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EDEXCEL A LEVEL MATHEMATICS: QUESTIONS BY TOPIC

Exponentials and Logarithms (6)

A Level Mathematics Topic 6

37 questions · 281 marks · Mark schemes included

Pearson Edexcel A Level Mathematics (9MA0), June 2018 - June 2025

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HOW TO USE THIS BOOKLET

Questions are grouped by specification sub-topic and run from the lowest mark tariff to the highest within each one.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme.

Diagrams, graphs, tables and data are printed exactly as they appeared in the original papers.

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6.2 GRADIENT OF E^{kx}		2 Qs · 20 marks	4
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6.4 LAWS OF LOGARITHMS		3 Qs · 19 marks	61
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6.5 SOLVING EQUATIONS OF THE FORM $A^x = B$		2 Qs · 5 marks	70
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QUESTION	SOURCE	MARKS	PAGE
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31. Q8	Paper 1: Pure Mathematics 1 November2021	9	109
32. Q12	Paper 1: Pure Mathematics 1 June2018	10	113
33. Q14	Paper 2: Pure Mathematics 2 June2018	10	117
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35. Q15 <i>also Topic 5</i>	Paper 2: Pure Mathematics 2 November2021	11	124
36. Q11	Paper 1: Pure Mathematics 1 June2025	8	129
37. Q9	Paper 2: Pure Mathematics 2 June2025	8	133



QUESTIONS

37 questions · 281 marks · 6 sub-topics



SUB-TOPIC 6.2

GRADIENT OF $E^{\wedge}KX$

Sub-topic 6.2 · 2 questions · 20 marks · lowest tariff first



6.

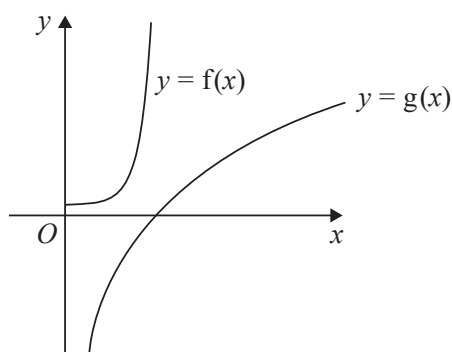


Figure 1

Figure 1 shows a sketch of the curves with equations $y = f(x)$ and $y = g(x)$ where

$$f(x) = e^{4x^2 - 1} \quad x > 0$$

$$g(x) = 8 \ln x \quad x > 0$$

(a) Find

(i) $f'(x)$

(ii) $g'(x)$

(2)

Given that $f'(x) = g'(x)$ at $x = \alpha$

(b) show that α satisfies the equation

$$4x^2 + 2 \ln x - 1 = 0$$

(2)

The iterative formula

$$x_{n+1} = \sqrt{\frac{1 - 2 \ln x_n}{4}}$$

is used with $x_1 = 0.6$ to find an approximate value for α

(c) Calculate, giving each answer to 4 decimal places,

(i) the value of x_2

(ii) the value of α

(3)







(Total for Question 6 is 7 marks)



15. A curve has equation $y = f(x)$, where

$$f(x) = \frac{7xe^x}{\sqrt{e^{3x} - 2}} \quad x > \ln \sqrt[3]{2}$$

(a) Show that

$$f'(x) = \frac{7e^x(e^{3x}(2-x) + Ax + B)}{2(e^{3x} - 2)^{\frac{3}{2}}}$$

where A and B are constants to be found.

(5)

(b) Hence show that the x coordinates of the turning points of the curve are solutions of the equation

$$x = \frac{2e^{3x} - 4}{e^{3x} + 4}$$

(2)

The equation $x = \frac{2e^{3x} - 4}{e^{3x} + 4}$ has two positive roots α and β where $\beta > \alpha$

A student uses the iteration formula

$$x_{n+1} = \frac{2e^{3x_n} - 4}{e^{3x_n} + 4}$$

in an attempt to find approximations for α and β

Diagram 1 shows a plot of part of the curve with equation $y = \frac{2e^{3x} - 4}{e^{3x} + 4}$ and part of the line with equation $y = x$

Using Diagram 1 on page 42

(c) draw a staircase diagram to show that the iteration formula starting with $x_1 = 1$ can be used to find an approximation for β

(1)

Use the iteration formula with $x_1 = 1$, to find, to 3 decimal places,

(d) (i) the value of x_2

(ii) the value of β

(3)

Using a suitable interval and a suitable function that should be stated

(e) show that $\alpha = 0.432$ to 3 decimal places.

(2)





Question 15 continued

Only use the copy of Diagram 1 if you need to redraw your answer to part (c).

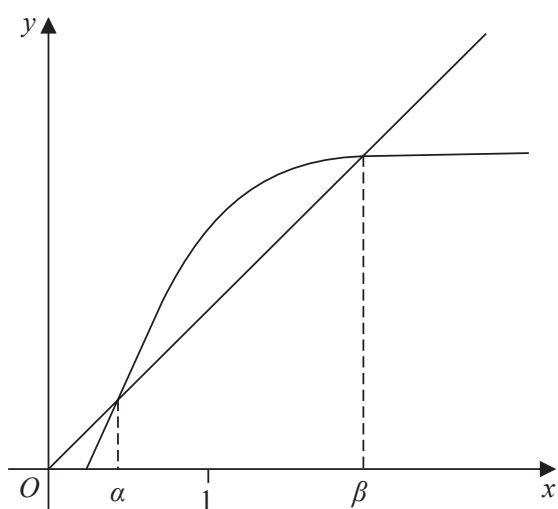
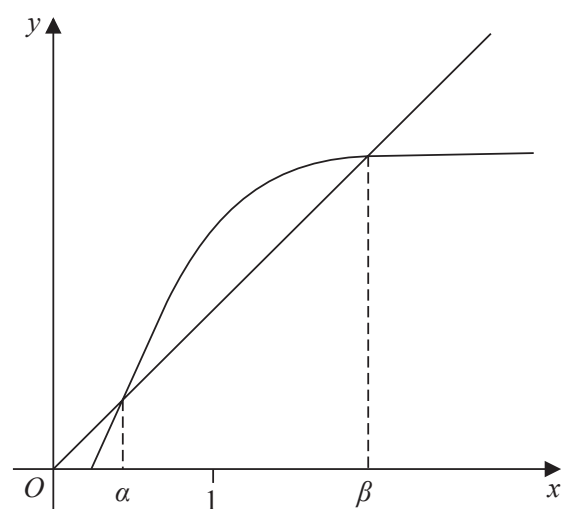


Diagram 1



copy of Diagram 1

Question 15 continued

SUB-TOPIC 6.3

LOGARITHMS AND LN X

Sub-topic 6.3 · 13 questions · 103 marks · lowest tariff first



(1)

$$\lim_{\delta x \rightarrow 0} \sum_{x=2.1}^{6.3} \frac{2}{x} \delta x = \ln k$$

(2)



(Total for Question 4 is 9 marks)



Question 4 continued

(Total for Question 4 is 4 marks)



(Total for Question 12 is 5 marks)













(Total for Question 12 is / marks)



11.

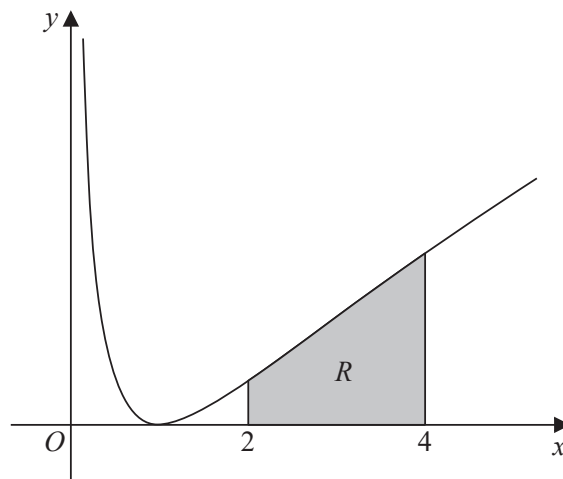


Figure 2

Figure 2 shows a sketch of part of the curve with equation

$$y = (\ln x)^2 \quad x > 0$$

The finite region R , shown shaded in Figure 2, is bounded by the curve, the line with equation $x = 2$, the x -axis and the line with equation $x = 4$

The table below shows corresponding values of x and y , with the values of y given to 4 decimal places.

x	2	2.5	3	3.5	4
y	0.4805	0.8396	1.2069	1.5694	1.9218

- (a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of R , giving your answer to 3 significant figures.

(3)

- (b) Use algebraic integration to find the exact area of R , giving your answer in the form

$$y = a(\ln 2)^2 + b \ln 2 + c$$

where a , b and c are integers to be found.

(5)





(Total for Question 11 is 6 marks)



Question 4 continued

Lined area for writing the answer to Question 4 continued.



(Total for Question 4 is 9 marks)







(Total for Question 10 is 10 marks)



7.

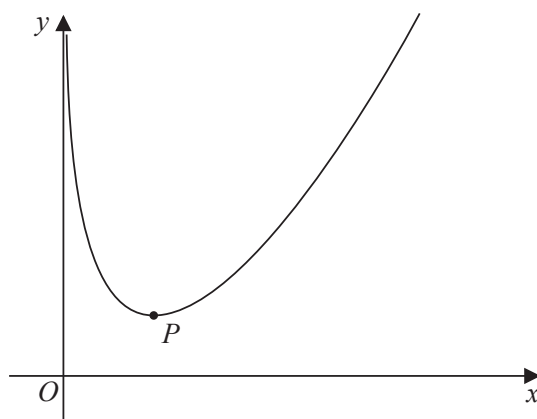


Figure 1

Figure 1 shows a sketch of the curve C with equation

$$y = \frac{4x^2 + x}{2\sqrt{x}} - 4\ln x \quad x > 0$$

(a) Show that

$$\frac{dy}{dx} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}} \quad (4)$$

The point P , shown in Figure 1, is the minimum turning point on C .

