \cos \theta$	B1	1.1b
	$\int x^{\frac{1}{2}} \sqrt{a-x} dx = \int \sqrt{a} \sin \theta \sqrt{a-a \sin^2 \theta} \times 2a \sin \theta \cos \theta \{d\theta\}$	M1	2.1
	$= \int \sqrt{a} \sin \theta \sqrt{a} \cos \theta \times 2a \sin \theta \cos \theta d\theta = 2a^2 \int \sin^2 \theta \cos^2 \theta d\theta$ $= 2a^2 \int \left(\frac{1}{2} \sin 2\theta\right)^2 d\theta$	dM1	3.1a
	Replaces or considers limits $\{x=0 \Rightarrow\} \theta=0, \{x=a \Rightarrow\} \theta=\frac{\pi}{2}$ $= \frac{1}{2} a^2 \int_0^{\frac{\pi}{2}} \sin^2 2\theta d\theta \quad *$	A1*	1.1b
		(4)	
(b)	$\dots \int \sin^2 2\theta d\theta \rightarrow \dots \int \frac{1-\cos 4\theta}{2} d\theta$	M1	1.1b
	$\rightarrow \dots \left(\frac{\theta}{2} - \frac{1}{8} \sin 4\theta\right)$	dM1	2.1
	$\mu \int \sin^2 2\theta d\theta \rightarrow \frac{\mu}{2} \left(\theta - \frac{1}{4} \sin 4\theta\right)$	A1	1.1b
	$\left\{ \int_0^a x^{\frac{1}{2}} \sqrt{a-x} dx \right\} = \frac{a^2}{4} \left(\frac{\pi}{2} - 0 - 0\right) = \frac{1}{8} \pi a^2$	A1	1.1b
		(4)	

(8 marks)

Notes

(a)

B1: $\frac{dx}{d\theta} = 2a \sin \theta \cos \theta$ or $dx = 2a \sin \theta \cos \theta d\theta$ o.e. seen or implied by their substitution.

Note that writing $x = a \sin^2 \theta = \frac{a}{2} (1 - \cos 2\theta) \rightarrow \frac{dx}{d\theta} = a \sin 2\theta$ is correct. Condone use of $\frac{dx}{da}$

M1: Attempts to substitute, fully replacing $x^{\frac{1}{2}}$ and $\sqrt{a-x}$ with θ 's and dx with their $dx = \dots$

Look for $x^{\frac{1}{2}} \sqrt{a-x} dx \rightarrow f(\theta)g(\theta)h(\theta)$ where

- $f(\theta)$ is an attempt at $\sqrt{a \sin^2 \theta}$ e.g. allow $a \sin \theta$ but just $a \sin^2 \theta$ is not condoned
- $g(\theta)$ is an attempt at $\sqrt{a - a \sin^2 \theta}$ but not $\sqrt{a} - \sqrt{a \sin^2 \theta}$ unless $\sqrt{a - a \sin^2 \theta}$ is attempted first
- $h(\theta) =$ their dx or their $\frac{dx}{d\theta}$ or $\frac{1}{\text{their } \frac{dx}{d\theta}}$ (in terms of θ only but condone da seen)

Condone slips provided the intention is clear, e.g. $x^{\frac{1}{2}} \rightarrow \sqrt{a} \sin^2 \theta$ but x must be eliminated. There is no need to replace the limits and condone the absence of $d\theta$ and/or the integral sign for this mark.



dM1: Attempts to use $\sin 2\theta = 2 \sin \theta \cos \theta$ to convert an integral of the form $\int \sin^2 \theta \cos^2 \theta \, d\theta$ or e.g. the form $\int \sin \theta \cos \theta \sin 2\theta \, d\theta$ to $\int \dots \sin^2 2\theta \, d\theta$

There is no need to replace the limits and condone the absence of $d\theta$ and/or the integral sign for this mark.

A1*: Replaces or considers limits $\{x=0 \Rightarrow\} \theta=0$, $\{x=a \Rightarrow\} \theta=\frac{\pi}{2}$ at some stage before the given answer and proceeds with no errors to the given answer. The replaced limits may appear with their integral symbol and do not have to be justified and do not have to appear on every line. Condone infrequent slips in notation, e.g. $\sin \theta^2$ in a line as long as it is not consistently poor. You must see the integral sign with the correct limits and the $d\theta$ together in the given answer.

(b)

M1: Adopts an appropriate strategy by using the double angle identity to obtain an integrable form

$$\dots \int \sin^2 2\theta \, d\theta \rightarrow \dots \int \frac{\pm 1 \pm \cos 4\theta}{2} \, d\theta \text{ which may be seen as}$$

$$\lambda \int \sin^2 2\theta \, d\theta \rightarrow \frac{\lambda}{2} \int \pm 1 \pm \cos 4\theta \, d\theta \text{ with the } \frac{1}{2} \text{ absorbed into their coefficient of the integral.}$$

dM1: Integrates into the form $\pm p\theta \pm q \sin 4\theta$

A1: Correct integration of $\mu \int \sin^2 2\theta \, d\theta$ to $\frac{\mu}{2} \left(\theta - \frac{1}{4} \sin 4\theta \right)$. Here μ may be 1.

Condone lack of limits here.

A1: Applies limits to the correct integral and proceeds to $\frac{1}{8} \pi a^2$ following correct work.

There is no need to see 0 substituted in and condone any omission of integral signs and/or $d\theta$

Note that $\frac{a^2}{4} \left[\theta + \frac{1}{4} \sin 4\theta \right]_0^{\frac{\pi}{2}} = \frac{1}{8} \pi a^2$ is incorrect and scores M1dM1A0A0

Use of $\sin^2 2\theta = \frac{\pm 1 \pm \cos k\theta}{2}$ with $k \neq 4$ scores M0dM0A0A0 but may lead to $\frac{1}{8} \pi a^2$

Condone use of x in place of θ e.g. $\frac{a^2}{4} \left[x - \frac{1}{4} \sin 4x \right]_0^{\frac{\pi}{2}} = \frac{1}{8} \pi a^2$

See overleaf for some alternative approaches.



Alternative 13(a) working backwards:

$$\frac{1}{2}a^2 \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta \qquad \frac{dx}{d\theta} = 2a \sin \theta \cos \theta \text{ score B1 (as in main scheme)}$$

$$= \frac{1}{2}a^2 \int_0^{\frac{\pi}{2}} (2 \sin \theta \cos \theta)(2 \sin \theta \cos \theta) \, d\theta$$

$$= a \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta \, dx$$

Score M1 here for using the double angle identity and replacing ... $\sin \theta \cos \theta \, d\theta$ with dx

$$= a \int_0^a \sqrt{\frac{x}{a}} \sqrt{1 - \frac{x}{a}} \, dx$$

Score dM1 here for a full attempt to replace all trig leading to everything in terms of x only

Must come from the form $\int \dots \sin \theta \cos \theta \, dx$

A1 fully correct with limits replaced / considered before the final line and the final line fully correct with limits, integral sign and dx as per the main scheme.

Alternative 13(b) via IBP Way 1:

$$\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta \qquad \begin{cases} u = \sin^2 2\theta & v' = 1 \\ u' = 4 \sin 2\theta \cos 2\theta & v = \theta \end{cases}$$

$$= \left[\theta \sin^2 2\theta \right]_0^{\frac{\pi}{2}} - 4 \int_0^{\frac{\pi}{2}} \theta \sin 2\theta \cos 2\theta \, d\theta$$

$$= \left[\theta \sin^2 2\theta \right]_0^{\frac{\pi}{2}} - 2 \int_0^{\frac{\pi}{2}} \theta \sin 4\theta \, d\theta \qquad \begin{cases} u = \theta & v' = \sin 4\theta \\ u' = 1 & v = -\frac{\cos 4\theta}{4} \end{cases}$$

$$= \left[\theta \sin^2 2\theta \right]_0^{\frac{\pi}{2}} - 2 \left(\left[-\frac{\theta \cos 4\theta}{4} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\frac{\cos 4\theta}{4} \, d\theta \right) \qquad \text{Score M1 here.}$$

$$= \left[\theta \sin^2 2\theta \right]_0^{\frac{\pi}{2}} - 2 \left[-\frac{\theta \cos 4\theta}{4} + \frac{\sin 4\theta}{16} \right]_0^{\frac{\pi}{2}} \qquad \text{Score dM1 here, A1 if correct (ignoring limits).}$$

$$= \left[\theta \sin^2 2\theta + \frac{\theta \cos 4\theta}{2} - \frac{\sin 4\theta}{8} \right]_0^{\frac{\pi}{2}}$$

$$= \left(0 + \frac{\pi}{4} - 0 \right) - (0 + 0 - 0) = \frac{\pi}{4} \rightarrow \left\{ \int_0^a x^{\frac{1}{2}} \sqrt{a-x} \, dx \right\} = \frac{1}{8} \pi a^2 \text{ score A1* (with no errors).}$$

Alternative 13(b) via IBP Way 2:

$$\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta \qquad \left\{ \begin{array}{ll} u = \sin 2\theta & v' = \sin 2\theta \\ u' = 2 \cos 2\theta & v = -\frac{\cos 2\theta}{2} \end{array} \right\}$$

$$= \left[-\frac{\sin 2\theta \cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos^2 2\theta \, d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta + \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta = \left[-\frac{\sin 2\theta \cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos^2 2\theta \, d\theta + \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta$$

$$\Rightarrow 2 \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta = \left[-\frac{\sin 2\theta \cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 1 \, d\theta \quad \text{Score M1 here.}$$

$$\Rightarrow 2 \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta = \left[-\frac{\sin 2\theta \cos 2\theta}{2} + \theta \right]_0^{\frac{\pi}{2}} \quad \text{Score dM1 here, A1 if correct including the 2 (ignoring limits).}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta = \frac{1}{2} \left(0 + \frac{\pi}{2} \right) - \frac{1}{2} (0 + 0) = \frac{\pi}{4}$$

$$\rightarrow \left\{ \int_0^a x^{\frac{1}{2}} \sqrt{a-x} \, dx = \right\} = \frac{1}{8} \pi a^2 \quad \text{score A1* (with no errors).}$$



Question	Scheme	Marks	AOs
14(a)	Attempts to use both $\sin(x - 60^\circ) = \pm \sin x \cos 60^\circ \pm \cos x \sin 60^\circ$ $\cos(x - 30^\circ) = \pm \cos x \cos 30^\circ \pm \sin x \sin 30^\circ$	M1	2.1
	Correct equation $2 \sin x \cos 60^\circ - 2 \cos x \sin 60^\circ = \cos x \cos 30^\circ + \sin x \sin 30^\circ$	A1	1.1b
	Either uses $\frac{\sin x}{\cos x} = \tan x$ and attempts to make $\tan x$ the subject E.g. $(2 \cos 60^\circ - \sin 30^\circ) \tan x = \cos 30^\circ + 2 \sin 60^\circ$ Or attempts $\sin 30^\circ$ etc with at least two correct and collects terms in $\sin x$ and $\cos x$ E.g. $\left(2 \times \frac{1}{2} - \frac{1}{2}\right) \sin x = \left(2 \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) \cos x$,	M1	2.1
	Proceeds to given answer showing all key steps E.g. $\frac{1}{2} \tan x = \frac{3\sqrt{3}}{2} \Rightarrow \tan x = 3\sqrt{3}$ *	A1*	1.1b
		(4)	
(b)	Deduces that $x = 2\theta + 60^\circ$	B1	2.2a
	$\tan(2\theta + 60^\circ) = 3\sqrt{3} \Rightarrow 2\theta + 60^\circ = 79.1^\circ, 259.1^\circ, \dots$	M1	1.1b
	Correct method to find one value of θ E.g. $\theta = \frac{79.1^\circ - 60^\circ}{2}$	dM1	1.1b
	$\theta = \text{awrt } 9.6^\circ, 99.6^\circ$ (See note)	A1	2.1
		(4)	
(8 marks)			
Notes:			

(a)

M1: Attempts to use both compound angle expansions to set up an equation in $\sin x$ and $\cos x$
The terms must be correct but condone sign errors and a slip on the multiplication of 2

A1: Correct equation $2 \sin x \cos 60^\circ - 2 \cos x \sin 60^\circ = \cos x \cos 30^\circ + \sin x \sin 30^\circ$ o.e.

Note that $\cos 60^\circ = \sin 30^\circ$ and $\cos 30^\circ = \sin 60^\circ$

Also allow this mark for candidates who substitute in their trigonometric values "early"

$$2 \sin x \times \frac{1}{2} - 2 \cos x \times \frac{\sqrt{3}}{2} = \cos x \times \frac{\sqrt{3}}{2} + \sin x \times \frac{1}{2} \text{ o.e.}$$

M1: Shows the necessary progress towards showing the given result.

There are three key moves, two of which must be shown for this mark.

- uses $\frac{\sin x}{\cos x} = \tan x$ to form an equation in just $\tan x$.
- uses exact numerical values for $\sin 30^\circ, \sin 60^\circ, \cos 30^\circ, \cos 60^\circ$ with at least two correct
- collects terms in $\sin x$ and $\cos x$ or alternatively in $\tan x$

A1*: Proceeds to the given answer with accurate work showing all necessary lines.