(b) Show that the x coordinate of P is a solution of

$$x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12} \right)^{\frac{2}{3}} \quad (3)$$

(c) Use the iteration formula

$$x_{n+1} = \left(\frac{4}{3} - \frac{\sqrt{x_n}}{12} \right)^{\frac{2}{3}} \quad \text{with } x_1 = 2$$

to find (i) the value of x_2 to 5 decimal places,

(ii) the x coordinate of P to 5 decimal places.

(3)



Question 7 continued

(Total for Question 7 is 10 marks)





Question 13 continued

(Total for Question 13 is 10 marks)







(Total for Question 13 is 11 marks)







(Total for Question 13 is 13 marks)

SUB-TOPIC 6.4

LAWS OF LOGARITHMS

Sub-topic 6.4 · 3 questions · 19 marks · lowest tariff first



3. Using the laws of logarithms, solve the equation

$$\log_3 (12y + 5) - \log_3 (1 - 3y) = 2$$

(3)



Question 3 continued

(Total for Question 3 is 3 marks)



9. Given that $a > b > 0$ and that a and b satisfy the equation

$$\log a - \log b = \log(a - b)$$

(a) show that

$$a = \frac{b^2}{b-1} \quad (3)$$

(b) Write down the full restriction on the value of b , explaining the reason for this restriction. (2)

Question 9 continued

(Total for Question 9 is 5 marks)





Question 11 continued

Lined area for writing the answer to Question 11.



SUB-TOPIC 6.5

SOLVING EQUATIONS OF THE FORM $A^X = B$

Sub-topic 6.5 · 2 questions · 5 marks · lowest tariff first



2. By taking logarithms of both sides, solve the equation

$$4^{3p-1} = 5^{210}$$

giving the value of p to one decimal place.

(3)

Question 2 continued

(Total for Question 2 is 3 marks)



3. Given that

$$3^x = 7^y$$

find the exact value of $\frac{x}{y}$

(2)

Question 3 continued

(Total for Question 3 is 2 marks)



SUB-TOPIC 6.6

LOGARITHMIC GRAPHS AND PARAMETER ESTIMATION

Sub-topic 6.6 · 1 question · 9 marks · lowest tariff first





Question 13 continued



Question 13 continued

(Total for Question 13 is 9 marks)



SUB-TOPIC 6.7

EXPONENTIAL MODELS

Sub-topic 6.7 · 16 questions · 125 marks · lowest tariff first



(Total for Question 4 is 4 marks)



9.

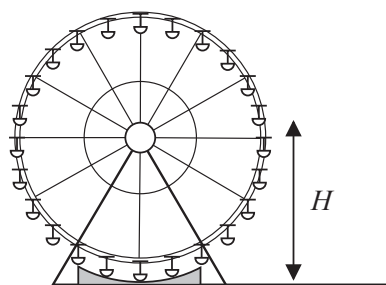


Figure 4

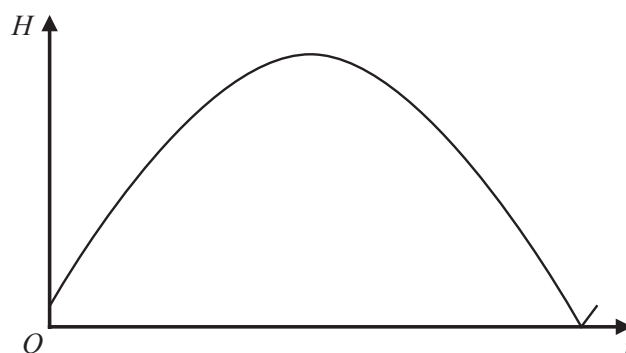


Figure 5

Figure 4 shows a sketch of a Ferris wheel.

The height above the ground, H m, of a passenger on the Ferris wheel, t seconds after the wheel starts turning, is modelled by the equation

$$H = |A \sin(bt + \alpha)^\circ|$$

where A , b and α are constants.

Figure 5 shows a sketch of the graph of H against t , for one revolution of the wheel.

Given that

- the maximum height of the passenger above the ground is 50 m
- the passenger is 1 m above the ground when the wheel starts turning
- the wheel takes 720 seconds to complete one revolution

(a) find a complete equation for the model, giving the exact value of A , the exact value of b and the value of α to 3 significant figures.

(4)

(b) Explain why an equation of the form

$$H = |A \sin(bt + \alpha)^\circ| + d$$

where d is a positive constant, would be a more appropriate model.

(1)

(Total for Question 3 is 5 marks)



(Total for Question 5 is 6 marks)



Question 6 continued

Lined area for writing the answer to Question 6 continued.



(Total for Question 3 is 7 marks)



Question 7 continued

Lined area for writing the answer to Question 7 continued.



Question 7 continued

Lined area for writing the answer to Question 7.

(Total for Question 7 is 7 marks)







(Total for Question 13 is 7 marks)



8.

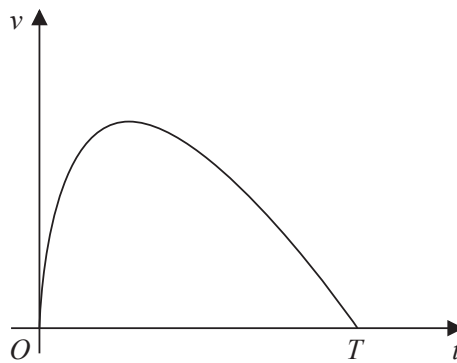


Figure 2

A car stops at two sets of traffic lights.

Figure 2 shows a graph of the speed of the car, $v \text{ ms}^{-1}$, as it travels between the two sets of traffic lights.

The car takes T seconds to travel between the two sets of traffic lights.

The speed of the car is modelled by the equation

$$v = (10 - 0.4t) \ln(t + 1) \quad 0 \leq t \leq T$$

where t seconds is the time after the car leaves the first set of traffic lights.

According to the model,

(a) find the value of T (1)

(b) show that the maximum speed of the car occurs when

$$t = \frac{26}{1 + \ln(t + 1)} - 1 \quad (4)$$

Using the iteration formula

$$t_{n+1} = \frac{26}{1 + \ln(t_n + 1)} - 1$$

with $t_1 = 7$

(c) (i) find the value of t_3 to 3 decimal places,
 (ii) find, by repeated iteration, the time taken for the car to reach maximum speed. (3)





(Total for Question 6 is 6 marks)







(Total for Question 10 is 8 marks)

8. A scientist is studying the growth of two different populations of bacteria.

The number of bacteria, N , in the **first** population is modelled by the equation

$$N = Ae^{kt} \quad t \geq 0$$

where A and k are positive constants and t is the time in hours from the start of the study.

Given that

- there were 1000 bacteria in this population at the start of the study
- it took exactly 5 hours from the start of the study for this population to double

(a) find a complete equation for the model.

(4)

(b) Hence find the rate of increase in the number of bacteria in this population exactly 8 hours from the start of the study. Give your answer to 2 significant figures.

(2)

The number of bacteria, M , in the **second** population is modelled by the equation

$$M = 500e^{1.4kt} \quad t \geq 0$$

where k has the value found in part (a) and t is the time in hours from the start of the study.

Given that T hours after the start of the study, the number of bacteria in the two different populations was the same,

(c) find the value of T .

(3)





(Total for Question 6 is 7 marks)





Question 12 continued

(Total for Question 12 is 10 marks)





Question 12 continued



(Total for Question 12 is 11 marks)



15. (a) Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.
Give the exact value of R and the value of α in radians to 3 decimal places.

(3)

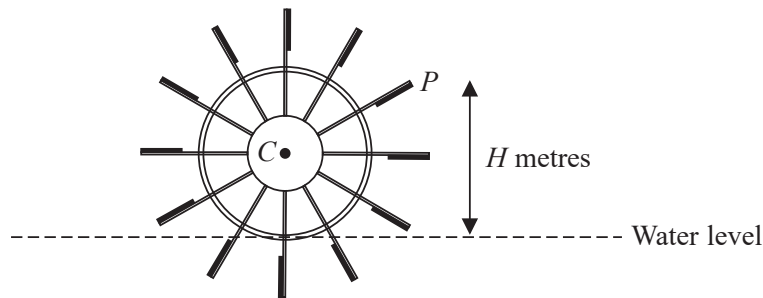


Figure 6

Figure 6 shows the cross-section of a water wheel.

The wheel is free to rotate about a fixed axis through the point C .

The point P is at the end of one of the paddles of the wheel, as shown in Figure 6.

The water level is assumed to be horizontal and of constant height.

The vertical height, H metres, of P above the water level is modelled by the equation

$$H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$$

where t is the time in seconds after the wheel starts rotating.

Using the model, find

- (b) (i) the maximum height of P above the water level,
(ii) the value of t when this maximum height first occurs, giving your answer to one decimal place.

(3)

In a single revolution of the wheel, P is below the water level for a total of T seconds.

According to the model,

- (c) find the value of T giving your answer to 3 significant figures.

(Solutions based entirely on calculator technology are not acceptable.)

(4)

In reality, the water level may not be of constant height.

- (d) Explain how the equation of the model should be refined to take this into account.