E.g. I $2 \sin x \cos 60^\circ - 2 \cos x \sin 60^\circ = \cos x \cos 30^\circ + \sin x \sin 30^\circ$

$$\sin x (2 \cos 60^\circ - \sin 30^\circ) = \cos x (\cos 30^\circ + 2 \sin 60^\circ)$$

1. collect terms

$$(2 \cos 60^\circ - \sin 30^\circ) \tan x = \cos 30^\circ + 2 \sin 60^\circ$$

2. $\frac{\sin x}{\cos x} = \tan x$ so M1

$$\tan x = \frac{\cos 30^\circ + 2 \sin 60^\circ}{2 \cos 60^\circ - \sin 30^\circ} = \frac{\frac{\sqrt{3}}{2} + \sqrt{3}}{1 - \frac{1}{2}} = 3\sqrt{3}$$

3..uses values and completes proof A1*

E.g II

$$2 \sin x \times \frac{1}{2} - 2 \cos x \times \frac{\sqrt{3}}{2} = \cos x \times \frac{\sqrt{3}}{2} + \sin x \times \frac{1}{2}$$

1.uses values

$$\Rightarrow \frac{1}{2} \sin x = \frac{3\sqrt{3}}{2} \cos x$$

2.collects terms so M1

$$\Rightarrow \tan x = 3\sqrt{3}$$

3. $\frac{\sin x}{\cos x} = \tan x$ completes proof A1*

(b) Hence
B1: Deduces that $x = 2\theta + 60^\circ$ o.e such as $\theta = \frac{x - 60^\circ}{2}$

 This is implied for sight of the equation $\tan(2\theta + 60^\circ) = 3\sqrt{3}$
M1: Proceeds from $\tan(2\theta \pm \alpha^\circ) = 3\sqrt{3} \Rightarrow 2\theta \pm \alpha^\circ = \text{one of } 79.1^\circ, 259.1^\circ, \dots$ where $\alpha \neq 0$

 One angle for $\arctan(3\sqrt{3})$ **must** be correct in degrees or radians(3sf). FYI radian answers 1.38, 4.52

dM1: Correct method to find one value of θ from their $2\theta \pm \alpha^\circ = 79.1^\circ$ to $\theta = \frac{79.1^\circ \mp \alpha^\circ}{2}$

 This is dependent upon one angle being correct, which must be in degrees, for $\arctan(3\sqrt{3})$

$$\tan(2\theta + 60^\circ) = 3\sqrt{3} \Rightarrow \theta = 9.6^\circ \text{ would imply B1 M1 dM1}$$

A1: $\theta = \text{awrt } 9.6^\circ, 99.6^\circ$ with no other values given in the range

Otherwise: Via the use of $\cos(2\theta + 30^\circ) = \cos 2\theta \cos 30^\circ - \sin 2\theta \sin 30^\circ$.

$$2 \sin 2\theta = \cos(2\theta + 30^\circ) \Rightarrow \tan 2\theta = \frac{\sqrt{3}}{5} \Rightarrow \theta = 9.6^\circ, 99.6^\circ$$

The order of the marks needs to match up to the main scheme so 0110 is possible.
B1: For achieving $\tan 2\theta = \frac{\sqrt{3}}{5}$ o.e so allow $\tan 2\theta = \text{awrt } 0.346$ or $\tan 2\theta = \frac{\cos 30^\circ}{2 + \sin 30^\circ}$

 Or via double angle identities $\sqrt{3} \tan^2 \theta + 10 \tan \theta - \sqrt{3} = 0$ o.e.

M1: Attempts to use the compound angle identities to reach a form $\tan 2\theta = k$ where k is a constant not $3\sqrt{3}$ (or expression in trig terms such as $\cos 30$ etc as seen above)

 Or via double angle identities reaches a 3TQ in $\tan \theta$
dM1: Correct order of operations from $\tan 2\theta = k$ leading to $\theta = \dots$

 Correctly solves their $\sqrt{3} \tan^2 \theta + 10 \tan \theta - \sqrt{3} = 0$ leading to $\theta = \dots$
A1: $\theta = \text{awrt } 9.6^\circ, 99.6^\circ$ with no other values given in the range.

 Note that $\tan(2\theta + 60^\circ) = 3\sqrt{3} \Rightarrow \theta = 9.6^\circ, 99.6^\circ$ is acceptable for full marks

Question	Scheme	Marks	AOs
10 (a)	$\cos 3A = \cos (2A + A) = \cos 2A \cos A - \sin 2A \sin A$	M1	3.1a
	$= (2 \cos^2 A - 1) \cos A - (2 \sin A \cos A) \sin A$	dM1	1.1b
	$= (2 \cos^2 A - 1) \cos A - 2 \cos A (1 - \cos^2 A)$	ddM1	2.1
	$= 4 \cos^3 A - 3 \cos A^*$	A1*	1.1b
		(4)	
(b)	$1 - \cos 3x = \sin^2 x \Rightarrow \cos^2 x + 3 \cos x - 4 \cos^3 x = 0$	M1	1.1b
	$\Rightarrow \cos x (4 \cos^2 x - \cos x - 3) = 0$ $\Rightarrow \cos x (4 \cos x + 3)(\cos x - 1) = 0$ $\Rightarrow \cos x = \dots$	dM1	3.1a
	Two of $-90^\circ, 0, 90^\circ$, awrt 139°	A1	1.1b
	All four of $-90^\circ, 0, 90^\circ$, awrt 139°	A1	2.1
		(4)	
	(8 marks)		

Notes:

(a)

Allow a proof in terms of x rather than A

M1: Attempts to use the compound angle formula for $\cos(2A + A)$ or $\cos(A + 2A)$

Condone a slip in sign

dM1: Uses correct double angle identities for $\cos 2A$ and $\sin 2A$

$\cos 2A = 2 \cos^2 A - 1$ must be used. If either of the other two versions are used expect to see an attempt to replace $\sin^2 A$ by $1 - \cos^2 A$ at a later stage.

Depends on previous mark.

ddM1: Attempts to get all terms in terms of $\cos A$ using correct and appropriate identities.

Depends on both previous marks.

A1*: A completely correct and rigorous proof including correct notation, no mixed variables, missing brackets etc.

Alternative right to left is possible:

$$4 \cos^3 A - 3 \cos A = \cos A (4 \cos^2 A - 3) = \cos A (2 \cos^2 A - 1 + 2(1 - \sin^2 A) - 2) = \cos A (\cos 2A - 2 \sin^2 A)$$

$$= \cos A \cos 2A - 2 \sin A \cos A \sin A = \cos A \cos 2A - \sin 2A \sin A = \cos(2A + A) = \cos 3A$$

Score M1: For $4 \cos^3 A - 3 \cos A = \cos A (4 \cos^2 A - 3)$

dM1: For $\cos A (2 \cos^2 A - 1 + 2(1 - \sin^2 A) - 2)$ (Replaces $4 \cos^2 A - 1$ by $2 \cos^2 A - 1$ and $2(1 - \sin^2 A)$)

ddM1: Reaches $\cos A \cos 2A - \sin 2A \sin A$

A1: $\cos(2A + A) = \cos 3A$

(b)

M1: For an attempt to produce an equation just in $\cos x$ using both part (a) and the identity $\sin^2 x = 1 - \cos^2 x$

Allow one slip in sign or coefficient when copying the result from part (a)

dM1: Dependent upon the preceding mark. It is for taking the cubic equation in $\cos x$ and making a valid attempt to solve. This could include factorisation or division of a $\cos x$ term followed by an attempt to solve the 3 term quadratic equation in $\cos x$ to reach at least one non zero value for $\cos x$.

May also be scored for solving the cubic equation in $\cos x$ to reach at least one non zero value for $\cos x$.

A1: Two of $-90^\circ, 0, 90^\circ$, awrt 139° **Depends on the first method mark.**

A1: All four of $-90^\circ, 0, 90^\circ$, awrt 139° with no extra solutions offered within the range.

Note that this is an alternative approach for obtaining the cubic equation in (b):



Question	Scheme	Marks	AOs
12	$1 - \cos 2\theta \equiv \tan \theta \sin 2\theta, \theta \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}$		
(a) Way 1	$\tan \theta \sin 2\theta = \left(\frac{\sin \theta}{\cos \theta}\right)(2 \sin \theta \cos \theta)$	M1	1.1b
	$= \left(\frac{\sin \theta}{\cancel{\cos \theta}}\right)(2 \sin \theta \cancel{\cos \theta}) = 2 \sin^2 \theta = 1 - \cos 2\theta *$	M1	1.1b
		A1*	2.1
		(3)	
(a) Way 2	$1 - \cos 2\theta = 1 - (1 - 2 \sin^2 \theta) = 2 \sin^2 \theta$	M1	1.1b
	$= \left(\frac{\sin \theta}{\cos \theta}\right)(2 \sin \theta \cos \theta) = \tan \theta \sin 2\theta *$	M1	1.1b
		A1*	2.1
		(3)	
	$(\sec^2 x - 5)(1 - \cos 2x) = 3 \tan^2 x \sin 2x, -\frac{\pi}{2} < x < \frac{\pi}{2}$		
(b) Way 1	$(\sec^2 x - 5) \tan x \sin 2x = 3 \tan^2 x \sin 2x$ or $(\sec^2 x - 5)(1 - \cos 2x) = 3 \tan x(1 - \cos 2x)$		
	Deduces $x = 0$	B1	2.2a
	Uses $\sec^2 x = 1 + \tan^2 x$ and cancels/factorises out $\tan x$ or $(1 - \cos 2x)$ e.g. $(1 + \tan^2 x - 3 \tan x - 5) \tan x = 0$ or $(1 + \tan^2 x - 3 \tan x - 5)(1 - \cos 2x) = 0$ or $1 + \tan^2 x - 5 = 3 \tan x$	M1	2.1
	$\tan^2 x - 3 \tan x - 4 = 0$	A1	1.1b
	$(\tan x - 4)(\tan x + 1) = 0 \Rightarrow \tan x = \dots$	M1	1.1b
	$x = -\frac{\pi}{4}, 1.326$	A1	1.1b
		A1	1.1b
		(6)	
(9 marks)			
Notes for Question 12			
(a)	Way 1		
M1:	Applies $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and $\sin 2\theta = 2 \sin \theta \cos \theta$ to $\tan \theta \sin 2\theta$		
M1:	Cancels as scheme (may be implied) and attempts to use $\cos 2\theta = 1 - 2 \sin^2 \theta$		
A1*:	For a correct proof showing all steps of the argument		
(a)	Way 2		
M1:	For using $\cos 2\theta = 1 - 2 \sin^2 \theta$		
Note:	If the form $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ or $\cos 2\theta = 2 \cos^2 \theta - 1$ is used, the mark cannot be awarded until $\cos^2 \theta$ has been replaced by $1 - \sin^2 \theta$		
M1:	Attempts to write their $2 \sin^2 \theta$ in terms of $\tan \theta$ and $\sin 2\theta$ using $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and $\sin 2\theta = 2 \sin \theta \cos \theta$ within the given expression		
A1*:	For a correct proof showing all steps of the argument		
Note:	If a proof meets in the middle; e.g. they show $\text{LHS} = 2 \sin^2 \theta$ and $\text{RHS} = 2 \sin^2 \theta$; then some indication must be given that the proof is complete. E.g. $1 - \cos 2\theta \equiv \tan \theta \sin 2\theta$, QED, box		

Notes for Question 12 Continued			
(b)			
B1:	Deduces that the given equation yields a solution $x = 0$		
M1:	For using the key step of $\sec^2 x = 1 + \tan^2 x$ and cancels/factorises out $\tan x$ or $(1 - \cos 2x)$ or $\sin 2x$ to produce a quadratic factor or quadratic equation in just $\tan x$		
Note:	Allow the use of $\pm \sec^2 x = \pm 1 \pm \tan^2 x$ for M1		
A1:	Correct 3TQ in $\tan x$. E.g. $\tan^2 x - 3\tan x - 4 = 0$		
Note:	E.g. $\tan^2 x - 4 = 3\tan x$ or $\tan^2 x - 3\tan x = 4$ are acceptable for A1		
M1:	For a correct method of solving their 3TQ in $\tan x$		
A1:	Any one of $-\frac{\pi}{4}$, awrt -0.785 , awrt 1.326 , -45° , awrt 75.964°		
A1:	Only $x = -\frac{\pi}{4}, 1.326$ cao stated in the range $-\frac{\pi}{2} < x < \frac{\pi}{2}$		
Note:	Alternative Method (Alt 1)		
	$(\sec^2 x - 5)\tan x \sin 2x = 3\tan^2 x \sin 2x$ or $(\sec^2 x - 5)(1 - \cos 2x) = 3\tan x(1 - \cos 2x)$		
	Deduces $x = 0$	B1	2.2a
	$\sec^2 x - 5 = 3\tan x \Rightarrow \frac{1}{\cos^2 x} - 5 = 3\left(\frac{\sin x}{\cos x}\right)$ $1 - 5\cos^2 x = 3\sin x \cos x$ $1 - 5\left(\frac{1 + \cos 2x}{2}\right) = \frac{3}{2}\sin 2x$ $-\frac{3}{2} - \frac{5}{2}\cos 2x = \frac{3}{2}\sin 2x$ $\{3\sin 2x + 5\cos 2x = -3\}$	Complete process (as shown) of using the identities for $\sin 2x$ and $\cos 2x$ to proceed as far as $\pm A \pm B \cos 2x = \pm C \sin 2x$	M1 2.1
		$-\frac{3}{2} - \frac{5}{2}\cos 2x = \frac{3}{2}\sin 2x$ o.e.	A1 1.1b
	$\sqrt{34}\sin(2x + 1.03) = -3$	Expresses their answer in the form $R\sin(2x + \alpha) = k$; $k \neq 0$ with values for R and α	M1 1.1b
	$\sin(2x + 1.03) = -\frac{3}{\sqrt{34}}$		
	$x = -\frac{\pi}{4}, 1.326$	A1 A1	1.1b 1.1b

Question	Scheme	Marks	AOs
12(a)(i)	$y \times \frac{dx}{dt} = 5 \sin 2t \times 6 \cos t \quad \text{or} \quad 5 \times 2 \sin t \cos t \times 6 \cos t$	M1	1.2



	$(\text{Area} =) \int 5 \sin 2t \times 6 \cos t \, dt = \int 5 \times 2 \sin t \cos t \times 6 \cos t \, dt$ or $\int 5 \sin 2t \times 6 \cos t \, dt = \int 60 \sin t \cos^2 t \, dt$	dM1	1.1b
	$(\text{Area} =) \int_0^{\frac{\pi}{2}} 60 \sin t \cos^2 t \, dt *$	A1*	2.1*
		(3)	
(a)(ii)	$\int 60 \sin t \cos^2 t \, dt = -20 \cos^3 t$	M1 A1	1.1b 1.1b
	$\text{Area} = \left[-20 \cos^3 t \right]_0^{\frac{\pi}{2}} = 0 - (-20) = 20 *$	A1*	2.1
		(3)	
(b)	$5 \sin 2t = 4.2 \Rightarrow \sin 2t = \frac{4.2}{5}$	M1	3.4
	$t = 0.4986..., 1.072...$	A1	1.1b
	Attempts to find the x values at both t values	dM1	3.4
	$t = 0.4986... \Rightarrow x = 2.869...$ $t = 1.072 \Rightarrow x = 5.269...$	A1	1.1b
	Width of path = 2.40 metres	A1	3.2a
		(5)	
(11 marks)			

Notes:

(a)(i)

M1: Attempts to multiply y by $\frac{dx}{dt}$ to obtain $A \sin 2t \cos t$ but may apply $\sin 2t = 2 \sin t \cos t$ here

dM1: Attempts to use $\sin 2t = 2 \sin t \cos t$ within an integral which may be implied by

e.g. $A \int \sin 2t \times \cos t \, dt = \int k \sin t \cos^2 t \, dt$

A1*: Fully correct work leading to the given answer.

This must include $\sin 2t = 2 \sin t \cos t$ or e.g. $5 \sin 2t = 10 \sin t \cos t$ seen explicitly in their proof and a correct intermediate line that includes an integral sign and the “dt”

Allow the limits to just “appear” in the final answer e.g. working need not be shown for the limits.

(a)(ii)

M1: Obtains $\int 60 \sin t \cos^2 t \, dt = k \cos^3 t$. This may be attempted via a substitution of $u = \cos t$ to obtain

$$\int 60 \sin t \cos^2 t \, dt = k u^3$$

A1: Correct integration $-20 \cos^3 t$ or equivalent e.g. $-20u^3$

A1*: Rigorous proof with all aspects correct including the correct limits and the $0 - (-20)$ and

not just: $-20 \cos^3 \frac{\pi}{2} - (-20 \cos^3 0) = 20$

(b)

M1: Uses the given model and attempts to find value(s) of t when $\sin 2t = \frac{4.2}{5}$. Look for $2t = \sin^{-1} \frac{4.2}{5} \Rightarrow t = \dots$

A1: At least one correct value for t , correct to 2 dp. FYI $t = 0.4986..., 1.072...$ or in degrees $t = 28.57..., 61.42...$



Question	Scheme	Marks	AOs
14(a)	$\sin(x + 30^\circ) = \sin x \cos 30^\circ \pm \cos x \sin 30^\circ$ Attempts to use $\cos(x + 30^\circ) = \cos x \cos 30^\circ \mp \sin x \sin 30^\circ$ $\Rightarrow \pm \sin x \cos 30^\circ \pm \cos x \sin 30^\circ \pm \sqrt{3}(\pm \cos x \cos 30^\circ \pm \sin x \sin 30^\circ)$	M1	2.1
	Correct expression $\sin x \cos 30^\circ + \cos x \sin 30^\circ + \sqrt{3}(\cos x \cos 30^\circ - \sin x \sin 30^\circ)$	A1	1.1b
	States or implies that $\sin 30^\circ = \frac{1}{2}$ and $\cos 30^\circ = \frac{\sqrt{3}}{2}$ $= \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x + \sqrt{3} \left(\frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \sin x \right) = 2 \cos x$ *	A1*	2.1
		(3)	
(b)	$2 \cos \theta = 3 \sin 2\theta \Rightarrow 2 \cos \theta = 6 \sin \theta \cos \theta$	M1	2.1
	$\sin \theta = \frac{1}{3}$	A1	1.1b
	$\theta = \arcsin \frac{1}{3} \Rightarrow \theta = \dots$	dM1	1.1b
	$(\theta =) \text{awrt } 19.5^\circ, \text{awrt } 160.5^\circ, 90^\circ$	A1	2.2a
		(4)	
(7 marks)			
Notes:			
(a) Condone a complete proof entirely in θ (or another variable) instead of x. Do not be concerned with the omission of degrees. M1: Attempts to use both compound angle expansions to set up an expression in $\sin x$ and $\cos x$ i.e. $\pm \sin x \cos 30^\circ \pm \cos x \sin 30^\circ \pm \sqrt{3}(\pm \cos x \cos 30^\circ \pm \sin x \sin 30^\circ)$ The terms must be correct but condone sign errors and a slip on the multiplication of the $\sqrt{3}$ if they attempt to multiply out the brackets. (The $\sqrt{3}$ may be omitted entirely) This mark may be implied by further work e.g. $\pm \frac{\sqrt{3}}{2} \sin x \pm \frac{1}{2} \cos x \pm \sqrt{3} \left(\pm \frac{\sqrt{3}}{2} \cos x \pm \frac{1}{2} \sin x \right)$ A1: Correct expression $\sin x \cos 30^\circ + \cos x \sin 30^\circ + \sqrt{3}(\cos x \cos 30^\circ - \sin x \sin 30^\circ)$ o.e. May be implied by $\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x + \sqrt{3} \left(\frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \sin x \right)$ (or implied if multiplied out) A1*: Proceeds to the given answer with no errors seen including invisible brackets (condone a missing trailing bracket). We must see the exact numerical values used for $\sin 30^\circ$ and $\cos 30^\circ$ before proceeding to the given answer. Minimum required $\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x + \sqrt{3} \left(\frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \sin x \right) = 2 \cos x$ which scores M1A1A1* $\sin x \cos 30^\circ + \cos x \sin 30^\circ + \sqrt{3}(\cos x \cos 30^\circ - \sin x \sin 30^\circ) = 2 \cos x$ scores M1A1A0* There should not be any notational or bracketing errors and no mixed or missing variables.			

Alternative for part (a)

M1: Writes the left hand side of the equation as

$$\frac{1}{2}\sin(x+30^\circ) + \frac{\sqrt{3}}{2}\cos(x+30^\circ) = \sin 30^\circ \sin(x+30^\circ) + \cos 30^\circ \cos(x+30^\circ)$$

A1: Correct expression for $\sin 30^\circ \sin(x+30^\circ) + \cos 30^\circ \cos(x+30^\circ) = \cos x$

A1*: Proceeds to the given answer $\sin(x+30^\circ) + \sqrt{3}\cos(x+30^\circ) = 2\cos(x+30^\circ-30^\circ) = 2\cos x$ with **no errors seen including invisible brackets** (condone a missing trailing bracket). There should not be any notational or bracketing errors and no mixed or missing variables.

Alternative part (a) – using the R-alpha method

M1: States e.g. $R\cos(x+30^\circ \pm \alpha) = \sqrt{3}\cos(x+30^\circ) \mp \sin(x+30^\circ)$

and attempts to find either R or α correctly. (may be implied)

A1: Achieves $2\cos(x+30^\circ-30^\circ)$

A1*: Achieves $2\cos x$ with no errors seen and both stages of working shown e.g.

- States $(R\cos(x+30^\circ \pm \alpha) =) \sqrt{3}\cos(x+30^\circ)\cos\alpha \mp \sin(x+30^\circ)\sin\alpha$ oe
- Shows $R = \sqrt{1+3} = 2$ and $\tan\alpha = \left(\frac{1}{\sqrt{3}}\right)$ o.e. e.g. using cosine or sine

(b) Condone the use of x for θ and mixed variables

M1: Sets $2\cos\theta = 3\sin 2\theta$ and proceeds to $A\cos\theta = B\sin\theta\cos\theta$. (allow $A=B$)

May be implied by $\sin\theta = k$ ($k \neq 0,1$)

A1: $\sin\theta = \frac{1}{3}$

dM1: Finds at least one of their values of θ for their $\sin\theta = k$ ($k \neq 0,1$) It is dependent on the previous method mark. You may need to check their value(s) (in degrees or radians) but may be implied by e.g. awrt 19° (or awrt 20°) or awrt 161° (or awrt 160°) (awrt 0.34 or awrt 2.8 in radians)

A1: Deduces that $(\theta =) 90^\circ$ as well as giving $(\theta =)$ awrt 19.5° , awrt 160.5° with no other values in the given range (ignore any found outside of the range)
The degree symbol is not required. (Note the angles are 19.4712206...and 160.528779...)
Answers in radians score A0.



Question	Scheme	Marks	AOs
6	(a) $f(x) = -3x^3 + 8x^2 - 9x + 10, x \in \mathbb{R}$		
(a)	(i) $\{f(2) = -24 + 32 - 18 + 10 \Rightarrow f(2) = 0\}$	B1	1.1b
	(ii) $\{f(x) = (x-2)(-3x^2 + 2x - 5) \text{ or } (2-x)(3x^2 - 2x + 5)\}$	M1	2.2a
		A1	1.1b
		(3)	
(b)	$-3y^6 + 8y^4 - 9y^2 + 10 = 0 \Rightarrow (y^2 - 2)(-3y^4 + 2y^2 - 5) = 0$		
	Gives a partial explanation by - explaining that $-3y^4 + 2y^2 - 5 = 0$ has no {real} solutions with a reason, e.g. $b^2 - 4ac = (2)^2 - 4(-3)(-5) = -56 < 0$ - or stating that $y^2 = 2$ has 2 {real} solutions or $y = \pm\sqrt{2}$ {only}	M1	2.4
	Complete proof that the given equation has exactly two {real} solutions	A1	2.1
		(2)	
(c)	$3\tan^3 \theta - 8\tan^2 \theta + 9\tan \theta - 10 = 0; 7\pi \leq \theta < 10\pi$		
	{Deduces that} there are 3 solutions	B1	2.2a
		(1)	

(6 marks)

Notes for Question 6

(a)(i)	
B1:	$f(2) = 0$ or 0 stated by itself in part (a)(i)
(a)(ii)	
M1:	Deduces that $(x-2)$ or $(2-x)$ is a factor and attempts to find the other quadratic factor by - using long division to obtain either $\pm 3x^2 \pm kx + \dots, k = \text{value} \neq 0$ or $\pm 3x^2 \pm \alpha x + \beta, \beta = \text{value} \neq 0, \alpha$ can be 0 - factorising to obtain their quadratic factor in the form $(\pm 3x^2 \pm kx \pm c), k = \text{value} \neq 0, c$ can be 0, or in the form $(\pm 3x^2 \pm \alpha x \pm \beta), \beta = \text{value} \neq 0, \alpha$ can be 0
A1:	$(x-2)(-3x^2 + 2x - 5), (2-x)(3x^2 - 2x + 5)$ or $-(x-2)(3x^2 - 2x + 5)$ stated together as a product
(b)	
M1:	See scheme
A1:	See scheme. Proof must be correct with no errors , e.g. giving an incorrect discriminant value
Note:	Correct calculation e.g. $(2)^2 - 4(-3)(-5), 4 - 60$ or -56 must be given for the first explanation
Note:	Note that M1 can be allowed for - a correct follow through calculation for the discriminant of their “$-3y^4 + 2y^2 - 5$” which would lead to a value < 0 together with an explanation that $-3y^4 + 2y^2 - 5 = 0$ has no {real} solutions - or for the omission of < 0
Note:	< 0 must also been stated in a discriminant method for A1
Note:	Do not allow A1 for incorrect working, e.g. $(2)^2 - 4(-3)(-5) = -54 < 0$
Note:	$y^2 = 2 \Rightarrow y = \pm 2$, so 2 solutions is not allowed for A1, but can be condoned for M1
Note:	Using the formula on $-3y^4 + 2y^2 - 5 = 0$ or $-3x^2 + 2x - 5 = 0$ gives y^2 or $x = \frac{-2 \pm \sqrt{-56}}{-6}$ or $\frac{-1 \pm \sqrt{-14}}{-3}$

Notes for Question 6 Continued	
Note:	Completing the square on $-3x^2 + 2x - 5 = 0$ gives $x^2 - \frac{2}{3}x + \frac{5}{3} = 0 \Rightarrow \left(x - \frac{1}{3}\right)^2 - \frac{1}{9} + \frac{5}{3} = 0 \Rightarrow x = \frac{1}{3} \pm \sqrt{\frac{-14}{9}}$
Note:	Do not recover work for part (b) in part (c)
(c)	
B1:	See scheme
Note:	Give B0 for stating $\theta = \text{awrt } 23.1, \text{awrt } 26.2, \text{awrt } 29.4$ without reference to 3 solutions