(1)









11.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The value of a car, £ V , is modelled by the equation

$$V = 1500 + Ae^{-kt}$$

MARK SCHEMES

official mark schemes



Question	Scheme	Marks	AOs
6(a)(i)	$(f'(x) =) 8xe^{4x^2-1}$ or e.g. $\frac{8xe^{4x^2}}{e}$ oe	B1	1.1b
(ii)	$(g'(x) =) \frac{8}{x}$ or e.g. $8x^{-1}$ oe	B1	1.2
		(2)	
(b)	$8xe^{4x^2-1} = \frac{8}{x} \Rightarrow e^{4x^2-1} = \frac{1}{x^2} \Rightarrow 4x^2 - 1 = \ln \frac{1}{x^2}$	M1	1.1b
	$4x^2 - 1 = \ln \frac{1}{x^2} \Rightarrow 4x^2 - 1 = -2 \ln x$ $\Rightarrow 4x^2 + 2 \ln x - 1 = 0^*$	A1*	2.1
		(2)	
(c)(i)	$x_1 = 0.6 \Rightarrow x_2 = \sqrt{\frac{1-2 \ln 0.6}{4}}$	M1	1.1b
	$(x_2 =) 0.7109$	A1	1.1b
(ii)	$(\alpha =) 0.6706$	B1 (A1 In ePEN)	1.1b
		(3)	
(7 marks)			
(a)(i) B1: Correct derivative in any form. "$f'(x) =$" is not required. Apply isw if necessary. (ii) B1: Correct derivative in any form. "$g'(x) =$" is not required. Apply isw if necessary. (b) M1: Eliminates e by setting their $f'(x) =$ their $g'(x)$ where $f'(x) = Axe^{4x^2-1}$ oe and $g'(x) = \frac{B}{x}$ oe with $A \times B > 0$ and proceeds via $e^{4x^2-1} = \frac{\dots}{x^2}$ or equivalent work (see below) to obtain $4x^2 - 1 = \ln \frac{\dots}{x^2}$ oe e.g. $\ln x + 4x^2 - 1 = \ln \frac{1}{x}$ Allow if they use α for x. Note that there are various alternatives for this mark but the derivatives must be of the form defined above and the processing must be correct with coefficient/sign slips only. <u>Examples of equivalent work:</u> $8xe^{4x^2-1} = \frac{8}{x} \Rightarrow x^2 e^{4x^2-1} = 1 \Rightarrow \ln x^2 + \ln e^{4x^2-1} = 0 \Rightarrow \ln e^{4x^2-1} = -\ln x^2 \Rightarrow 4x^2 - 1 = -2 \ln x$ $\frac{8xe^{4x^2}}{e} = \frac{8}{x} \Rightarrow \frac{1}{e} e^{4x^2} = \frac{1}{x^2} \Rightarrow e^{4x^2} = \frac{e}{x^2} \Rightarrow \ln e^{4x^2} = \ln \frac{e}{x^2} \Rightarrow 4x^2 = \ln \frac{e}{x^2} = 1 - 2 \ln x$ A1*: Obtains the printed answer with sufficient working and no errors. Sufficient work would require the "e" eliminated before the given answer. Must follow correct derivatives in part (a). Condone $4x^2 + 2 \ln x - 1 = 0$ and condone $4\alpha^2 + 2 \ln \alpha - 1 = 0$ or $4\alpha^2 + 2 \ln \alpha - 1 = 0$			

Note that if both derivatives in (a) **are correct** we will allow fully correct work using the equation in (b) to work backwards to verify that $pf'(x) = qg'(x)$ for M1 then obtains

$f'(x) = g'(x)$ with a minimal conclusion for A1

If either derivative in (a) is incorrect or missing, candidates who work backwards score no marks in (b).

(c)(i)/(ii)

M1: Attempts to use the iterative formula with $x_1 = 0.6$

Award this mark for e.g. $(x_2 =) \sqrt{\frac{1 - 2 \ln 0.6}{4}}$ or may be implied by awrt 0.71 provided no incorrect working is seen.

Candidates sometimes find x_3 (or possibly subsequent terms) rather than x_2 in which case the M1 can be implied. (See table below for first few iterations)

A1: ($x_2 =$) awrt 0.7109

Sight of ($x_2 =$) awrt 0.7109 scores M1A1

B1(A1 on ePEN): ($\alpha =$) 0.6706 (4dp)

Must be this value and **not** awrt 0.6706

For reference:

x_1	0.6
x_2	0.7109239143
x_3	0.6485329086
x_4	0.6830236199
x_5	0.6637868021
x_6	0.6744606223
.	.
.	.
.	.
α	0.6706416243

Question	Scheme	Marks	AOs
15(a)	$\dots xe^x + \dots e^x$	M1	1.1b
	$k(xe^x + e^x)$	A1	1.1b
	$\frac{d}{dx}(\sqrt{e^{3x}-2}) = \frac{1}{2} \times 3e^{3x}(e^{3x}-2)^{-\frac{1}{2}}$	B1	1.1b
	$(f'(x) =) \frac{(e^{3x}-2)^{\frac{1}{2}}(7xe^x + 7e^x) - \frac{3}{2}e^{3x}(e^{3x}-2)^{-\frac{1}{2}} \times 7xe^x}{e^{3x}-2}$	dM1	2.1
	$f'(x) = \frac{7e^x(e^{3x}(2-x) - 4x - 4)}{2(e^{3x}-2)^{\frac{3}{2}}}$	A1	1.1b
		(5)	
(b)	$e^{3x}(2-x) - 4x - 4 = 0 \Rightarrow x(\dots e^{3x} \pm \dots) = \dots e^{3x} \pm \dots$	M1	1.1b
	$\Rightarrow x = \frac{2e^{3x}-4}{e^{3x}+4} \quad *$	A1*	2.1
		(2)	
(c)	Draws a vertical line $x=1$ up to the curve then across to the line $y=x$ then up to the curve finishing at the root (need to see a minimum of 2 vertical and horizontal lines tending to the root)	B1	2.1
		(1)	
(d)(i)	$x_2 = \frac{2e^3-4}{e^3+4} = 1.5017756\dots$	M1	1.1b
	$x_2 = \text{awrt } 1.502$	A1	1.1b
(ii)	$\beta = 1.968$	dB1	2.2b
		(3)	
(e)	$h(x) = \frac{2e^{3x}-4}{e^{3x}+4} - x$ $h(0.4315) = -0.000297\dots \quad h(0.4325) = 0.000947\dots$	M1	3.1a
	Both calculations correct and e.g. states: - There is a change of sign - e.g $f'(x)$ is continuous $\alpha = 0.432$ (to 3dp)	A1cao	2.4
		(2)	
(13 marks)			
Notes			
(a)			
M1:	Attempts the product rule on xe^x (or may be $7xe^x$) achieving an expression of the form $\dots xe^x \pm \dots e^x$. If it is clear that the quotient rule has been applied instead which may be quoted then M0.		
A1:	$k(xe^x + e^x)$ (e.g. $7(xe^x + e^x)$) or equivalent which may be unsimplified (may be implied by further work)		
B1:	$\left(\frac{d}{dx}(\sqrt{e^{3x}-2})\right) = \frac{1}{2} \times 3e^{3x}(e^{3x}-2)^{-\frac{1}{2}}$ (simplified or unsimplified)		

dM1: Attempts to use the quotient rule. It is dependent on the previous method mark.

Score for achieving an expression of the form

$$(f'(x) =) \frac{(e^{3x} - 2)^{\frac{1}{2}} ("7"xe^x + "7"e^x) - "\frac{3}{2}"e^{3x}(e^{3x} - 2)^{-\frac{1}{2}} \times "7"xe^x}{e^{3x} - 2} \text{ or equivalent (do not be}$$

concerned by the constants for their “7” or their “ $\frac{3}{2}$ ” which may be both 1)

If it is clear that the quotient rule has been applied the wrong way round then score M0.

Alternatively, applies the product rule. Score for achieving an expression of the form

$$(f'(x) =) (e^{3x} - 2)^{-\frac{1}{2}} ("7"xe^x + "7"e^x) - "\frac{3}{2}"e^{3x}(e^{3x} - 2)^{-\frac{3}{2}} \times "7"xe^x \text{ or equivalent (do not be}$$

concerned by the constants for their “7” or their “ $\frac{3}{2}$ ” which may be both 1)

Do not condone invisible brackets.

A1: $(f'(x) =) \frac{7e^x(e^{3x}(2-x) - 4x - 4)}{2(e^{3x} - 2)^{\frac{3}{2}}}$ following a fully correct differentiated expression.

You may need to check to see if (a) is continued after other parts for evidence of this.

Condone the lack of $f'(x) =$ on the left hand side or allow the use of $\frac{dy}{dx}$ or y' instead.

Alternative (a) attempt using the triple product rule

$$\begin{aligned} \text{e.g. } \frac{d}{dx} \left(7xe^x (e^{3x} - 2)^{-\frac{1}{2}} \right) &= 7e^x (e^{3x} - 2)^{-\frac{1}{2}} + 7xe^x (e^{3x} - 2)^{-\frac{1}{2}} + 7xe^x \times \left(-\frac{1}{2} \right) \times 3e^{3x} (e^{3x} - 2)^{-\frac{3}{2}} \\ &\Rightarrow \frac{(7e^x + 7xe^x)(e^{3x} - 2) + 7xe^x \times \left(-\frac{1}{2} \right) \times 3e^{3x}}{(e^{3x} - 2)^{\frac{3}{2}}} = \frac{7e^x \left(e^{3x} - 2 + xe^{3x} - 2x - \frac{3}{2}xe^{3x} \right)}{(e^{3x} - 2)^{\frac{3}{2}}} \Rightarrow \frac{7e^x (e^{3x}(2-x) - 4x - 4)}{2(e^{3x} - 2)^{\frac{3}{2}}} \end{aligned}$$

M1: Attempts the product rule on $xe^x \rightarrow \dots xe^x \pm \dots e^x$ which may be seen within the expression $\dots e^x (e^{3x} - 2)^{-\frac{1}{2}} \pm \dots xe^x (e^{3x} - 2)^{-\frac{1}{2}} + \dots$ simplified or unsimplified.

A1: $k(xe^x + e^x)$ which may be seen within the expression $k \left(e^x (e^{3x} - 2)^{-\frac{1}{2}} + xe^x (e^{3x} - 2)^{-\frac{1}{2}} \right) + \dots$ simplified or unsimplified.

B1: $\left(-\frac{1}{2} \right) \times 3e^{3x} (e^{3x} - 2)^{-\frac{3}{2}}$ which may be seen within the expression $\dots + k \left(xe^x \times \left(-\frac{1}{2} \right) \times 3e^{3x} (e^{3x} - 2)^{-\frac{3}{2}} \right)$ simplified or unsimplified.

dM1: A complete method using all three products (which may appear all on one line). Do not condone invisible brackets.