Question	Scheme	Marks	AOs
14(a)	e.g. $2 \frac{\sin \theta}{\cos \theta} (8 \cos \theta + 23(1 - \cos^2 \theta)) = 8 \times 2 \sin \theta \cos \theta \sec^2 \theta$	B1	1.2
	$2 \tan \theta (8 \cos \theta + 23 \sin^2 \theta) = 8 \sin 2\theta \sec^2 \theta$ $\Rightarrow 2 \sin \theta \cos \theta (8 \cos \theta + 23(1 - \cos^2 \theta)) = 8 \sin 2\theta$ $\sin 2\theta (8 \cos \theta + 23(1 - \cos^2 \theta)) = 8 \sin 2\theta$ $\sin 2\theta (23 \cos^2 \theta - 8 \cos \theta - 15) = 0$	M1A1	2.1 2.2a
		(3)	
(b)	$\sin 2x(23 \cos^2 x - 8 \cos x - 15) = 0$		
	$\sin 2x = 0 \Rightarrow x = 360^\circ \text{ or } 540^\circ$	B1	2.2a
	$23 \cos^2 x - 8 \cos x - 15 \Rightarrow \cos x = -\frac{15}{23}$	M1	1.1b
	$\cos x = -\frac{15}{23} \Rightarrow x = \dots$	dM1	1.1b
	$x = 360^\circ, 540^\circ \text{ and awrt } 491^\circ \text{ only}$	A1	2.3
		(4)	
(7 marks)			
Notes			
(a) Allow use of e.g. x but the final mark requires the equation to be in terms of θ B1(M1 on EPEN): For recalling and using at least one correct trigonometric identity in the given equation. e.g. one of: $\sin^2 \theta + \cos^2 \theta = 1$, $1 + \tan^2 \theta = \sec^2 \theta$, $\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\sin 2\theta = 2 \sin \theta \cos \theta$ This may be seen explicitly or may be implied by their working by e.g. $\tan \theta \cos \theta = \sin \theta$ or they might multiply both sides by $\cos^2 \theta$ leaving $8 \sin 2\theta$ on the rhs implying $1 + \tan^2 \theta = \sec^2 \theta$ M1: For manipulating the equation using trigonometric identities (condoning sign slips only in the identities and arithmetic slips) to obtain an expression of the form: $A \sin 2\theta \cos^2 \theta + B \sin 2\theta \cos \theta + C \sin 2\theta (=0)$ or $\sin 2\theta (A \cos^2 \theta + B \cos \theta + C) (=0)$ with $A, B, C \neq 0$ A1: $\sin 2\theta (23 \cos^2 \theta - 8 \cos \theta - 15) = 0$ oe e.g. $\sin 2\theta (-23 \cos^2 \theta + 8 \cos \theta + 15) = 0$ cao Note that this is not a given answer so condone notational slips e.g. $\cos \theta^2$ for $\cos^2 \theta$ provided the intention is clear but the final equation must have no notational errors. Note that the “= 0” is not required for the M1 but is required for the A1 Note: some candidates arrive at the correct final answer fortuitously following errors in their work.			
(b) Allow all marks in (b) to score if the correct equation is obtained fortuitously in part (a) Also allow use of θ instead of x throughout in part (b). Correct answers, no working scores max 1000 B1: For one of $x = 360(^\circ)$ or $x = 540(^\circ)$ Condone $x = 2\pi$ or $x = 3\pi$ for this mark. The degrees symbol is not required. This may come from $\cos x = 1$ M1: Attempts to solve their 3TQ from part (a) or a “made up” 3TQ (which may only be seen in (b)) leading to a value for $\cos x$. The general guidance for solving a 3 term quadratic equation can be applied. Allow solution(s) from a calculator which may be implied by at least one correct value for their 3TQ. Must be a value for $\cos x$ and not e.g. x. dM1: Attempts to find one of their angles in the range $360 < x < 540$ (but not 450) for their $\cos x = k$ where $k < 1$ May be implied by their value(s) but must be in degrees. Requires them to state a value for $\cos x$. Must be checked (you can check $\cos(\text{their } x) = \text{their } k$ (1sf)) A1: $x = 360^\circ, 540^\circ$ and awrt 491° only with no other values in range (including 450). The degrees symbol is not required awrt 491 must come from $\cos x = -\frac{15}{23}$			

Question	Scheme	Marks	AOs
6 (a)	$5 \sin 2\theta = 9 \tan \theta \Rightarrow 10 \sin \theta \cos \theta = 9 \times \frac{\sin \theta}{\cos \theta}$ $A \cos^2 \theta = B \quad \text{or} \quad C \sin^2 \theta = D \quad \text{or} \quad P \cos^2 \theta \sin \theta = Q \sin \theta$	M1	3.1a
	For a correct simplified equation in one trigonometric function Eg $10 \cos^2 \theta = 9 \quad 10 \sin^2 \theta = 1$ oe	A1	1.1b
	Correct order of operations For example $10 \cos^2 \theta = 9 \Rightarrow \theta = \arccos(\pm) \sqrt{\frac{9}{10}}$	dM1	2.1
	Any one of the four values awrt $\theta = \pm 18.4^\circ, \pm 161.6^\circ$	A1	1.1b
	All four values $\theta =$ awrt $\pm 18.4^\circ, \pm 161.6^\circ$	A1	1.1b
	$\theta = 0^\circ, \pm 180^\circ$	B1	1.1b
		(6)	
(b)	Attempts to solve $x - 25^\circ = -18.4^\circ$	M1	1.1b
	$x = 6.6^\circ$	A1ft	2.2a
		(2)	
(8 marks)			

(a)

M1: Scored for the whole strategy of attempting to form an equation in one function of the form given in the scheme. For this to be awarded there must be an attempt at using $\sin 2\theta = \dots \sin \theta \cos \theta$, $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and possibly $\pm 1 \pm \sin^2 \theta = \pm \cos^2 \theta$ to form an equation in one "function" usually $\sin^2 \theta$ or $\cos^2 \theta$

Allow for this mark equations of the form $P \cos^2 \theta \sin \theta = Q \sin \theta$ oe

A1: Uses the correct identities $\sin 2\theta = 2 \sin \theta \cos \theta$ and $\tan \theta = \frac{\sin \theta}{\cos \theta}$ to form a correct simplified equation in one trigonometric function. It is usually one of the equations given in the scheme, but you may see equivalent correct equations such as $10 = 9 \sec^2 \theta$ which is acceptable, but in almost all cases it is for a correct equation in $\sin \theta$ or $\cos \theta$

dM1: Uses the correct order of operations for their equation, usually in terms of just $\sin \theta$ or $\cos \theta$, to find at least one value for θ (Eg. square root before invcos). It is dependent upon the previous M.

Note that some candidates will use $\cos^2 \theta = \frac{\pm \cos 2\theta \pm 1}{2}$ and the same rules apply.

Look for correct order of operations.

A1: Any one of the four values awrt $\pm 18.4^\circ, \pm 161.6^\circ$. Allow awrt 0.32 (rad) or 2.82 (rad)

A1: All four values awrt $\pm 18.4^\circ, \pm 161.6^\circ$ and no other values apart from $0^\circ, \pm 180^\circ$

B1: $\theta = 0^\circ, \pm 180^\circ$ This can be scored independent of method.

(b)

M1: Attempts to solve $x - 25^\circ = \theta$ where θ is a solution of their part (a)

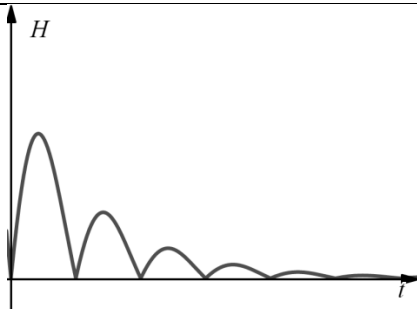
A1ft: For awrt $x = 6.6^\circ$ but you may ft on their $\theta + 25^\circ$ where $-25 < \theta < 0$

If multiple answers are given, the correct value for their θ must be chosen

Question	Scheme	Marks	AOs
7	(i) $4\sin x = \sec x$, $0 \leq x < \frac{\pi}{2}$; (ii) $5\sin\theta - 5\cos\theta = 2$, $0 \leq \theta < 360^\circ$		
(i) Way 1	For $\sec x = \frac{1}{\cos x}$	B1	1.2
	$\{4\sin x = \sec x \Rightarrow\} 4\sin x \cos x = 1 \Rightarrow 2\sin 2x = 1 \Rightarrow \sin 2x = \frac{1}{2}$	M1	3.1a
	$x = \frac{1}{2}\arcsin\left(\frac{1}{2}\right)$ or $\frac{1}{2}\left(\pi - \arcsin\left(\frac{1}{2}\right)\right) \Rightarrow x = \frac{\pi}{12}, \frac{5\pi}{12}$	dM1	1.1b
		A1	1.1b
		(4)	
(i) Way 2	For $\sec x = \frac{1}{\cos x}$	B1	1.2
	$\{4\sin x = \sec x \Rightarrow\} 4\sin x \cos x = 1 \Rightarrow 16\sin^2 x \cos^2 x = 1$ $16\sin^2 x(1 - \sin^2 x) = 1$ $16(1 - \cos^2 x)\cos^2 x = 1$ $16\sin^4 x - 16\sin^2 x + 1 = 0$ $16\cos^4 x - 16\cos^2 x + 1 = 0$ $\sin^2 x$ or $\cos^2 x = \frac{16 \pm \sqrt{192}}{32} \left\{ = \frac{2 \pm \sqrt{3}}{4} \text{ or } 0.933..., 0.066... \right\}$	M1	3.1a
	$x = \arcsin\left(\sqrt{\frac{2 \pm \sqrt{3}}{4}}\right)$ or $x = \arccos\left(\sqrt{\frac{2 \pm \sqrt{3}}{4}}\right) \Rightarrow x = \frac{\pi}{12}, \frac{5\pi}{12}$	dM1	1.1b
		A1	1.1b
		(4)	
(ii)	Complete strategy, i.e. - Expresses $5\sin\theta - 5\cos\theta = 2$ in the form $R\sin(\theta - \alpha) = 2$, finds both R and α, and proceeds to $\sin(\theta - \alpha) = k$, $k < 1$, $k \neq 0$ - Applies $(5\sin\theta - 5\cos\theta)^2 = 2^2$, followed by applying both $\cos^2\theta + \sin^2\theta = 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$ to proceed to $\sin 2\theta = k$, $k < 1$, $k \neq 0$	M1	3.1a
	$R = \sqrt{50}$ $\tan \alpha = 1 \Rightarrow \alpha = 45^\circ$		
	$(5\sin\theta - 5\cos\theta)^2 = 2^2 \Rightarrow$ $25\sin^2\theta + 25\cos^2\theta - 50\sin\theta\cos\theta = 4$ $\Rightarrow 25 - 25\sin 2\theta = 4$	M1	1.1b
	$\sin(\theta - 45^\circ) = \frac{2}{\sqrt{50}}$	A1	1.1b
	dependent on the first M mark e.g. $\theta = \arcsin\left(\frac{2}{\sqrt{50}}\right) + 45^\circ$ e.g. $\theta = \frac{1}{2}\left(\arcsin\left(\frac{21}{25}\right)\right)$	dM1	1.1b
	$\theta = \text{awrt } 61.4^\circ, \text{ awrt } 208.6^\circ$	A1	2.1
	Note: Working in radians does not affect any of the first 4 marks		
		(5)	
(9 marks)			

Question	Scheme	Marks	AOs
7	(ii) $5\sin\theta - 5\cos\theta = 2$, $0 \leq \theta < 360^\circ$		
(ii) Alt 1	Complete strategy, i.e. Attempts to apply $(5\sin\theta)^2 = (2 + 5\cos\theta)^2$ or $(5\sin\theta - 2)^2 = (5\cos\theta)^2$ followed by applying $\cos^2\theta + \sin^2\theta = 1$ and solving a quadratic equation in either $\sin\theta$ or $\cos\theta$ to give at least one of $\sin\theta = k$ or $\cos\theta = k$, $k < 1$, $k \neq 0$	M1	3.1a
	e.g. $25\sin^2\theta = 4 + 20\cos\theta + 25\cos^2\theta$ $\Rightarrow 25(1 - \cos^2\theta) = 4 + 20\cos\theta + 25\cos^2\theta$	M1	1.1b
	or e.g. $25\sin^2\theta - 20\sin\theta + 4 = 25\cos^2\theta$ $\Rightarrow 25\sin^2\theta - 20\sin\theta + 4 = 25(1 - \sin^2\theta)$		
	$50\cos^2\theta + 20\cos\theta - 21 = 0$		
	$\cos\theta = \frac{-20 \pm \sqrt{4600}}{100}$, o.e.	A1	1.1b
	dependent on the first M mark e.g. $\theta = \arccos\left(\frac{-2 + \sqrt{46}}{10}\right)$	dM1	1.1b
	$\theta = \text{awrt } 61.4^\circ$, awrt 208.6°	A1	2.1
		(5)	
Notes for Question 7			
(i)			
B1:	For recalling that $\sec x = \frac{1}{\cos x}$		
M1:	Correct strategy of Way 1: applying $\sin 2x = 2\sin x \cos x$ and proceeding to $\sin 2x = k$, $k \leq 1$, $k \neq 0$ Way 2: squaring both sides, applying $\cos^2 x + \sin^2 x = 1$ and solving a quadratic equation in either $\sin^2 x$ or $\cos^2 x$ to give $\sin^2 x = k$ or $\cos^2 x = k$, $k \leq 1$, $k \neq 0$		
dM1:	Uses the correct order of operations to find at least one value for x in either radians or degrees		
A1:	Clear reasoning to achieve both $x = \frac{\pi}{12}$, $\frac{5\pi}{12}$ and no other values in the range $0 \leq x < \frac{\pi}{2}$		
Note:	Give dM1 for $\sin 2x = \frac{1}{2} \Rightarrow$ any of $\frac{\pi}{12}$, $\frac{5\pi}{12}$, 15° , 75° , awrt 0.26 or awrt 1.3		
Note:	Give special case, SC B1M0M0A0 for writing down any of $\frac{\pi}{12}$, $\frac{5\pi}{12}$, 15° or 75° with no working		

Notes for Question 7 Continued	
(ii)	
M1:	See scheme
Note:	Alternative strategy: Expresses $5\sin\theta - 5\cos\theta = 2$ in the form $R\cos(\theta + \alpha) = -2$, finds both R and α , and proceeds to $\cos(\theta + \alpha) = k$, $ k < 1$, $k \neq 0$
M1:	Either - uses $R\sin(\theta - \alpha)$ to find the values of both R and α - attempts to apply $(5\sin\theta - 5\cos\theta)^2 = 2^2$, uses $\cos^2\theta + \sin^2\theta = 1$ and proceeds to find an equation of the form $\pm\lambda \pm \mu\sin 2\theta = \pm\beta$ or $\pm\mu\sin 2\theta = \pm\beta$; $\mu \neq 0$ - attempts to apply $(5\sin\theta)^2 = (2 + 5\cos\theta)^2$ or $(5\sin\theta - 2)^2 = (5\cos\theta)^2$ and uses $\cos^2\theta + \sin^2\theta = 1$ to form an equation in $\cos\theta$ only or $\sin\theta$ only
A1:	For $\sin(\theta - 45^\circ) = \frac{2}{\sqrt{50}}$, o.e., $\cos(\theta + 45^\circ) = -\frac{2}{\sqrt{50}}$, o.e. or $\sin 2\theta = \frac{21}{25}$, o.e. or $\cos\theta = \frac{-20 \pm \sqrt{4600}}{100}$, o.e. or $\cos\theta = \text{awrt } 0.48, \text{ awrt } -0.88$ or $\sin\theta = \frac{20 \pm \sqrt{4600}}{100}$, o.e., or $\sin\theta = \text{awrt } 0.88, \text{ awrt } -0.48$
Note:	$\sin(\theta - 45^\circ)$, $\cos(\theta + 45^\circ)$, $\sin 2\theta$ must be made the subject for A1
dM1:	dependent on the first M mark Uses the correct order of operations to find at least one value for x in either degrees or radians
Note:	dM1 can also be given for $\theta = 180^\circ - \arcsin\left(\frac{2}{\sqrt{50}}\right) + 45^\circ$ or $\theta = \frac{1}{2}\left(180^\circ - \arcsin\left(\frac{21}{25}\right)\right)$
A1:	Clear reasoning to achieve both $\theta = \text{awrt } 61.4^\circ, \text{ awrt } 208.6^\circ$ and no other values in the range $0 \leq \theta < 360^\circ$
Note:	Give M0M0A0M0A0 for writing down any of $\theta = \text{awrt } 61.4^\circ, \text{ awrt } 208.6^\circ$ with no working
Note:	Alternative solutions: (to be marked in the same way as Alt 1): $5\sin\theta - 5\cos\theta = 2 \Rightarrow 5\tan\theta - 5 = 2\sec\theta \Rightarrow (5\tan\theta - 5)^2 = (2\sec\theta)^2$ $\Rightarrow 25\tan^2\theta - 50\tan\theta + 25 = 4\sec^2\theta \Rightarrow 25\tan^2\theta - 50\tan\theta + 25 = 4(1 + \tan^2\theta)$ $\Rightarrow 21\tan^2\theta - 50\tan\theta + 21 = 0 \Rightarrow \tan\theta = \frac{50 \pm \sqrt{736}}{42} = \frac{25 \pm 2\sqrt{46}}{21} = 1.8364..., 0.5445...$ $\Rightarrow \theta = \text{awrt } 61.4^\circ, \text{ awrt } 208.6^\circ \text{ only}$ $5\sin\theta - 5\cos\theta = 2 \Rightarrow 5 - 5\cot\theta = 2\text{cosec}\theta \Rightarrow (5 - 5\cot\theta)^2 = (2\text{cosec}\theta)^2$ $\Rightarrow 25 - 50\cot\theta + 25\cot^2\theta = 4\text{cosec}^2\theta \Rightarrow 25 - 50\cot\theta + 25\cot^2\theta = 4(1 + \cot^2\theta)$ $\Rightarrow 21\cot^2\theta - 50\cot\theta + 21 = 0 \Rightarrow \cot\theta = \frac{50 \pm \sqrt{736}}{42} = \frac{25 \pm 2\sqrt{46}}{21} = 1.8364..., 0.5445...$ $\Rightarrow \theta = \text{awrt } 61.4^\circ, \text{ awrt } 208.6^\circ \text{ only}$

Question	Scheme	Marks	AOs
12 (a)	$f(x) = 10e^{-0.25x} \sin x$		
	$\Rightarrow f'(x) = -2.5e^{-0.25x} \sin x + 10e^{-0.25x} \cos x$ oe	M1 A1	1.1b 1.1b
	$f'(x) = 0 \Rightarrow -2.5e^{-0.25x} \sin x + 10e^{-0.25x} \cos x = 0$	M1	2.1
	$\frac{\sin x}{\cos x} = \frac{10}{2.5} \Rightarrow \tan x = 4^*$	A1*	1.1b
		(4)	
(b)		M1 A1	1.1b 1.1b
		(2)	
(c)	Solves $\tan x = 4$ and substitutes answer into $H(t)$	M1	3.1a
	$H(4.47) = 10e^{-0.25 \times 4.47} \sin 4.47 $	M1	1.1b
	awrt 3.18 (metres)	A1	3.2a
		(3)	
(d)	The times between each bounce should not stay the same when the heights of each bounce is getting smaller	B1	3.5b
		(1)	
(10 marks)			

(a)

M1: For attempting to differentiate using the product rule condoning slips, for example the power of e .

 So for example score expressions of the form $\pm \dots e^{-0.25x} \sin x \pm \dots e^{-0.25x} \cos x$ M1

 Sight of $vdu - u dv$ however is M0

A1: $f'(x) = -2.5e^{-0.25x} \sin x + 10e^{-0.25x} \cos x$ which may be unsimplified

M1: For clear reasoning in setting their $f'(x) = 0$, factorising/ cancelling out the $e^{-0.25x}$ term leading to a trigonometric equation in only $\sin x$ and $\cos x$

 Do not allow candidates to substitute $x = \arctan 4$ into $f'(x)$ to score this mark.

A1*: Shows the steps $\frac{\sin x}{\cos x} = \frac{10}{2.5}$ or equivalent leading to $\Rightarrow \tan x = 4^*$. $\frac{\sin x}{\cos x}$ must be seen.

(b)

M1: Draws at least two "loops". The height of the second loop should be lower than the first loop.

Condone the sight of rounding where there should be cusps

A1: At least 4 loops with decreasing heights and no rounding at the cusps.

The intention should be that the graph should 'sit' on the x-axis but be tolerant.

It is possible to overwrite Figure 3, but all loops must be clearly seen.

(c)

M1: Understands that to solve the problem they are required to substitute an answer to $\tan t = 4$ into $H(t)$ This can be awarded for an attempt to substitute $t = \text{awrt } 1.33$ or $t = \text{awrt } 4.47$ into $H(t)$ $H(t) = 6.96$ implies the use of $t = 1.33$ Condone for this mark only, an attempt to substitute $t = \text{awrt } 76^\circ$ or $\text{awrt } 256^\circ$ into $H(t)$ **M1:** Substitutes $t = \text{awrt } 4.47$ into $H(t) = \left| 10e^{-0.25t} \sin t \right|$. Implied by awrt 3.2**A1:** Awrt 3.18 metres. Condone the lack of units. If two values are given the correct one must be seen to have been chosen

It is possible for candidates to sketch this on their graphical calculators and gain this answer. If there is no incorrect working seen and 3.18 is given, then award 111 for such an attempt.

(d)

B1: Makes reference to the fact that the time between each bounce should not stay the same when the heights of each bounce is getting smaller.

Look for "time (or gap) between the bounces will change"

‘bounces would not be equal times apart’

‘bounces would become more frequent’

But do not accept ‘the times between each bounce would be longer or slower’

Do not accept explanations such as there are other factors that would affect this such as "wind resistance", friction etc

Question	Scheme	Marks	AOs
15(a)	Uses the common ratio $\frac{5+2\sin\theta}{12\cos\theta} = \frac{6\tan\theta}{5+2\sin\theta}$ o.e.	M1	3.1a
	Cross multiplies and uses $\tan\theta \times \cos\theta = \sin\theta$ $(5+2\sin\theta)^2 = 6 \times 12 \sin\theta$	dM1	1.1b
	Proceeds to given answer $25 + 20\sin\theta + 4\sin^2\theta = 72\sin\theta$ $\Rightarrow 4\sin^2\theta - 52\sin\theta + 25 = 0$ *	A1*	2.1
		(3)	
(a) Alt	(a) Alternative example:		
	Uses the common ratio $12r\cos\theta = 5+2\sin\theta$, $12r^2\cos\theta = 6\tan\theta$ $\Rightarrow 12\cos\theta \left(\frac{5+2\sin\theta}{12\cos\theta} \right)^2 = 6\tan\theta$	M1	3.1a
	Multiplies up and uses $\tan\theta \times \cos\theta = \sin\theta$ $(5+2\sin\theta)^2 = 6\tan\theta \times 12\cos\theta = 72\sin\theta$	dM1	1.1b
	Proceeds to given answer $25 + 20\sin\theta + 4\sin^2\theta = 72\sin\theta$ $\Rightarrow 4\sin^2\theta - 52\sin\theta + 25 = 0$ *	A1*	2.1
		(3)	
(b)	$4\sin^2\theta - 52\sin\theta + 25 = 0 \Rightarrow \sin\theta = \frac{1}{2} \left(\frac{25}{4} \right)$	M1	1.1b
	$\theta = \frac{5\pi}{6}$	A1	1.2
		(2)	
(c)	Attempts a value for either a or r e.g. $a = 12\cos\theta = 12 \times -\frac{\sqrt{3}}{2}$ or $r = \frac{5+2\sin\theta}{12\cos\theta} = \frac{5+2 \times \frac{1}{2}}{12 \times -\frac{\sqrt{3}}{2}}$	M1	3.1a
	" a " = $-6\sqrt{3}$ and " r " = $-\frac{1}{\sqrt{3}}$ o.e.	A1	1.1b
	Uses $S_{\infty} = \frac{a}{1-r} = \frac{-6\sqrt{3}}{1+\frac{1}{\sqrt{3}}}$	dM1	2.1
	Rationalises denominator $S_{\infty} = \frac{-6\sqrt{3}}{1+\frac{1}{\sqrt{3}}} = \frac{-18}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$	ddM1	1.1b
	$(S_{\infty})9(1-\sqrt{3})$	A1	2.1
		(5)	
(10 marks)			
Notes:			

(a)

M1: For the key step in using the ratio of $\frac{a_2}{a_1} = \frac{a_3}{a_2}$
dM1: Cross multiplies and uses $\tan\theta \times \cos\theta = \sin\theta$

Alternative:

M1: Expresses the 2nd and 3rd terms in terms of the first term and the common ratio and eliminates “r”

dM1: Multiplies up and uses $\tan \theta \times \cos \theta = \sin \theta$

A1*: Proceeds to the given answer including the “= 0” with no errors and sufficient working shown.

Other approaches may be seen in (a) and can be marked in a similar way e.g. M1 for correctly obtaining an equation in θ using the GP, M1 for applying $\tan \theta \times \cos \theta = \sin \theta$ or equivalent and eliminating fractions, A1 as above

$$\text{Example: } u_2 = \frac{u_1 \times u_3}{u_2} \Rightarrow 5 + 2 \sin \theta = \frac{12 \cos \theta \times 6 \tan \theta}{5 + 2 \sin \theta} \quad \text{M1}$$

$$\Rightarrow (5 + 2 \sin \theta)^2 = 72 \sin \theta \quad \text{dM1}$$

$$25 + 20 \sin \theta + 4 \sin^2 \theta = 72 \sin \theta \quad \text{A1}$$

$$\Rightarrow 4 \sin^2 \theta - 52 \sin \theta + 25 = 0 \quad *$$

(b)

M1: Attempts to solve $4 \sin^2 \theta - 52 \sin \theta + 25 = 0$. Must be clear they have found $\sin \theta$ and not e.g. just x from $4x^2 - 52x + 25 = 0$. Working does not need to be seen but see general guidance for solving a 3TQ if necessary. Note that the $\frac{25}{2}$ does not need to be seen.

A1: $\theta = \frac{5\pi}{6}$ and no other values unless they are rejected or the $\frac{5\pi}{6}$ clearly selected here and not in (c)

A minimum requirement in (b) is e.g. $\sin \theta = \frac{1}{2}$, $\theta = \frac{5\pi}{6}$

Do **not** allow 150° for $\frac{5\pi}{6}$

PTO for the notes to part (c)

(c) Allow full marks in (c) if e.g. $\theta = \frac{\pi}{6}$ is their answer to (b) but $\theta = \frac{5\pi}{6}$ is used here.

or if e.g. $\theta = \frac{5\pi}{6}$ is their answer to (b) but $\theta = \frac{\pi}{6}$ is used here allow the M marks only.

M1: For attempting a value (exact or decimal) for either a or r using **their** θ

$$\text{E.g. } a = 12 \cos \theta = \left(12 \times -\frac{\sqrt{3}}{2}\right) \text{ or } r = \frac{5 + 2 \sin \theta}{12 \cos \theta} = \left(\frac{5 + 2 \times \frac{1}{2}}{12 \times -\frac{\sqrt{3}}{2}}\right) \text{ oe e.g. } r = \frac{6 \tan \theta}{5 + 2 \sin \theta} = \left(\frac{6 \times -\frac{1}{\sqrt{3}}}{5 + 2 \times \frac{1}{2}}\right)$$

A1: Finds both $a = -6\sqrt{3}$ and $r = -\frac{1}{\sqrt{3}}$ which can be left unsimplified but $\sin \theta = \frac{1}{2}$, $\cos \theta = -\frac{\sqrt{3}}{2}$ and $\tan \theta = -\frac{\sqrt{3}}{3}$ (if required) must have been used.

dm1: Uses both **values** of " a " and " r " with the equation $S_{\infty} = \frac{a}{1-r} = \frac{-6\sqrt{3}}{1+\frac{1}{\sqrt{3}}}$ to create an expression

involving surds where a and r have come from appropriate work and $|r| < 1$

Depends on the first method mark.

ddM1: Rationalises denominator. The denominator must be of the form $p \pm q\sqrt{3}$ oe e.g. $p + \frac{q}{\sqrt{3}}$

Depends on both previous method marks.

$$\text{Note that stating e.g. } \frac{k}{p+q\sqrt{3}} \times \frac{p-q\sqrt{3}}{p-q\sqrt{3}} \text{ or } \frac{k}{p+\frac{q}{\sqrt{3}}} \times \frac{p-\frac{q}{\sqrt{3}}}{p-\frac{q}{\sqrt{3}}} \text{ is sufficient.}$$

A1: Obtains $(S_{\infty}) 9(1-\sqrt{3})$

Note that full marks are available in (c) for the use of $\theta = 150^\circ$

Note also that marks may be implied in (c) by e.g.

$$\begin{aligned} S_{\infty} &= \frac{a}{1-r} = \frac{12 \cos \theta}{1 - \frac{5+2 \sin \theta}{12 \cos \theta}} = \frac{144 \cos^2 \theta}{12 \cos \theta - 5 - 2 \sin \theta} = \frac{144 \cos^2 \frac{5\pi}{6}}{12 \cos \frac{5\pi}{6} - 5 - 2 \sin \frac{5\pi}{6}} \\ &= \frac{108}{-6-6\sqrt{3}} = \frac{108}{-6-6\sqrt{3}} \times \frac{-6+6\sqrt{3}}{-6+6\sqrt{3}} = \frac{-648+648\sqrt{3}}{-72} = 9(1-\sqrt{3}) \end{aligned}$$