A1: As above in main scheme notes.

(b) Note that if they do not have values $A = -4$, $B = -4$ in (a) (which may be seen later) then maximum score is M1A0*

M1: Sets their $e^{3x}(2-x) - 4x - 4$ equal to zero, collects terms in x on one side of the equation and non x terms on the other and attempts to factorise the side with x as a common factor. Condone sign slips only for this mark. Allow A and B to be used instead of “-4” and “-4”

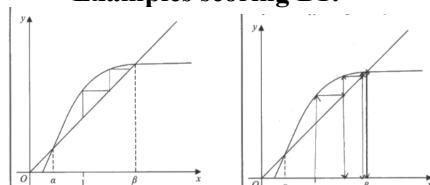


A1*: Achieves the given answer with no errors including invisible brackets. If they do not reach the printed answer then it is A0. If they subsequently write $x_{n+1} = \frac{2e^{3x_n} - 4}{e^{3x_n} + 4}$ then isw

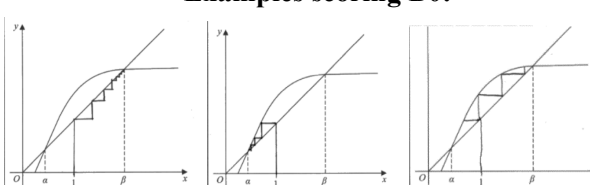
(c)

B1: Starting at $x_1 = 1$ look for at least 2 sets of vertical and horizontal lines drawn (may be dashes) tending to β . Condone a lack of arrows on the lines but the sequence of lines should finish at the point of intersection where the root is. Condone the initial vertical line not starting from the x -axis. Mark the intention to draw horizontal and vertical lines. If they have any lines to the left of $x = 1$ this is B0. If they use both diagrams and do not indicate which one they want marking, then the “copy of Diagram 1” should be marked.

Examples scoring B1:



Examples scoring B0:



(d)(i)

M1: Substitutes 1 into the iterative formula. The values embedded in the formula is sufficient for this mark. May be implied by awrt 1.50

A1: awrt 1.502 isw

(d)(ii)

dB1: 1.968 cao (which can only be scored if M1 is scored in (d)(i))

SC: If (d)(i) is rounded to 1.50 then allow 1.97 in (d)(ii) to score M1A0dB1 for (d)

(e)

M1: Attempts to substitute $x = 0.4315$ and 0.4325 into a suitable function and gets one value correct (rounded or truncated to 1sf). It is allowable to use a tighter interval that contains the root $0.4317388728...$

If no function is stated then may be implied by their answers to e.g. $f'(0.4315)$, $f'(0.4325)$

You will need to check their calculation is correct.

Other possible functions include:

- $h(x) = x - \frac{2e^{3x} - 4}{e^{3x} + 4}$ (other way round to MS) $h(0.4315) = 0.0002974...$, $h(0.4325) = -0.0009479...$

- their $f'(x) = \pm \left(\frac{7e^x (e^{3x} (2 - x) - 4x - 4)}{2(e^{3x} - 2)^{\frac{3}{2}}} \right)$

(If correct A and B then $f'(0.4315) = \mp 0.005789...$, $f'(0.4325) = \pm 0.01831...$)

- their $g(x) = \pm (e^{3x} (2 - x) - 4x - 4)$

(If correct A and B then $g(0.4315) = \mp 0.002275...$, $g(0.4325) = \pm 0.007261...$)

A1: Requires

- Both calculations correct (rounded or truncated to 1sf)
- A statement that there is a change in sign and that their **function** is continuous (must refer to the function used for the substitution (which is not $f(x)$))
Accept equivalent statements for $f'(0.4315) < 0$, $f'(0.4325) > 0$ e.g.
 $f'(0.4315) \times f'(0.4325) < 0$, “one negative one positive”. A minimum is “change of sign and continuous” but do **not** allow this mark if the comment about continuity is clearly incorrect e.g. “because x is continuous” or “because the interval is continuous”
- A minimal conclusion e.g. “hence $\alpha = 0.432$ ”, “so rounds to 0.432”. Do not allow “hence root”

Question	Scheme	Marks	AOs
4 (a)	$\lim_{\delta x \rightarrow 0} \sum_{x=2.1}^{6.3} \frac{2}{x} \delta x = \int_{2.1}^{6.3} \frac{2}{x} dx$	B1	1.2
		(1)	
(b)	$= [2 \ln x]_{2.1}^{6.3} = 2 \ln 6.3 - 2 \ln 2.1$	M1	1.1b
	$= \ln 9$ CSO	A1	1.1b
		(2)	
(3 marks)			
Notes:			

Mark (a) and (b) as one

(a)

B1: States that $\int_{2.1}^{6.3} \frac{2}{x} dx$ or equivalent such as $2 \int_{2.1}^{6.3} x^{-1} dx$ but must include the limits and the dx .

Condone $dx \leftrightarrow \delta x$ as it is very difficult to tell one from another sometimes

(b)

M1: Know that $\int \frac{1}{x} dx = \ln x$ and attempts to apply the limits (either way around)

Condone $\int \frac{2}{x} dx = p \ln x$ (including $p = 1$) or $\int \frac{2}{x} dx = p \ln qx$ as long as the limits are applied.

Also be aware that $\int \frac{2}{x} dx = \ln x^2$, $\int \frac{2}{x} dx = 2 \ln|x| + c$ and $\int \frac{2}{x} dx = 2 \ln cx$ o.e. are also correct

$[p \ln x]_{2.1}^{6.3} = p \ln 6.3 - p \ln 2.1$ is sufficient evidence to award this mark

A1: CSO $\ln 9$. Also answer $= \ln 3^2$ so $k = 9$ is fine. Condone $\ln|9|$

The method mark must have been awarded. Do not accept answers such as $\ln \frac{39.69}{4.41}$

Note that solutions appearing from "rounded" decimal work when taking $\ln$ s should not score the final mark. It is a "show that" question

E.g. $[2 \ln x]_{2.1}^{6.3} = 2 \ln 6.3 - 2 \ln 2.1 = 2.197 = \ln k \Rightarrow k = e^{2.197} = 8.998 = 9$



Question	Scheme	Marks	AOs
4 (a)	Attempts $f(3) =$ and $f(4) =$ where $f(x) = \pm(2\ln(8-x) - x)$	M1	2.1
	$f(3) = (2\ln(5) - x) = (+)0.22$ and $f(4) = (2\ln(4) - 4) = -1.23$ <u>Change of sign</u> and function <u>continuous</u> in interval $[3, 4] \Rightarrow$ <u>Root</u> *	A1*	2.4
		(2)	
(b)	For annotating the graph by drawing a cobweb diagram starting at $x_1 = 4$ It should have at least two spirals	M1	2.4
	Deduces that the iteration formula can be used to find an approximation for α because the cobweb spirals inwards for the cobweb diagram	A1	2.2a
		(2)	
(4 marks)			

Notes:**(a)**

M1: Attempts $f(3) =$ and $f(4) =$ where $f(x) = \pm(2\ln(8-x) - x)$ or alternatively **compares** $2\ln 5$ to 3 and $2\ln 4$ to 4. This is not routine and cannot be scored by substituting 3 and 4 in both functions

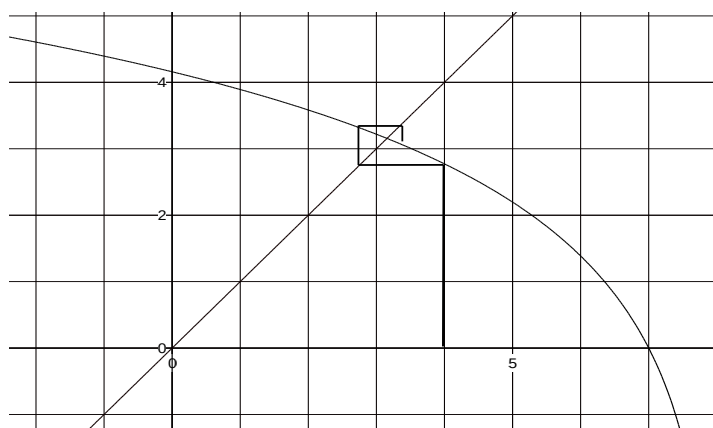
A1: Both values (calculations) correct to at least 1 sf with correct explanation and conclusion. (See underlined statements)

When comparing terms, allow reasons to be $2\ln 8 = 3.21 > 3$, $2\ln 4 = 2.77 < 4$ or similar

(b)

M1: For an attempt at using a cobweb diagram. Look for 5 or more correct straight lines. It may not start at 4 but it must show an understanding of the method. **If there is no graph then it is M0 A0**

A1: For a correct attempt starting at 4 and deducing that the iteration **can be used** as the iterations **converge to the root**. You must statement that it can be used with a suitable reason. Suitable reasons could be "it spirals inwards", it gets closer to the root", it converges "



Question	Scheme	Marks	AOs
12	$\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \times \frac{1}{x} \, dx$	M1	1.1b
	$= \frac{x^4}{4} \ln x - \frac{x^4}{16} (+c)$	M1 A1	1.1b 1.1b
	$\int_1^{e^2} x^3 \ln x \, dx = \left[\frac{x^4}{4} \ln x - \frac{x^4}{16} \right]_1^{e^2} = \left(\frac{e^8}{4} \ln e^2 - \frac{e^8}{16} \right) - \left(-\frac{1^4}{16} \right)$	M1	2.1
	$= \frac{7}{16} e^8 + \frac{1}{16}$	A1	1.1b
		(5)	
(5 marks)			
Notes:			

M1: Integrates by parts the right way round.