Scores M1A1 implied dm1 ddM1 A1

See next page for some other cases in (c) and how to mark them:

$$S_{\infty} = \frac{a}{1-r} = \frac{12 \cos \frac{5\pi}{6}}{1 - \frac{5 + 2 \sin \frac{5\pi}{6}}{12 \cos \frac{5\pi}{6}}} \quad \text{or e.g.} \quad S_{\infty} = \frac{a}{1-r} = \frac{12 \cos \frac{\pi}{6}}{1 - \frac{5 + 2 \sin \frac{\pi}{6}}{12 \cos \frac{\pi}{6}}}$$

And nothing else

scores M1A0dM1ddM0A0

$$S_{\infty} = \frac{a}{1-r} = \frac{12 \cos \frac{5\pi}{6}}{1 - \frac{5 + 2 \sin \frac{5\pi}{6}}{12 \cos \frac{5\pi}{6}}} = 9(1 - \sqrt{3})$$

Scores M1A1dM1ddM0A0

$$S_{\infty} = \frac{a}{1-r} = \frac{12 \cos \frac{\pi}{6}}{1 - \frac{5 + 2 \sin \frac{\pi}{6}}{12 \cos \frac{\pi}{6}}} = 9(1 + \sqrt{3})$$

Scores M1A0dM1ddM0A0

$$S_{\infty} = 9(1 - \sqrt{3}) \text{ with no working scores no marks}$$



Question	Scheme	Marks	AOs
15	$(\sin x - \cos x)^2 < 1 \Rightarrow \sin^2 x - 2 \sin x \cos x + \cos^2 x < 1$ o.e.	M1	1.1b
	Examples: $1 - 2 \sin x \cos x < 1$, $1 - \sin 2x < 1$, $-2 \sin x \cos x < 0$, $-\sin 2x < 0$	A1	2.2a
	As x is obtuse then $-2 \sin x \cos x$ is positive because $\sin x > 0$ and $\cos x < 0$ so we have a contradiction. Therefore $\sin x - \cos x \geq 1$ *	A1*	2.4
(3 marks)			

Notes

Condone poor notation e.g. $\sin x^2$ or e.g. $-2 \sin \theta \cos x < 1$ for the first two marks only.

M1: Expands $(\sin x - \cos x)^2$ to obtain $\sin^2 x \pm k \sin x \cos x + \cos^2 x$ where $k = 1$ or 2 o.e. May be implied.

A1: Uses a correct identity $\sin^2 x + \cos^2 x = 1$ or e.g. $-\sin^2 x - \cos^2 x = -1$ to obtain a correct inequality in any form that does not include the $\sin^2 x$ and $\cos^2 x$ terms. Condone e.g. $-2 \sin \cos x < 0$

A1*: Fully correct work which includes

- a convincing argument that explains why their inequality is not true
- a statement that indicates there is a contradiction
- a conclusion that $\sin x - \cos x \geq 1$ (there is no need to repeat “when x is obtuse”)
- no contradictory statements
- no mixed/missed variables, e.g., $-2 \sin \theta \cos x < 1$ or $1 - \sin 2 < 1$

Examples:

From $-2 \sin x \cos x < 0$:

In the second quadrant $-2 \sin x \cos x$ is $- \times + \times = +$
“(this is a contradiction)” or equivalent (therefore) $\sin x - \cos x \geq 1$

or

As x is obtuse, $\sin x > 0$, $\cos x < 0$ so $-2 \sin x \cos x > 0$
“(this is a contradiction)” or equivalent (therefore) $\sin x - \cos x \geq 1$

From $-\sin 2x < 0$:

As x is obtuse, $2x$ is reflex o.e. (i.e. $\pi < 2x < 2\pi$) so $-\sin 2x > 0$
“(this is) wrong” or equivalent (therefore) $\sin x - \cos x \geq 1$

From $1 - \sin 2x < 1$:

As x is obtuse, $2x$ is reflex o.e. (i.e. $180 < 2x < 360$) so $\sin 2x < 0$ so $1 - \sin 2x > 1$
“(this is a contradiction)” or equivalent (therefore) $\sin x - \cos x \geq 1$

From $\sin 2x > 0$:

As x is obtuse, $2x$ is reflex o.e. (i.e. $180 < 2x < 360$) so $\sin 2x < 0$
“(this is) incorrect” or equivalent (therefore) $\sin x - \cos x \geq 1$

Note that you may condone the absence of a statement referring to the fact that $(\sin x - \cos x)^2 < 1$ is only valid since $\sin x - \cos x > 0$ when x is obtuse.



Question	Scheme	Marks	AOs
12	$\frac{\sin(x+h) - \sin x}{h}$	B1	2.1
	$\frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$	M1	1.1b
		A1	1.1b
	(As $h \rightarrow 0$), $\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \rightarrow 0 \times \sin x + 1 \times \cos x$	dM1	2.1
	so $\frac{dy}{dx} = \cos x$ *	A1*	2.5

(5 marks)

Notes

Throughout the question allow the use of $h = \delta x$ if used consistently

There is no requirement to see "gradient of chord" written down.

B1: Gives the correct fraction such as $\frac{\sin(x+h) - \sin x}{x+h-x}$ or $\frac{\sin x - \sin(x+h)}{-h}$ or $\frac{\sin(x+h) - \sin(x-h)}{2h}$ or $\frac{\sin(x-h) - \sin x}{x-h-x}$. Condone invisible brackets. May be implied by $\frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$

M1: Uses the compound angle formula for $\sin(x \pm h)$ to give $\sin x \cos h \pm \cos x \sin h$

A1: Achieves $\frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$ or equivalent (may be implied by further work).
Allow invisible brackets to be recovered.

dM1: **It is dependent on both the B and the M marks being awarded.**

Complete attempt to apply the given limits to the gradient of their chord. They must isolate

$\left(\frac{\cos h - 1}{h} \right)$ and replace with 0 and isolate $\left(\frac{\sin h}{h} \right)$ and replace with 1.

e.g. $\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) = \sin x \times 0 + \cos x \times 1$

Accept as a minimum $\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) = \cos x$ (implying the application of the limits)

If they do not fully show $\left(\frac{\cos h - 1}{h} \right)$ and $\left(\frac{\sin h}{h} \right)$ being isolated but proceed from

e.g. $\frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$ to $0 \times \sin x + \cos x$ (or e.g. $0 + \cos x$) then this can be implied and

score dM1

$\frac{\sin x(\cos h - 1) + \cos x \sin h}{h} = \cos x$ is dM0

Condone if limit notation remains within their expression after the limits have been applied.

e.g. $\lim_{h \rightarrow 0} (\sin x \times 0 + \cos x \times 1)$

Alternatively, condone use of the small angle approximations such that

$\frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \rightarrow \frac{-\frac{h^2}{2} \sin x + h \cos x}{h} = -\frac{h}{2} \sin x + \cos x$ and replaces $\frac{h}{2}$ with 0



A1*: Uses correct mathematical language of limiting arguments to show that $\frac{dy}{dx} = \cos x$ with no errors seen. (cso)

We need to see $h \rightarrow 0$ at some point in their solution and linking $\frac{dy}{dx}$ with $\cos x$ e.g.

- $\frac{dy}{dx} = \dots = \lim_{h \rightarrow 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right) = \cos x$
- $\frac{dy}{dx} = \dots = \lim_{h \rightarrow 0} \left(-\frac{h}{2} \sin x + \cos x \right) = 0 \times \sin x + \cos x = \cos x$ (using small angle approximations)
- $\frac{dy}{dx} = \dots = \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} = \sin x \times 0 + 1 \times \cos x = \cos x$ as $h \rightarrow 0$

Condone $f'(x)$ or y' in place of $\frac{dy}{dx}$

Give final A0 for no evidence of limiting arguments:

e.g. when $h = 0$ $\frac{dy}{dx} = \dots = \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) = \sin x \times 0 + \cos x \times 1 = \cos x$ is A0

Do not allow the final A1 for just stating $\frac{\sin h}{h} = 1$ and $\frac{\cos h - 1}{h} = 0$ and attempting to apply these (without seeing e.g. $h \rightarrow 0$ at some point in their solution)

If they work in another variable (e.g. θ) then withhold the final mark. If they have mixed variables within some of their statements, then allow recovery but withhold the final mark.

Withhold this mark if there has been incorrect bracketing or invisible brackets when isolating $\sin x (\cos h - 1)$ e.g. $\frac{\sin x \cos h - 1 + \cos x \sin h}{h}$ but accept terms written as e.g. $\sin x \frac{\cos h - 1}{h}$ which do not require brackets. Condone a missing trailing bracket if the intention is clear.

Question	Scheme	Marks	AOs
9	$\frac{d}{d\theta}(\cos \theta) = -\sin \theta$; as $h \rightarrow 0$, $\frac{\sin h}{h} \rightarrow 1$ and $\frac{\cos h - 1}{h} \rightarrow 0$		
	$\frac{\cos(\theta + h) - \cos \theta}{h}$	B1	2.1
	$= \frac{\cos \theta \cos h - \sin \theta \sin h - \cos \theta}{h}$	M1	1.1b
		A1	1.1b
	$= -\frac{\sin h}{h} \sin \theta + \left(\frac{\cos h - 1}{h}\right) \cos \theta$		
	As $h \rightarrow 0$, $-\frac{\sin h}{h} \sin \theta + \left(\frac{\cos h - 1}{h}\right) \cos \theta \rightarrow -1 \sin \theta + 0 \cos \theta$	dM1	2.1
	so $\frac{d}{d\theta}(\cos \theta) = -\sin \theta$ *	A1*	2.5
		(5)	
(5 marks)			
Notes for Question 9			
B1:	Gives the correct fraction such as $\frac{\cos(\theta + h) - \cos \theta}{h}$ or $\frac{\cos(\theta + \delta\theta) - \cos \theta}{\delta\theta}$ Allow $\frac{\cos(\theta + h) - \cos \theta}{(\theta + h) - \theta}$ o.e. Note: $\cos(\theta + h)$ or $\cos(\theta + \delta\theta)$ may be expanded		
M1:	Uses the compound angle formula for $\cos(\theta + h)$ to give $\cos \theta \cos h \pm \sin \theta \sin h$		
A1:	Achieves $\frac{\cos \theta \cos h - \sin \theta \sin h - \cos \theta}{h}$ or equivalent		
dM1:	dependent on both the B and M marks being awarded Complete attempt to apply the given limits to the gradient of their chord		
Note:	They must isolate $\frac{\sin h}{h}$ and $\left(\frac{\cos h - 1}{h}\right)$, and replace $\frac{\sin h}{h}$ with 1 and replace $\left(\frac{\cos h - 1}{h}\right)$ with 0		
A1*:	cso. Uses correct mathematical language of limiting arguments to prove $\frac{d}{d\theta}(\cos \theta) = -\sin \theta$		
Note:	Acceptable responses for the final A mark include: $\frac{d}{d\theta}(\cos \theta) = \lim_{h \rightarrow 0} \left(-\frac{\sin h}{h} \sin \theta + \left(\frac{\cos h - 1}{h}\right) \cos \theta \right) = -1 \sin \theta + 0 \cos \theta = -\sin \theta$ - Gradient of chord $= -\frac{\sin h}{h} \sin \theta + \left(\frac{\cos h - 1}{h}\right) \cos \theta$. As $h \rightarrow 0$, gradient of chord tends to the gradient of the curve, so derivative is $-\sin \theta$ - Gradient of chord $= -\frac{\sin h}{h} \sin \theta + \left(\frac{\cos h - 1}{h}\right) \cos \theta$. As $h \rightarrow 0$, gradient of curve is $-\sin \theta$		
Note:	Give final A0 for the following example which shows no limiting arguments : when $h = 0$, $\frac{d}{d\theta}(\cos \theta) = -\frac{\sin h}{h} \sin \theta + \left(\frac{\cos h - 1}{h}\right) \cos \theta = -1 \sin \theta + 0 \cos \theta = -\sin \theta$		
Note:	Do not allow the final A1 for stating $\frac{\sin h}{h} = 1$ or $\left(\frac{\cos h - 1}{h}\right) = 0$ and attempting to apply these		
Note:	In this question $\delta\theta$ may be used in place of h		
Note:	Condone $f'(\theta)$ where $f(\theta) = \cos \theta$ or $\frac{dy}{d\theta}$ where $y = \cos \theta$ used in place of $\frac{d}{d\theta}(\cos \theta)$		

Notes for Question 9 Continued	
Note:	Condone x used in place of θ if this is done consistently
Note:	Give final A0 for $\frac{d}{d\theta}(\cos x) = \lim_{h \rightarrow 0} \left(-\frac{\sin h}{h} \sin \theta + \left(\frac{\cos h - 1}{h} \right) \cos \theta \right) = -1 \sin \theta + 0 \cos \theta = -\sin \theta$ $\frac{d}{d\theta} = \dots$ - Defining $f(x) = \cos \theta$ and applying $f'(x) = \dots$ $\frac{d}{dx}(\cos \theta)$
Note:	Give final A1 for a correct limiting argument in x, followed by $\frac{d}{d\theta}(\cos \theta) = -\sin \theta$ e.g. $\frac{d}{d\theta}(\cos x) = \lim_{h \rightarrow 0} \left(-\frac{\sin h}{h} \sin x + \left(\frac{\cos h - 1}{h} \right) \cos x \right) = -1 \sin x + 0 \cos x = -\sin x$ $\Rightarrow \frac{d}{d\theta}(\cos \theta) = -\sin \theta$
Note:	Applying $h \rightarrow 0$, $\sin h \rightarrow h$, $\cos h \rightarrow 1$ to give e.g. $\lim_{h \rightarrow 0} \left(\frac{\cos \theta \cos h - \sin \theta \sin h - \cos \theta}{h} \right) = \left(\frac{\cos \theta(1) - \sin \theta(h) - \cos \theta}{h} \right) = \frac{-\sin \theta(h)}{h} = -\sin \theta$ is final M0 A0 for incorrect application of limits
Note:	$\lim_{h \rightarrow 0} \left(\frac{\cos \theta \cos h - \sin \theta \sin h - \cos \theta}{h} \right) = \lim_{h \rightarrow 0} \left(-\frac{\sin h}{h} \sin \theta + \left(\frac{\cos h - 1}{h} \right) \cos \theta \right)$ $= \lim_{h \rightarrow 0} (-1) \sin \theta + 0 \cos \theta = -\sin \theta. \text{ So for } \lim_{h \rightarrow 0} \text{ not removing } \lim_{h \rightarrow 0}$ when the limit was taken is final A0
Note:	Alternative Method: Considers $\frac{\cos(\theta + h) - \cos(\theta - h)}{(\theta + h) - (\theta - h)}$ which simplifies to $\frac{-2 \sin \theta \sin h}{2h}$

Question	Scheme	Marks	AOs
8 (a)	$D = 5 + 2\sin(30 \times 6.5)^\circ = \text{awrt } 4.48\text{m}$ with units	B1	3.4
		(1)	
(b)	$3.8 = 5 + 2\sin(30t)^\circ \Rightarrow \sin(30t)^\circ = -0.6$	M1	1.1b
		A1	1.1b
	$t = 10.77$	dM1	3.1a
	10:46 a.m. or 10:47 a.m.	A1	3.2a
		(4)	

(5 marks)

Notes:
(a)

B1: Scored for using the model ie. substituting $t = 6.5$ into $D = 5 + 2\sin(30t)^\circ$ and stating $D = \text{awrt } 4.48\text{m}$. The units must be seen somewhere in (a). So allow when $D = 4.482.. = 4.5\text{ m}$. Allow the mark for a correct answer without any working.

(b)

M1: For using $D = 3.8$ and proceeding to $\sin(30t)^\circ = k$, $|k| \leq 1$

A1: $\sin(30t)^\circ = -0.6$ This may be implied by any correct answer for t such as $t = 7.2$

If the A1 implied, the calculation must be performed in degrees.

dM1: For finding **the first value** of t for their $\sin(30t)^\circ = k$ after $t = 8.5$.

You may well see other values as well which is not an issue for this dM mark

(Note that $\sin(30t)^\circ = -0.6 \Rightarrow 30t = 216.9^\circ$ as well but this gives $t = 7.2$)

For the correct $\sin(30t)^\circ = -0.6 \Rightarrow 30t = 323.1 \Rightarrow t = \text{awrt } 10.8$

For the incorrect $\sin(30t)^\circ = +0.6 \Rightarrow 30t = 396.9 \Rightarrow t = \text{awrt } 13.2$

So award this mark if you see $30t = \text{inv sin their } -0.6$ to give the first value of t where $30t > 255$

A1: Allow 10:46 a.m. (12 hour clock notation) or 10:46 (24 hour clock notation) oe

Allow 10:47 a.m. (12 hour clock notation) or 10:47 (24 hour clock notation) oe

DO NOT allow 646 minutes or 10 hours 46 minutes.



Question	Scheme	Marks	AOs
9(a)	Deduces that $A = \pm 50$ or $b = \frac{1}{4}$	B1	3.4
	Deduces that $A = \pm 50$ and $b = \frac{1}{4}$	B1	3.4
	Uses $t = 0, H = 1 \Rightarrow \alpha = \dots$ E.g. $1 = 50 \sin(\alpha)^\circ \Rightarrow \alpha = \dots$	M1	3.4
	$H = \left \pm 50 \sin\left(\frac{1}{4}t + 1.15\right)^\circ \right $	A1	3.3
		(4)	
(b)	E.g. the minimum height above the ground of the passenger on the original model was 0 m or Adding “d” means the passenger does not touch the ground.	B1	3.5b
		(1)	
(5 marks)			
Notes:			

(a) Note that B0B1 is not possible

B1: Uses the equation of the given model to deduce that $A = \pm 50$ **or** $b = \frac{1}{4}$ o.e.

May be seen embedded within their equation.

B1: Uses the equation of the given model to deduce that $A = \pm 50$ **and** $b = \frac{1}{4}$ o.e.

May be seen embedded within their equation.

M1: Uses $t = 0$ and $H = 1$ in the equation of the model to find a value for α .

Follow through on their value for A . Allow for $\pm 1 = 50 \sin(\alpha)^\circ \Rightarrow \alpha = \dots$ where α is in degrees or radians.

Note that in radians $\sin^{-1}\left(\frac{1}{50}\right) \approx \frac{1}{50}$ (0.0200...) which may appear incorrect but is in fact ok.

Also in degrees a value of e.g. 1.14 (truncated) would indicate the method.

A1: Writes down the correct full equation of the model: $H = \left| \pm 50 \sin\left(\frac{1}{4}t + 1.15\right)^\circ \right|$ o.e.

Condone omission of degrees symbol and allow awrt 1.15 for α .

Allow if a correct equation is seen anywhere in their solution.

(b)

B1: Gives a suitable explanation with no contradictory statements.

Condone “so that pod/capsule/seat/passenger/ferris wheel/it etc. will not hit/touch the ground”

Responses that focus on the starting point of the model are likely to score B0

Question	Scheme	Marks	AOs
6 (a)	$R = \sqrt{5}$	B1	1.1b
	$\tan \alpha = 2 \Rightarrow \alpha = \dots$	M1	1.1b
	$\alpha = 1.107$	A1	1.1b
		(3)	
	$\theta = 5 + \sqrt{5} \sin\left(\frac{\pi t}{12} + 1.107 - 3\right)$		
(b)	$(5 + \sqrt{5})^\circ\text{C}$ or awrt 7.24°C	B1ft	2.2a
		(1)	
(c)	$\frac{\pi t}{12} + 1.107 - 3 = \frac{\pi}{2} \Rightarrow t =$	M1	3.1b
	$t =$ awrt 13.2	A1	1.1b
	Either 13:14 or 1:14 pm or 13 hours 14 minutes after midnight.	A1	3.2a
		(3)	
(7 marks)			
Notes:			

(a)

B1: $R = \sqrt{5}$ only.

M1: Proceeds to a value of α from $\tan \alpha = \pm 2$, $\tan \alpha = \pm \frac{1}{2}$, $\sin \alpha = \pm \frac{2}{\sqrt{5}}$ OR $\cos \alpha = \pm \frac{1}{\sqrt{5}}$

It is implied by either awrt 1.11 (radians) or 63.4 (degrees)

A1: $\alpha =$ awrt 1.107

(b)

B1ft: Deduces that the maximum temperature is $(5 + \sqrt{5})^\circ\text{C}$ or awrt 7.24°C Remember to isw
Condone a lack of units. Follow through on their value of R so allow $(5 + "R")^\circ\text{C}$

(c)

M1: An complete strategy to find t from $\frac{\pi t}{12} \pm 1.107 - 3 = \frac{\pi}{2}$.

Follow through on their 1.107 but the angle must be in radians.

It is possible via degrees but only using $15t \pm 63.4 - 171.9 = 90$

A1: awrt $t = 13.2$

A1: The question asks for the time of day so accept either 13:14, 1:14 pm, 13 hours 14 minutes after midnight, 13h 14, or 1 hour 14 minutes after midday. If in doubt use review

.....
It is possible to attempt parts (b) and (c) via differentiation but it is unlikely to yield correct results.

$$\frac{d\theta}{dt} = \frac{\pi}{12} \cos\left(\frac{\pi t}{12} - 3\right) - \frac{2\pi}{12} \sin\left(\frac{\pi t}{12} - 3\right) = 0 \Rightarrow \tan\left(\frac{\pi t}{12} - 3\right) = \frac{1}{2} \Rightarrow t = 13.23 = 13:14 \text{ scores M1 A1 A1}$$

$$\frac{d\theta}{dt} = \cos\left(\frac{\pi t}{12} - 3\right) - 2 \sin\left(\frac{\pi t}{12} - 3\right) = 0 \Rightarrow \tan\left(\frac{\pi t}{12} - 3\right) = \frac{1}{2} \Rightarrow t = 13.23 = 13:14 \text{ they can score M1 A0 A1 (SC)}$$

A value of $t = 1.23$ implies the minimum value has been found and therefore incorrect method M0.

.....

Question	Scheme	Marks	AOs														
13(a)	$a = 60$	B1	3.1b														
	$2 = "60"-b(-20)^2 \Rightarrow b = \dots$	M1	3.4														
	$H = 60 - 0.145(t - 20)^2$	A1	3.3														
		(3)															
(b)	Height = 2 m	B1	3.4														
		(1)															
(c)	$\alpha = 180$ or $\beta = 31$	M1	3.4														
	$H = 29 \cos(9t + 180)^\circ + 31$	A1	3.3														
		(2)															
(d)	e.g. “The model allows for more than one circuit”	B1	3.5a														
		(1)															
(7 marks)																	
Notes																	
(a)																	
B1:	$a = 60$ (may be seen in their final equation of the model or implied by 60 substituted for a in the model)																
M1:	Attempts to find b by substituting in $t = 0$, $H = 2$ and their a and proceeding to a value for b . May be seen as two simultaneous equations formed: $2 = a - b(-20)^2$ and $60 = a - b(20 - 20)^2$ proceeding to a value for b																
A1:	$H = 60 - 0.145(t - 20)^2$ or equivalent such as $H = -\frac{29}{200}t^2 + 5.8t + 2$ or $H = 60 - \frac{29}{200}(t - 20)^2$ isw once a correct equation for the model is seen. Must be in terms of H and t . If they just state $a = 60$, $b = 0.145$ then A0 A correct answer with no working seen scores full marks.																
(b)																	
B1:	2 cao (condone lack of units) This can be scored even if their model in (a) is incorrect (they may have used symmetry to determine this value)																
(c)																	
M1:	$(\alpha =) 180$ or $(\beta =) 31$ Condone $(\alpha =) \pi$																
A1:	$H = 29 \cos(9t + 180)^\circ + 31$ or equivalent e.g. $H = -29 \cos(9t) + 31$ isw once a correct equation for the model is seen. Must be in terms of H and t . If they just state $\alpha = 180$, $\beta = 31$ then A0. A correct equation with no working seen scores both marks. Does not require the degree symbol.																
(d)																	
B1:	Score for a reason which makes reference to any of - the alternative model allows repetition (allow phrases e.g. “multiple cycles”, “repeated circuits”, “cyclical”, “periodic”, “loops around”, “the original model can only go up and down once”) - the alternative model after 2 minutes the carriage will be back at the start (e.g. “at 2 mins, $H = 2$”) - the original/quadratic model after 40 seconds (or any time after this) will be negative (e.g. “the height will be negative which cannot happen”) - the original model after 2 minutes would not be back at the start Do not allow vague responses on their own e.g. “the original model is a parabola” If calculations are used then they must be correct using a correct model (allow rounded or truncated) Look for a valid reason and ignore reference to anything else as long as it does not contradict																
	t	0	5	10	15	20	25	30	35	40	45	50	55	60	80	100	120
	h	2	27	46	56	60	56	46	27	2	-31	-71	-118	-172	-462	-868	-1390

Question	Scheme	Marks	AOs
10(a)	$\frac{dH}{dt} = \frac{H \cos 0.25t}{40} \Rightarrow \int \frac{1}{H} dH = \int \frac{\cos 0.25t}{40} dt$	M1	3.1a
	$\ln H = \frac{1}{10} \sin 0.25t (+c)$	M1 A1	1.1b 1.1b
	Substitutes $t = 0, H = 5 \Rightarrow c = \ln(5)$	dM1	3.4
	$\ln\left(\frac{H}{5}\right) = \frac{1}{10} \sin 0.25t \Rightarrow H = 5e^{0.1 \sin 0.25t} *$	A1*	2.1
		(5)	
(b)	Max height = $5e^{0.1} = 5.53 \text{ m}$ (Condone lack of units)	B1	3.4
		(1)	
(c)	Sets $0.25t = \frac{5\pi}{2}$	M1	3.1b
	31.4	A1	1.1b
		(2)	
(8 marks)			

(a)

M1: Separates the variables to reach $\int \frac{1}{H} dH = \int \frac{\cos 0.25t}{40} dt$ or equivalent.

The integral signs need to be present on both sides and the dH AND dt need to be in the correct positions.

M1: Integrates both sides to reach $\ln H = A \sin 0.25t$ or equivalent with or without the $+c$

A1: $\ln H = \frac{1}{10} \sin 0.25t + c$ or equivalent with or without the $+c$. Allow two constants, one either side

If the 40 was on the lhs look for $40 \ln H = 4 \sin 0.25t + c$ or equivalent.

dM1: Substitutes $t = 0, H = 5 \Rightarrow c = \dots$ There needs to have been a single $+c$ to find.

It is dependent upon the previous M mark. You may allow even if you don't explicitly see " $t = 0, H = 5$ " as it may be implied from their previous line

If the candidate has attempted to change the subject and made an error/ slip then condone it for this M but not the final A. Eg. $40 \ln H = 4 \sin 0.25t + c \Rightarrow H^{40} = e^{4 \sin 0.25t} + e^c \Rightarrow 5^{40} = 1 + e^c \Rightarrow c = \dots$

Also many students will be attempting to get to the given answer so condone the method of finding $c = \dots$
These students will lose the A1* mark

A1*: Proceeds via $\ln H = \frac{1}{10} \sin 0.25t + \ln 5$ or equivalent to the given answer $H = 5e^{0.1 \sin 0.25t}$ with at least one correct intermediate line and no incorrect work.