Look for $kx^4 \ln x - \int kx^4 \times \frac{1}{x} \, dx$ o.e. with $k > 0$. Condone a missing dx

M1: Uses a correct method to integrate an expression of the form $\int kx^4 \times \frac{1}{x} \, dx \rightarrow c x^4$

A1: $\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \frac{x^4}{16} (+c)$ which may be left unsimplified

M1: Attempts to substitute 1 and e^2 into an expression of the form $\pm px^4 \ln x \pm qx^4$, subtracts and uses $\ln e^2 = 2$ (which may be implied).

A1: $\frac{7}{16} e^8 + \frac{1}{16}$ o.e. Allow $0.4375 e^8 + 0.0625$ or uncanceled fractions. NOT ISW: $7e^8 + 1$ is A0

.....
You may see attempts where substitution has been attempted.

E.g. $u = \ln x \Rightarrow x = e^u$ and $\frac{dx}{du} = e^u$

M1: Attempts to integrate the correct way around condoning slips on the coefficients

$$\int x^3 \ln x \, dx = \int e^{4u} u \, du = \frac{e^{4u}}{4} u - \int \frac{e^{4u}}{4} \, du$$

M1 A1: $\int x^3 \ln x \, dx = \frac{e^{4u}}{4} u - \frac{e^{4u}}{16} (+c)$

M1 A1: Substitutes 0 and 2 into an expression of the form $\pm pue^{4u} \pm qe^{4u}$ and subtracts

.....
It is possible to use integration by parts "the other way around"

To do this, candidates need to know or use $\int \ln x \, dx = x \ln x - x$

$$\text{FYI } I = \int x^3 \ln x \, dx = x^3 (x \ln x - x) - \int (x \ln x - x) \times 3x^2 \, dx = x^3 (x \ln x - x) - 3I + \frac{3}{4} x^4$$

$$\text{Hence } 4I = x^4 \ln x - \frac{1}{4} x^4 \Rightarrow I = \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4$$

Score M1 for a full attempt at line 1 (condoning bracketing and coefficient slips) followed by M 1 for line 2 where terms in I o.e. to form the answer.

Question	Scheme	Marks	AOs
13(a)	$k = e^2 \quad \text{or} \quad x \neq e^2$	B1	2.2a
		(1)	



(b)	$g'(x) = \frac{(\ln x - 2) \times \frac{3}{x} - (3 \ln x - 7) \times \frac{1}{x}}{(\ln x - 2)^2} = \frac{1}{x(\ln x - 2)^2}$ or $g'(x) = \frac{d}{dx} \left(3 - (\ln(x) - 2)^{-1} \right) = (\ln x - 2)^{-2} \times \frac{1}{x} = \frac{1}{x(\ln x - 2)^2}$ or $g'(x) = (\ln x - 2)^{-1} \times \frac{3}{x} - (3 \ln x - 7)(\ln x - 2)^{-2} \times \frac{1}{x} = \frac{1}{x(\ln x - 2)^2}$	M1 A1	1.1b 2.1
	As $x > 0$ (or $1/x > 0$) AND $\ln x - 2$ is squared so $g'(x) > 0$	A1cso	2.4
		(3)	
(c)	Attempts to solve either $3 \ln x - 7 \dots 0$ or $\ln x - 2 \dots 0$ or $3 \ln a - 7 \dots 0$ or $\ln a - 2 \dots 0$ where $\dots$ is “=” or “>” to reach a value for x or a but may be seen as an inequality e.g. $x > \dots$ or $a > \dots$	M1	3.1a
	$0 < a < e^2, \quad a > e^{\frac{7}{3}}$	A1	2.2a
		(2)	
(6 marks)			

Notes:

(a)

B1: Deduces $k = e^2$ or $x \neq e^2$ Condone $k = \text{awrt } 7.39$ or $x \neq \text{awrt } 7.39$

(b)

M1: Attempts to differentiate via the quotient rule and with $\ln x \rightarrow \frac{1}{x}$ so allow for:

$$\frac{d}{dx}(g(x)) = \frac{(\ln x - 2) \times \frac{\alpha}{x} - (3 \ln x - 7) \times \frac{\beta}{x}}{(\ln x - 2)^2}, \quad \beta > 0$$

But a correct rule may be implied by their u, v, u', v' followed by applying $\frac{vu' - uv'}{v^2}$ etc.

Alternatively attempts to write $g(x) = \frac{3 \ln(x) - 7}{\ln(x) - 2} = 3 - (\ln(x) - 2)^{-1}$ and attempts the chain rule so allow for:

$$3 - (\ln(x) - 2)^{-1} \rightarrow (\ln(x) - 2)^{-2} \times \frac{\alpha}{x}$$

Alternatively writes $g(x) = (3 \ln(x) - 7)(\ln(x) - 2)^{-1}$ and attempts the product rule so allow for:

$$g'(x) = (\ln x - 2)^{-1} \times \frac{\alpha}{x} - (3 \ln x - 7)(\ln x - 2)^{-2} \times \frac{\beta}{x}$$

In general condone missing brackets for the M mark. E.g. if they quote $u = 3 \ln x - 7$ and $v = \ln x - 2$ and don't make the differentiation easier, they can be awarded this mark for applying the correct rule. Also allow this mark if they quote the correct quotient rule but only have v rather than v^2 in the denominator.

A1: $\frac{1}{x(\ln x - 2)^2}$ Allow $\frac{\frac{1}{x}}{(\ln x - 2)^2}$ i.e. we need to see the numerator simplified to $1/x$

Note that some candidates establish the correct numerator and correct denominator independently and provided they obtain the correct expressions, this mark can be awarded.

But allow a correctly expanded denominator.

A1cso: States that as $x > 0$ **AND** $\ln x - 2$ is squared so $g'(x) > 0$



Question	Scheme	Marks	AOs
9(a) Way 1	$x = (t + 3)^2 - 25$	M1	1.1b
	$\Rightarrow x + 25 = (t + 3)^2 \Rightarrow (x + 25)^{\frac{1}{2}} = (t + 3) \Rightarrow y = \dots$	M1	2.1
	$y = 6 \ln (x + 25)^{\frac{1}{2}} \Rightarrow y = 3 \ln (x + 25)$	A1cso	1.1b
		(3)	
	(a) Way 2		
	$y = 6 \ln (t + 3) = 3 \ln (t + 3)^2$	M1	1.1b
	$y = 3 \ln (t + 3)^2 = 3 \ln (t^2 + 6t + 9) = 3 \ln (x + 16 + 9)$	M1	2.1
	$y = 3 \ln (x + 25)$	A1cso	1.1b
	(a) Way 3		
	$y = 6 \ln (t + 3) \Rightarrow \frac{y}{6} = \ln (t + 3) \Rightarrow t + 3 = e^{\frac{y}{6}} \Rightarrow t = e^{\frac{y}{6}} - 3$	M1	1.1b
	$x = \left(e^{\frac{y}{6}} - 3\right)^2 + 6\left(e^{\frac{y}{6}} - 3\right) - 16 \Rightarrow y = \dots$ or $x = \left(e^{\frac{y}{6}} - 3 + 8\right)\left(e^{\frac{y}{6}} - 3 - 2\right) \Rightarrow y = \dots$	M1	2.1
	$y = 3 \ln (x + 25)$	A1cso	1.1b
	(a) Way 4		
	$x = (t + 3)^2 - 25$	M1	1.1b
	$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\left(\frac{6}{t+3}\right)}{2t+6} \Rightarrow \frac{3}{(t+3)^2} = \frac{3}{x+25} \Rightarrow y = 3 \ln (x + 25)(+c)$	M1	2.1
	e.g. $t = 0 \Rightarrow x = -16, y = 6 \ln 3 \Rightarrow 6 \ln 3 = 3 \ln (9) \Rightarrow c = 0$ $y = 3 \ln (x + 25)$	A1cso	1.1b
(b)	$x = 0, y = 3 \ln 25$ oe e.g. $6 \ln 5$	B1ft	2.2a
	$\frac{dy}{dx} = \frac{3}{x + "25"} \Rightarrow \frac{dy}{dx} = \frac{3}{0 + "25"} \quad \left(= \frac{3}{25} \right)$ or $\frac{dy}{dx} = \frac{\left(\frac{6}{t+3}\right)}{2t+6} \Rightarrow \frac{6}{2+3} \quad \left(= \frac{6}{50} = \frac{3}{25} \right)$	M1	2.1
	$y - "3 \ln 25" = \frac{3}{25} "x \{-0\}"$	dM1	3.1a
	$25y - 3x = 150 \ln 5$	A1	2.2a
		(4)	
	(7 marks)		
Notes			
Choose the mark scheme that best matches their chosen method.			

(a)

Way 1

M1: Attempts to complete the square. Award for sight of $x = (t + 3)^2 \pm \dots$ where $\dots \neq 0$

M1: Rearranges their $x = (t + 3)^2 - 25$ to either $(t + 3) = \dots$ or $(t + 3)^2 = \dots$ and then substitutes correctly their expression into the parametric equation for y . So e.g., $t = \sqrt{x + 25} - 3 \rightarrow y = 6 \ln(\sqrt{x + 25} - 3)$ is M0.

A1cso: $y = 3 \ln(x + 25)$ including brackets with all stages of working shown.

The “ $y =$ ” must appear at some point.

Way 2

M1: Attempts to use the power rule for logarithms $y = 6 \ln(t + 3) = \dots \ln(t + 3)^2$ where $\dots \neq 6$

M1: Writes $y = 6 \ln(t + 3)$ as $3 \ln(t + 3)^2$ and then multiplies out and substitutes correctly in for t to obtain a Cartesian equation for C

A1cso: $y = 3 \ln(x + 25)$ including brackets with all stages of working shown.

The “ $y =$ ” must appear at some point.

Way 3

M1: Attempts to make t the subject for $y = 6 \ln(t + 3)$ to obtain $t = e^{\frac{y}{6}} \pm \dots$ where $\dots \neq 0$

M1: Substitutes $t = e^{\frac{y}{6}} \pm \dots$ correctly into $x = t^2 + 6t - 16$ and rearranges to make y the subject.

A1cso: $y = 3 \ln(x + 25)$ including brackets with all stages of working shown.

The “ $y =$ ” must appear at some point.

Way 4

M1: Attempts to complete the square. Award for sight of $x = (t + 3)^2 \pm \dots$ where $\dots \neq 0$

M1: Attempts to find $\frac{dy}{dx}$ where $\frac{dy}{dx} = \frac{\left(\frac{\dots}{t+3}\right)}{at+b}$, $a, b \neq 0$ and uses the completed square form to find $\frac{dy}{dx}$ in terms of x and then integrates to obtain a Cartesian equation for C

A1cso: A complete method using any correct point on the curve to show that $c = 0$ and obtain $y = 3 \ln(x + 25)$ with all stages of working shown. The “ $y =$ ” must appear at some point.

Note that a common incorrect approach in (a) is:

$$x = t^2 + 6t - 16 = (t - 2)(t + 8) \Rightarrow x = t - 2 \Rightarrow t = x + 2 \Rightarrow y = 6 \ln(x + 5)$$

which scores no marks.



(b)

B1ft: Deduces $y = 3 \ln 25$ or e.g. $y = 6 \ln 5$ but allow follow through on their Cartesian equation with $x = 0$ and apply isw after a correct value or ft value for y

M1: Attempts to find $\frac{dy}{dx}$ when $x = 0$ so score for obtaining $\frac{dy}{dx} = \frac{\dots}{x + "25"}$ and substituting in $x = 0$

Allow this mark if they use the letters A and B e.g. $\frac{dy}{dx} = \frac{\dots}{x + B} = \frac{\dots}{0 + B}$ or allow a “made up” A and B .

or

Attempts to find $\frac{dy}{dx}$ when $t = 2$ by finding $\frac{dy}{dx} = \frac{\left(\frac{6}{t+3}\right)}{2t+6} \Rightarrow \frac{\frac{6}{5}}{2 \times 2 + 6} \left(= \frac{6}{50} = \frac{3}{25} \right)$

For the derivative look for $\frac{dy}{dx} = \frac{\left(\frac{\dots}{t+3}\right)}{at+b}$ or e.g. $\left(\frac{\dots}{t+3}\right) \times \frac{1}{at+b}$ $a, b \neq 0$

NOTE if candidates find $\frac{dy}{dx} = \frac{\left(\frac{6}{t+3}\right)}{2t+6} = \frac{6(2t+6)}{t+3} = 12$ we will give BOD that $t = 2$ has been used unless

there is clear evidence that $t = 2$ has not been used.

dM1: Attempts to find the equation of the tangent. Score for sight of $y - "3 \ln 25" = \frac{3}{25} (x - \{-0\})$ or if they use

$y = mx + c$ they must proceed as far as $c = \dots$ **It is dependent on the previous method mark.**

Must have numeric A and B now.

A1: $25y - 3x = 150 \ln 5$ or any integer multiple of this equation in the form $ax + by = c \ln 5$



Question	Scheme	Marks	AOs
12(a)	$u = 1 + \sqrt{x} \Rightarrow x = (u-1)^2 \Rightarrow \frac{dx}{du} = 2(u-1)$ or $u = 1 + \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$	B1	1.1b
	$\int \frac{x}{1+\sqrt{x}} dx = \int \frac{(u-1)^2}{u} 2(u-1) du$ or $\int \frac{x}{1+\sqrt{x}} dx = \int \frac{x}{u} \times 2x^{\frac{1}{2}} du = \int \frac{2x^{\frac{3}{2}}}{u} du = \int \frac{2(u-1)^3}{u} du$	M1	2.1
	$\int_0^{16} \frac{x}{1+\sqrt{x}} dx = \int_1^5 \frac{2(u-1)^3}{u} du$	A1	1.1b
		(3)	
(b)	$2 \int \frac{u^3 - 3u^2 + 3u - 1}{u} du = 2 \int \left(u^2 - 3u + 3 - \frac{1}{u} \right) du = \dots$	M1	3.1a
	$= (2) \left[\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln u \right]$	A1	1.1b
	$= 2 \left[\frac{5^3}{3} - \frac{3(5)^2}{2} + 3(5) - \ln 5 - \left(\frac{1}{3} - \frac{3}{2} + 3 - \ln 1 \right) \right]$	dM1	2.1
	$= \frac{104}{3} - 2 \ln 5$	A1	1.1b
		(4)	
(7 marks)			
Notes			
(a) B1: Correct expression for $\frac{dx}{du}$ or $\frac{du}{dx}$ (or u') or dx in terms of du or du in terms of dx M1: Complete method using the given substitution. This needs to be a correct method for their $\frac{dx}{du}$ or $\frac{du}{dx}$ leading to an integral in terms of u only (ignore any limits if present) so for each case you need to see: $\frac{dx}{du} = f(u) \rightarrow \int \frac{x}{1+\sqrt{x}} dx = \int \frac{(u-1)^2}{u} f(u) du$ $\frac{du}{dx} = g(x) \rightarrow \int \frac{x}{1+\sqrt{x}} dx = \int \frac{x}{u} \times \frac{du}{g(x)} = \int h(u) du.$ In this case you can condone slips with coefficients e.g. allow $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \rightarrow \int \frac{x}{1+\sqrt{x}} dx = \int \frac{x}{u} \times \frac{x^{\frac{1}{2}}}{2} du = \int h(u) du$			

$$\text{but not } \frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}} \rightarrow \int \frac{x}{1+\sqrt{x}} dx = \int \frac{x}{u} \times \frac{x^{-\frac{1}{2}}}{2} du = \int h(u) du$$

A1: All correct with correct limits and no errors. The “ du ” must be present but may have been omitted along the way but it must have been seen at least once before the final answer. The limits can be seen as part of the integral or stated separately.

(b)

M1: Realises the requirement to cube the bracket and divide through by u and makes progress with the integration to obtain at least 3 terms of the required form e.g. 3 from ku^3 , ku^2 , ku , $k \ln u$

A1: Correct integration. This mark can be scored with the “2” still outside the integral or even if it has been omitted. But if the “2” has been combined with the integrand, the integration must be correct.

dM1: Completes the process by applying their “changed” limits and subtracts the right way round

Depends on the first method mark.

A1: Cao (Allow equivalents for $\frac{104}{3}$ e.g. $\frac{208}{6}$)

Question	Scheme	Marks	AOs
11(a)	$h = 0.5$	B1	1.1b
	$A \approx \frac{1}{2} \times \frac{1}{2} \{0.4805 + 1.9218 + 2(0.8396 + 1.2069 + 1.5694)\}$	M1	1.1b
	$= 2.41$	A1	1.1b
		(3)	
(b)	$\int (\ln x)^2 \, dx = x(\ln x)^2 - \int x \times \frac{2 \ln x}{x} \, dx$	M1 A1	3.1a 1.1b
	$= x(\ln x)^2 - 2 \int \ln x \, dx = x(\ln x)^2 - 2 \left(x \ln x - \int dx \right)$ $= x(\ln x)^2 - 2 \int \ln x \, dx = x(\ln x)^2 - 2x \ln x + 2x$	dM1	2.1
	$\int_2^4 (\ln x)^2 \, dx = \left[x(\ln x)^2 - 2x \ln x + 2x \right]_2^4$ $= 4(\ln 4)^2 - 2 \times 4 \ln 4 + 2 \times 4 - \left(2(\ln 2)^2 - 2 \times 2 \ln 2 + 2 \times 2 \right)$ $= 4(2 \ln 2)^2 - 16 \ln 2 + 8 - 2(\ln 2)^2 + 4 \ln 2 - 4$	ddM1	2.1
	$= 14(\ln 2)^2 - 12 \ln 2 + 4$	A1	1.1b
		(5)	
(8 marks)			
Notes			

(a)

B1: Correct strip width. May be implied by $\frac{1}{2} \times \frac{1}{2} \{ \dots \}$ or $\frac{1}{4} \times \{ \dots \}$

M1: Correct application of the trapezium rule.

Look for $\frac{1}{2} \times "h" \{0.4805 + 1.9218 + 2(0.8396 + 1.2069 + 1.5694)\}$ condoning slips in the digits.

The bracketing must be correct but it is implied by awrt 2.41

A1: 2.41 only. This is not awrt

(b)

M1: Attempts parts the correct way round to achieve $\alpha x(\ln x)^2 - \beta \int \ln x dx$ o.e.

May be unsimplified (see scheme). Watch for candidates who know or learn $\int \ln x dx = x \ln x - x$

who may write $\int (\ln x)^2 dx = \int (\ln x)(\ln x) dx = \ln x(x \ln x - x) - \int \frac{x \ln x - x}{x} dx$

A1: Correct expression which may be unsimplified

dM1: Attempts parts again to (only condone coefficient errors) to achieve $\alpha x(\ln x)^2 - \beta x \ln x \pm \gamma x$ o.e.

ddM1: Applies the limits 4 and 2 to an expression of the form $\pm \alpha x(\ln x)^2 \pm \beta x \ln x \pm \gamma x$, subtracts and applies $\ln 4 = 2 \ln 2$ at least once. Both M's must have been awarded

A1: Correct answer

It is possible to do $\int (\ln x)^2 dx$ via a substitution $u = \ln x$ but it is very similar.