DO NOT condone c's going to c's when they should be e^c or A

Accept as a minimum $\ln H = \frac{1}{10} \sin 0.25t + \ln 5 \Rightarrow H = e^{\frac{1}{10} \sin 0.25t + \ln 5}$ or $H = e^{\frac{1}{10} \sin 0.25t} \times e^{+\ln 5}$ before sight of the given answer

If the only error was to omit the integration signs on line 1, thus losing the first M1, allow the candidate to have access to this mark following a correct intermediate line (see above).

If they attempt to change the subject first then the constant of integration must have been adapted if the A1*

is to be awarded. $\ln H = \frac{1}{10} \sin 0.25t + c \Rightarrow H = e^{\frac{1}{10} \sin 0.25t + c} \Rightarrow H = Ae^{\frac{1}{10} \sin 0.25t}$

The dM1 and A1* under this method are awarded at virtually the same time.

Also, for the final two marks, you may see a proof from $\int_0^H \frac{40}{H} dH = \int_5^t \cos 0.25t dt$

.....
There is an alternative via the use of an integrating factor.

(b)

B1: States that the maximum height is 5.53 m Accept $5e^{0.1}$ Condone a lack of units here, but penalise if incorrect units are used.

(c)

M1: For identifying that it would reach the maximum height for the 2nd time when $0.25t = \frac{5\pi}{2}$ or 450

A1: Accept awrt 31.4 or 10π Allow if units are seen

Question	Scheme	Marks	AOs
12(a)	$K = 500$	B1	1.1b
	$\tan \alpha = \frac{480}{140} \Rightarrow \alpha = \dots$	M1	1.1b
	$\alpha = \text{awrt } 73.74^\circ \text{ or } 500 \cos(\theta + 73.74)^\circ$	A1	1.1b
		(3)	
(b)(i)	$R = 1000 + 500 \cos(30t + 73.74)^\circ$ or $R = 1000 + 140 \cos(30t)^\circ - 480 \sin(30t)^\circ$	B1ft	3.3
(b)(ii)	$\{R_{\min} =\} 500$	B1ft	3.4
		(2)	
(c)	$t = 3.5 \Rightarrow R = "1000" + "500" \cos(30(3.5) + "73.74")^\circ = \dots$	M1	3.4
	$R = \text{awrt } 500.1 \dots$ so the model is reliable	A1	3.5a
		(2)	
(d)	$\sin(30t + 70)^\circ = -1 \Rightarrow 30t + 70 = 270 \Rightarrow 30t = \dots \text{ (or } t = \dots)$	M1	3.4
	$30t = 200 \left(\text{or } t = \frac{20}{3} \right)$	A1	1.1b
	$R = "1000" + "500" \cos\left(30\left(\frac{20}{3}\right) + "73.74" \right)^\circ$ or $R = "1000" + 140 \cos("200")^\circ - 480 \sin("200")^\circ$	dM1	3.4
	$R = 1032 \text{ (or } 1033)$	A1	1.1b
		(4)	

(11 marks)**Notes**

Note: Candidates working in radians are able to score all the M and B marks in this question. Condone the absence of the degrees symbol throughout the whole question.

(a)

B1: Correct value for K . Condone $R = 500$

M1: Award for $\tan \alpha = \pm \frac{480}{140} \Rightarrow \alpha = \dots$, $\tan \alpha = \pm \frac{140}{480} \Rightarrow \alpha = \dots$, $\sin \alpha = \pm \frac{480}{"500"} \Rightarrow \alpha = \dots$ or
 $\cos \alpha = \pm \frac{140}{"500"} \Rightarrow \alpha = \dots$

Note $\alpha = \text{awrt } 1.3 \text{ (rad)}$ implies this mark.**A1:** $\alpha = \text{awrt } 73.74\{\circ\}$ or correct expression $500 \cos(\theta + 73.74)\{\circ\}$

(b)(i) Note: mark parts (b)(i) and (b)(ii) together.

B1ft: Correct equation of the model in either form including the $R =$ following through on their numerical K ($0 < K \leq 750$) and their numerical α .Allow for e.g. $R = 1500 - "500" + "500" \cos(30t + "73.74")\{\circ\}$ or for $R = 1500 - "500" + 140 \cos(30t)^\circ - 480 \sin(30t)^\circ$ but not e.g. $R = 1500 - K + K \cos(30t + \alpha)^\circ$ $R = 1000 + 140 \cos 30t - 480 \sin 30t$ (without the brackets) is correct.Allow this mark if they have truncated or rounded an otherwise correct α (to 3s.f.)

(b)(ii)

B1ft: 500 or follow through on (their A – their K) or $(1500 - 2 \times \text{their } K)$ provided it is non-negative and less than 1500. It must be clear this is their answer to (b)(ii) so expect to see e.g. (b) or $R_{\min} =$ or an indication it is the minimum.

(c) Note: if θ is used in place of $30t$ then they must revert back to $30t$ correctly to access the marks.

M1: Substitutes $t = 3.5$ into their model for the number of rabbits (you may need to check if no method is shown)

or substitutes $t = 3.5$ into their $\cos(30t + \alpha)^\circ$

Condone substitution of a value of t in the range $3 \leq t \leq 4.5$ for this mark.

A1: $R = \text{awrt } 500.1 \dots$ **or** 500 (not awrt) following substitution of $t = 3.5$, suggesting that the model is valid/reliable/appropriate/good.

or $\cos(30(3.5) + 73.74)^\circ \approx -1$ suggesting that the model is valid/reliable/appropriate/good.

Allow this mark if they have truncated or rounded an otherwise correct α (to 3s.f.)

Alt:

M1: Minimum occurs when $A + K \cos(30t + \alpha)^\circ = R_{\min} \Rightarrow \cos(30t + \alpha)^\circ = \lambda$ with $|\lambda| \leq 1$ leading to $t = \dots$

May just see $\cos(30t + 73.74)^\circ = -1 \Rightarrow t = \dots$ (or their $\cos(30t + \alpha)^\circ = -1 \Rightarrow t = \dots$)

$30t + \alpha = 180 \Rightarrow t = \dots$ implies this mark. Condone $30t + \alpha = \pi \Rightarrow t = \dots$ for this mark.

A1: $t = 3.54 \dots$ (i.e. the middle of April) so the model is valid/reliable/appropriate/good.

Do not condone incorrect statements, e.g., $t = 3.54 \dots$ i.e. the middle of March so close to middle of April. If using $A + K \cos(30t + \alpha)^\circ = R_{\min}$ then $R_{\min}$ must be = their A – their K

Allow this mark if they have truncated or rounded an otherwise correct α (to 3s.f.)

Alt 2 using differentiation (Condoned)

M1: Condone finding the minimum using $\dots \sin(30t + 73.74)^\circ = 0 \Rightarrow t = \dots$ (or their $\sin(30t + \alpha)^\circ = 0 \Rightarrow t = \dots$)

$30t + \alpha = 180 \Rightarrow t = \dots$ implies this mark. Condone $30t + \alpha = \pi \Rightarrow t = \dots$ for this mark.

A1: $t = 3.54 \dots$ (i.e. the middle of April) suggesting that the model is valid/reliable/appropriate.

Do not condone incorrect statements, e.g., $t = 3.54 \dots$ i.e. the middle of March so close to middle of April.

The complete derivative for $\frac{dR}{dt}$ does not need to be seen.

Allow this mark if they have truncated or rounded an otherwise correct α (to 3s.f.)

(d) Note: if θ is used in place of $30t$ then they must revert back to $30t$ correctly to access the marks.

M1: Realises that $\sin(30t + 70)^\circ = -1$, reaches $30t + 70 = 270$ or -90 and attempts to find t (or $30t$)

Condone attempts using differentiation. The minimum occurs when $\cos(30t + 70)^\circ = 0 \Rightarrow$

$30t + 70 = 270 \Rightarrow 30t = \dots$ (or $t = \dots$). They must use 270 or -90 and **not** 90 to achieve the

minimum. Condone $30t + \alpha = \frac{3\pi}{2} \Rightarrow t = \dots$ for this mark but not $30t + \alpha = \frac{\pi}{2} \Rightarrow t = \dots$

A1: Correct value for $30t$ (or t) Accept rounded or truncated values to at least 3s.f. e.g. 6.66 or 6.67

dm1: Substitutes their value of $t > 0$ (or $30t > 0$) coming from $30t + 70 = 270$ into their model for R

A1: Correct number of rabbits. Allow 1032 or 1033 but must be whole numbers and not just 1030.

Allow this mark if they have truncated or rounded an otherwise correct α (to 3s.f.)



Question	Scheme	Marks	AOs
15(a)	$R = \sqrt{5}$	B1	1.1b
	$\tan \alpha = \frac{1}{2}$ or $\sin \alpha = \frac{1}{\sqrt{5}}$ or $\cos \alpha = \frac{2}{\sqrt{5}} \Rightarrow \alpha = \dots$	M1	1.1b
	$\alpha = 0.464$	A1	1.1b
		(3)	
(b)(i)	$3 + 2\sqrt{5}$	B1ft	3.4
(ii)	$\cos(0.5t + 0.464) = 1 \Rightarrow 0.5t + 0.464 = 2\pi$ $\Rightarrow t = \dots$	M1	3.4
	$t = 11.6$	A1	1.1b
		(3)	
(c)	$3 + 2\sqrt{5} \cos(0.5t + 0.464) = 0$ $\cos(0.5t + 0.464) = -\frac{3}{2\sqrt{5}}$	M1	3.4
	$\cos(0.5t + 0.464) = -\frac{3}{2\sqrt{5}} \Rightarrow 0.5t + 0.464 = \cos^{-1}\left(-\frac{3}{2\sqrt{5}}\right)$ $\Rightarrow t = 2\left(\cos^{-1}\left(-\frac{3}{2\sqrt{5}}\right) - 0.464\right)$	dM1	1.1b
	So the time required is e.g.: $2(3.977\dots - 0.464) - 2(2.306\dots - 0.464)$	dM1	3.1b
	$= 3.34$	A1	1.1b
		(4)	
(d)	e.g. the "3" would need to vary	B1	3.5c
		(1)	
(11 marks)			
Notes			
(a) B1: $R = \sqrt{5}$ only. M1: Proceeds to a value for α from $\tan \alpha = \pm \frac{1}{2}$ or $\sin \alpha = \pm \frac{1}{\sqrt{5}}$ or $\cos \alpha = \pm \frac{2}{\sqrt{5}}$ It is implied by either awrt 0.464 (radians) or awrt 26.6 (degrees) A1: $\alpha =$ awrt 0.464 (b)(i) B1ft: For $(3 + 2\sqrt{5})$ m or awrt 7.47 m and remember to isw. Condone lack of units. Follow through on their R value so allow $3 + 2 \times$ Their R . (Allow in decimals with at least 3sf accuracy) (b)(ii) M1: Uses $0.5t \pm "0.464" = 2\pi$ to obtain a value for t Follow through on their 0.464 but this angle must be in radians. It is possible in degrees but only using $0.5t \pm "26.6" = 360$ A1: Awrt 11.6			

Alternative for (b):

$$H = 3 + 4 \cos(0.5t) - 2 \sin(0.5t) \Rightarrow \frac{dH}{dt} = -2 \sin(0.5t) - \cos(0.5t) = 0$$

$$\Rightarrow \tan(0.5t) = -\frac{1}{2} \Rightarrow 0.5t = 2.677..., 5.819... \Rightarrow t = 5.36, 11.6$$

$$t = 11.6 \Rightarrow H = 7.47$$

Score as follows:

M1: For a complete method:

Attempts $\frac{dH}{dt}$ and attempts to solve $\frac{dH}{dt} = 0$ for t A1: For $t = \text{awrt } 11.6$ B1ft: For awrt 7.47 or $3 + 2 \times \text{Their } R$

(c)

M1: Uses the model and sets $3 + 2\sqrt{5} \cos(\dots) = 0$ and proceeds to $\cos(\dots) = k$ where $|k| < 1$.Allow e.g. $3 + 2\sqrt{5} \cos(\dots) < 0$ dM1: Solves $\cos(0.5t \pm 0.464) = k$ where $|k| < 1$ to obtain at least one value for t This requires e.g. $2\left(\pi + \cos^{-1}(k) \pm \tan^{-1}\left(\frac{1}{2}\right)\right)$ or e.g. $2\left(\pi - \cos^{-1}(k) \pm \tan^{-1}\left(\frac{1}{2}\right)\right)$ **Depends on the previous method mark.**dM1: A fully correct strategy to find the required duration. E.g. finds 2 consecutive values of t when $H = 0$ and subtracts. Alternatively finds t when H is minimum and uses the times found correctly to find the required duration.**Depends on the previous method mark.****Examples:**

Second time at water level – first time at water level:

$$2\left(\pi + \cos^{-1}\left(\frac{3}{2\sqrt{5}}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right) - 2\left(\pi - \cos^{-1}\left(\frac{3}{2\sqrt{5}}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right) = 7.02685... - 3.68492...$$

 $2 \times (\text{first time at minimum point} - \text{first time at water level}):$

$$2\left(2\left(\pi - \tan^{-1}\left(\frac{1}{2}\right)\right) - 2\left(\pi - \cos^{-1}\left(\frac{3}{2\sqrt{5}}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right)\right) = 2(5.35589... - 3.68492...)$$

Note that both of these examples equate to $4\cos^{-1}\left(\frac{3}{2\sqrt{5}}\right)$ which is not immediately obvious**but may be seen as an overall method.****There may be other methods – if you are not sure if they deserve credit send to review.**

A1: Correct value. Must be 3.34 (not awrt).

Special Cases in (c):

Note that if candidates have an incorrect α and have e.g. $3 + 2\sqrt{5} \cos(0.5t - 0.464)$, this has no impact on the final answer. So for candidates using $3 + 2\sqrt{5} \cos(0.5t \pm \alpha)$ in (c) allow all the marks including the A mark as a correct method should always lead to 3.34

Some values to look for:

$$0.5t \pm 0.464 = \pm 2.306, \pm 3.977, \pm 8.598, \pm 10.26$$

(d)

B1: Correct refinement e.g. As in scheme. If they suggest a specific function to replace the “3” then it must be sensible e.g. a trigonometric function rather than e.g. a quadratic/linear one.