M1 A1, dM1: $\int u^2 e^u du = u^2 e^u - \int 2u e^u du = u^2 e^u - 2ue^u \pm 2e^u$

ddM1: Applies appropriate limits and uses $\ln 4 = 2 \ln 2$ at least once to an expression of the form $u^2 e^u - \beta u e^u \pm \gamma e^u$ Both M's must have been awarded

Question	Scheme	Marks	AOs
4(a)	$f'(x) = 2x + \frac{4x-4}{2x^2-4x+5}$	M1 A1	1.1b 1.1b
	$2x + \frac{4x-4}{2x^2-4x+5} = 0 \Rightarrow 2x(2x^2-4x+5) + 4x-4 = 0$	dM1	1.1b
	$2x^3 - 4x^2 + 7x - 2 = 0^*$	A1*	2.1
		(4)	
(b)	(i) $x_2 = \frac{1}{7}(2 + 4(0.3)^2 - 2(0.3)^3)$	M1	1.1b
	$x_2 = 0.3294$	A1	1.1b
	(ii) $x_4 = 0.3398$	A1	1.1b
		(3)	
(c)	$h(x) = 2x^3 - 4x^2 + 7x - 2$ $h(0.3415) = 0.00366... \quad h(0.3405) = -0.00130...$	M1	3.1a
	States: - there is a change of sign $f'(x)$ is continuous $\alpha = 0.341$ to 3dp	A1	2.4
		(2)	
(9 marks)			
Notes			

(a)

M1: Differentiates $\ln(2x^2 - 4x + 5)$ to obtain $\frac{g(x)}{2x^2 - 4x + 5}$ where $g(x)$ could be 1

A1: For $f'(x) = 2x + \frac{4x - 4}{2x^2 - 4x + 5}$

dM1: Sets their $f'(x) = ax + \frac{g(x)}{2x^2 - 4x + 5} = 0$ and uses "correct" algebra, condoning slips, to obtain a

cubic equation. E.g Look for $ax(2x^2 - 4x + 5) \pm g(x) = 0$ o.e. , condoning slips, followed by some attempt to simplify

A1*: Achieves $2x^3 - 4x^2 + 7x - 2 = 0$ with no errors. (The dM1 mark must have been awarded)

(b)(i)

M1: Attempts to use the iterative formula with $x_1 = 0.3$. If no method is shown award for $x_2 = \text{awrt } 0.33$

A1: $x_2 = \text{awrt } 0.3294$ Note that $\frac{1153}{3500}$ is correct

Condone an incorrect suffix if it is clear that a correct value has been found

(b)(ii)

A1: $x_4 = \text{awrt } 0.3398$ Condone an incorrect suffix if it is clear that a correct value has been found

(c)

M1: Attempts to substitute $x = 0.3415$ and $x = 0.3405$ into a suitable function and gets one value correct (rounded or truncated to 1 sf). It is allowable to use a tighter interval that contains the root 0.340762654

Examples of suitable functions are $2x^3 - 4x^2 + 7x - 2$, $x - \frac{1}{7}(4x^2 - 2x^3 + 2)$ and $f'(x)$ as this has been

found in part (a) with $f'(0.3405) = -0.00067\dots$, $f'(0.3415) = (+) 0.0018$

There must be sufficient evidence for the function, which would be for example, a statement such as $h(x) = 2x^3 - 4x^2 + 7x - 2$ or sight of embedded values that imply the function, not just a value or values

even if both are correct. Condone $h(x)$ being mislabelled as f

$h(0.3415) = 2 \times 0.3415^3 - 4 \times 0.3415^2 + 7 \times 0.3415 - 2$

A1: Requires

- both calculations correct (rounded or truncated to 1sf)
- a statement that there is a change in sign and that the function is continuous
- a minimal conclusion e.g. $\checkmark$, proven, $\alpha = 0.341$, root



Question	Scheme	Marks	AOs
10 (a)	$x = u^2 + 1 \Rightarrow dx = 2u du$ oe	B1	1.1b
	Full substitution $\int \frac{3dx}{(x-1)(3+2\sqrt{x-1})} = \int \frac{3 \times 2u du}{(u^2+1-1)(3+2u)}$	M1	1.1b
	Finds correct limits e.g. $p = 2, q = 3$	B1	1.1b
	$= \int \frac{3 \times 2 \cancel{u} du}{u^2(3+2u)} = \int \frac{6 du}{u(3+2u)} *$	A1*	2.1
		(4)	
(b)	$\frac{6}{u(3+2u)} = \frac{A}{u} + \frac{B}{3+2u} \Rightarrow A = \dots, B = \dots$	M1	1.1b
	Correct PF. $\frac{6}{u(3+2u)} = \frac{2}{u} - \frac{4}{3+2u}$	A1	1.1b
	$\int \frac{6 du}{u(3+2u)} = 2 \ln u - 2 \ln(3+2u) \quad (+c)$	dM1 A1ft	3.1a 1.1b
	Uses limits $u = "3", u = "2"$ with some correct $\ln$ work leading to $k \ln b$ E.g. $(2 \ln 3 - 2 \ln 9) - (2 \ln 2 - 2 \ln 7) = 2 \ln \frac{7}{6}$	M1	1.1b
	$\ln \frac{49}{36}$	A1	2.1
		(6)	
(10 marks)			
Notes: Mark (a) and (b) together as one complete question			

(a)**B1:** $dx = 2u du$ or exact equivalent. E.g. $\frac{dx}{du} = 2u, \frac{du}{dx} = \frac{1}{2}(x-1)^{-\frac{1}{2}}$ **M1:** Attempts a full substitution of $x = u^2 + 1$, including $dx \rightarrow \dots u du$ to form an integrand in terms of u .
Condone slips but there should be an attempt to use the correct substitution on the denominator.**B1:** Finds correct limits either states $p = 2, q = 3$ or sight of embedded values as limits to the integral**A1*:** Clear reasoning including one fully correct intermediate line, including the integral signs, leading to the given expression ignoring limits. So B1, M1, B0, A1 is possible if the limits are incorrect, omitted or left as 5 and 10.**(b)****M1:** Uses correct form of PF leading to values of A and B .**A1:** Correct PF $\frac{6}{u(3+2u)} = \frac{2}{u} - \frac{4}{3+2u}$ (Not scored for just the correct values of A and B)**dM1:** This is an overall problem solving mark. It is for using the correct PF form and integrating using $\ln$ s.
Look for $P \ln u + Q \ln(3+2u)$ **A1ft:** Correct integration for their $\frac{A}{u} + \frac{B}{3+2u} \rightarrow A \ln u + \frac{B}{2} \ln(3+2u)$ with or without modulus signs**M1:** Uses their 2 and 3 as limits, with at least one correct application of the addition law or subtraction law leading to the form $k \ln b$ or $\ln a$. PF's must have been attempted. Condone bracketing slips. Alternatively changing the u 's back to x 's and use limits of 5 and 10.

$$\text{or } {}^7C_4 a^3 2^4, \frac{7!}{4!3!} a^3 2^4, \binom{7}{4} a^3 2^4, 35a^3 2^4, 560a^3 \text{ etc.}$$

$$\text{or } {}^7C_3 a^3 (2x)^4, \frac{7!}{4!3!} a^3 (2x)^4, \binom{7}{3} a^3 (2x)^4, 35a^3 (2x)^4, 560a^3 x^4, \binom{7}{3} a^3 16x^4 \text{ etc.}$$

$$\text{or } {}^7C_3 a^3 2^4, \frac{7!}{4!3!} a^3 2^4, \binom{7}{3} a^3 2^4, 35a^3 2^4, 560a^3$$

You can condone missing brackets around the “2x” so allow e.g. $\frac{7!}{4!3!} a^3 2x^4$

An alternative is to attempt to expand $a^7 \left(1 + \frac{2x}{a}\right)^7$ to give $a^7 \left(\dots \frac{7 \times 6 \times 5 \times 4}{4!} \left(\frac{2x}{a}\right)^4 \dots \right)$

Allow M1 for e.g. $a^7 \left(\dots \frac{7 \times 6 \times 5 \times 4}{4!} \left(\frac{2x}{a}\right)^4 \dots \right), a^7 \left(\dots \binom{7}{4} \left(\frac{2x}{a}\right)^4 \dots \right), a^7 \left(\dots 35 \left(\frac{2x}{a}\right)^4 \dots \right)$ etc.

but condone missing brackets around the $\frac{2x}{a}$

Note that ${}^7C_3, \binom{7}{3}$ etc. are equivalent to ${}^7C_4, \binom{7}{4}$ etc. and are equally acceptable.

If the candidate attempts $(a + 2x)(a + 2x)(a + 2x) \dots$ etc. then it must be a complete method to reach the required term. Send to review if necessary.

dM1: For “560” $a^3 = 15120 \Rightarrow a = \dots$ Condone slips on copying the 15120 but their “560” must be an attempt at

${}^7C_4 \times 2$ or ${}^7C_4 \times 2^4$ and must be attempting the cube root of $\frac{15120}{560}$. **Depends on the first mark.**

A1: $a = 3$ and no other values i.e. ± 3 scores A0

Note that this is fairly common:

$${}^7C_4 a^3 2x^4 = 70a^3 x^4 \Rightarrow 70a^3 = 15120 \Rightarrow a^3 = 216 \Rightarrow a = 6$$

and scores M1 dM1 A0

7(a)	$\ln x \rightarrow \frac{1}{x}$	B1	1.1a
	Method to differentiate $\frac{4x^2 + x}{2\sqrt{x}}$ – see notes	M1	1.1b
	E.g. $2 \times \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2} \times \frac{1}{2}x^{-\frac{1}{2}}$	A1	1.1b
	$\frac{dy}{dx} = 3\sqrt{x} + \frac{1}{4\sqrt{x}} - \frac{4}{x} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$ *	A1*	2.1
		(4)	
(b)	$12x^2 + x - 16\sqrt{x} = 0 \Rightarrow 12x^{\frac{3}{2}} + x^{\frac{1}{2}} - 16 = 0$	M1	1.1b
	E.g. $12x^{\frac{3}{2}} = 16 - \sqrt{x}$	dM1	1.1b
	$x^{\frac{3}{2}} = \frac{4}{3} - \frac{\sqrt{x}}{12} \Rightarrow x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{\frac{2}{3}}$ *	A1*	2.1
		(3)	
(c)	$x_2 = \sqrt[3]{\left(\frac{4}{3} - \frac{\sqrt{2}}{12}\right)^2}$	M1	1.1b
	$x_2 = \text{awrt } 1.13894$	A1	1.1b
	$x = 1.15650$	A1	2.2a
		(3)	
(10 marks)			

Notes:

(a)

B1: Differentiates $\ln x \rightarrow \frac{1}{x}$ seen or implied

M1: Correct method to differentiate $\frac{4x^2 + x}{2\sqrt{x}}$:

Look for $\frac{4x^2 + x}{2\sqrt{x}} \rightarrow \dots x^{\frac{3}{2}} + \dots x^{\frac{1}{2}}$ being then differentiated to $Px^{\frac{1}{2}} + \dots$ **or** $\dots + Qx^{-\frac{1}{2}}$

Alternatively uses the quotient rule on $\frac{4x^2 + x}{2\sqrt{x}}$.

Condone slips but if rule is not quoted expect $\left(\frac{dy}{dx}\right) = \frac{2\sqrt{x}(Ax + B) - (4x^2 + x)Cx^{-\frac{1}{2}}}{(2\sqrt{x})^2} (A, B, C > 0)$

But a correct rule may be implied by their u, v, u', v' followed by applying $\frac{vu' - uv'}{v^2}$ etc.

Alternatively uses the product rule on $(4x^2 + x)(2\sqrt{x})^{-1}$

Condone slips but expect $\left(\frac{dy}{dx}\right) = Ax^{-\frac{1}{2}}(Bx + C) + D(4x^2 + x)x^{-\frac{3}{2}} (A, B, C > 0)$

In general condone missing brackets for the M mark. If they quote $u = 4x^2 + x$ and $v = 2\sqrt{x}$ and don't make the differentiation easier, they can be awarded this mark for applying the correct rule. Also allow this mark if they quote the correct quotient rule but only have v rather than v^2 in the denominator.

A1: Correct differentiation of $\frac{4x^2 + x}{2\sqrt{x}}$ although may not be simplified.



Question	Scheme	Marks	AOs
13	$C: y = x \ln x$; l is a normal to C at $P(e, e)$ Let x_A be the x -coordinate of where l cuts the x -axis		
	$\frac{dy}{dx} = \ln x + x\left(\frac{1}{x}\right) \quad \{= 1 + \ln x\}$	M1	2.1
		A1	1.1b
	$x = e, m_T = 2 \Rightarrow m_N = -\frac{1}{2} \Rightarrow y - e = -\frac{1}{2}(x - e)$ $y = 0 \Rightarrow -e = -\frac{1}{2}(x - e) \Rightarrow x = \dots$	M1	3.1a
	l meets x -axis at $x = 3e$ (allow $x = 2e + e \ln e$)	A1	1.1b
	{Areas:} either $\int_1^e x \ln x \, dx = [\dots]_1^e = \dots$ or $\frac{1}{2}((\text{their } x_A) - e)e$	M1	2.1
	$\left\{ \int x \ln x \, dx = \right\} \frac{1}{2} x^2 \ln x - \int \frac{1}{x} \left(\frac{x^2}{2} \right) \{dx\}$	M1	2.1
	$\left\{ = \frac{1}{2} x^2 \ln x - \int \frac{1}{2} x \{dx\} \right\} = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2$	dM1	1.1b
		A1	1.1b
	$\text{Area}(R_1) = \int_1^e x \ln x \, dx = [\dots]_1^e = \dots$; $\text{Area}(R_2) = \frac{1}{2}((\text{their } x_A) - e)e$ and so, $\text{Area}(R) = \text{Area}(R_1) + \text{Area}(R_2) \quad \{= \frac{1}{4}e^2 + \frac{1}{4} + e^2\}$	M1	3.1a
	$\text{Area}(R) = \frac{5}{4}e^2 + \frac{1}{4}$	A1	1.1b
(10)			
Notes for Question 13			
M1:	Differentiates by using the product rule to give $\ln x + x(\text{their } g'(x))$, where $g(x) = \ln x$		
A1:	Correct differentiation of $y = x \ln x$, which can be un-simplified or simplified		
M1:	Complete strategy to find the x coordinate where their normal to C at $P(e, e)$ meets the x -axis i.e. Sets $y = 0$ in $y - e = m_N(x - e)$ to find $x = \dots$		
Note:	m_T is found by using calculus and $m_N \neq m_T$		
A1:	l meets x -axis at $x = 3e$, allowing un-simplified values for x such as $x = 2e + e \ln e$		
Note:	Allow $x = \text{awrt } 8.15$		
M1:	Scored for either - Area under curve $= \int_1^e x \ln x \, dx = [\dots]_1^e = \dots$, with limits of e and 1 and some attempt to substitute these and subtract or Area under line $= \frac{1}{2}((\text{their } x_A) - e)e$, with a valid attempt to find x_A		
M1:	Integration by parts the correct way around to give $Ax^2 \ln x - \int B \left(\frac{x^2}{x} \right) \{dx\}$; $A \neq 0, B > 0$		
dM1:	dependent on the previous M mark Integrates the second term to give $\pm \lambda x^2$; $\lambda \neq 0$		
A1:	$\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2$		
M1:	Complete strategy of finding the area of R by finding the sum of two key areas. See scheme.		
A1:	$\frac{5}{4}e^2 + \frac{1}{4}$		

Notes for Question 13 Continued	
Note:	$\text{Area}(R_2)$ can also be found by integrating the line l between limits of e and their x_A i.e. $\text{Area}(R_2) = \int_e^{\text{their } x_A} \left(-\frac{1}{2}x + \frac{3}{2}e \right) dx = \left[\dots \right]_e^{\text{their } x_A} = \dots$
Note:	<u>Calculator approach with no algebra, differentiation or integration seen:</u> - Finding l cuts through the x-axis at awrt 8.15 is 2nd M1 2nd A1 - Finding area between curve and the x-axis between $x=1$ and $x=e$ to give awrt 2.10 is 3rd M1 - Using the above information (must be seen) to apply $\text{Area}(R) = 2.0972\dots + 7.3890\dots = 9.4862\dots$ is final M1 Therefore, a maximum of 4 marks out of the 10 available.



Question	Scheme	Marks	AOs
13 (a)	(i) Explains $2x - q = 0$ when $x = 2$ oe Hence $q = 4$ *	B1*	2.4
	(ii) Substitutes $\left(3, \frac{1}{2}\right)$ into $y = \frac{p-3x}{(2x-4)(x+3)}$ and solves	M1	1.1b
	$\frac{1}{2} = \frac{p-9}{(2) \times (6)} \Rightarrow p-9=6 \Rightarrow p=15$ *	A1*	2.1
		(3)	
(b)	Attempts to write $\frac{15-3x}{(2x-4)(x+3)}$ in PF's and integrates using lns between 3 and another value of x.	M1	3.1a
	$\frac{15-3x}{(2x-4)(x+3)} = \frac{A}{(2x-4)} + \frac{B}{(x+3)}$ leading to A and B	M1	1.1b
	$\frac{15-3x}{(2x-4)(x+3)} = \frac{1.8}{(2x-4)} - \frac{2.4}{(x+3)}$ or $\frac{0.9}{(x-2)} - \frac{2.4}{(x+3)}$ oe	A1	1.1b
	$I = \int \frac{15-3x}{(2x-4)(x+3)} dx = m \ln(2x-4) + n \ln(x+3) + (c)$	M1	1.1b
	$I = \int \frac{15-3x}{(2x-4)(x+3)} dx = 0.9 \ln(2x-4) - 2.4 \ln(x+3)$ oe	A1ft	1.1b
	Deduces that Area Either $\int_3^5 \frac{15-3x}{(2x-4)(x+3)} dx$ Or $[.....]_3^5$	B1	2.2a
	Uses correct ln work seen at least once for $\ln 6 = \ln 2 + \ln 3$ or $\ln 8 = 3 \ln 2$ $[0.9 \ln(6) - 2.4 \ln(8)] - [0.9 \ln(2) - 2.4 \ln(6)]$ $= 3.3 \ln 6 - 7.2 \ln 2 - 0.9 \ln 2$	dM1	2.1
	$= 3.3 \ln 3 - 4.8 \ln 2$	A1	1.1b
		(8)	
(11marks)			

(a)**B1*:** Is able to link $2x - q = 0$ and $x = 2$ to explain why $q = 4$ Eg "The asymptote $x = 2$ is where $2x - q = 0$ so $4 - q = 0 \Rightarrow q = 4$ ""The curve is not defined when $2 \times 2 - q = 0 \Rightarrow q = 4$ "There **must be some words** explaining why $q = 4$ and in most cases, you should see a reference to either "the asymptote $x = 2$ ", "the curve is not defined at $x = 2$ ", 'the denominator is 0 at $x = 2$ '**M1:** Substitutes $\left(3, \frac{1}{2}\right)$ into $y = \frac{p-3x}{(2x-4)(x+3)}$ and solvesAlternatively substitutes $\left(3, \frac{1}{2}\right)$ into $y = \frac{15-3x}{(2x-4)(x+3)}$ and shows $\frac{1}{2} = \frac{6}{(2) \times (6)}$ oe**A1*:** Full proof showing all necessary steps $\frac{1}{2} = \frac{p-9}{(2) \times (6)} \Rightarrow p-9=6 \Rightarrow p=15$ In the alternative there would have to be some recognition that these are equal eg ✓ hence $p = 15$ **(b)****M1:** Scored for an overall attempt at using PF's and integrating with lns seen with sight of limits 3 and another value of x .

Question	Scheme	Marks	AOs
13(a)	$3^{-2} \left(1 + \frac{x}{3}\right)^{-2} = 3^{-2}(1 + \dots x + \dots x^2)$	M1	1.1b
	$(-2)\left(\frac{x}{3}\right)$ or $\frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2$	M1	1.1b
	$\left(1 + \frac{x}{3}\right)^{-2} = 1 + (-2)\left(\frac{x}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2$	A1	1.1b
	$3^{-2} \left(1 + \frac{x}{3}\right)^{-2} = \frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}$	A1	2.1
		(4)	

(a)

M1: Attempts a binomial expansion by taking out a factor of 3^{-2} or $\frac{1}{3^2}$ or $\frac{1}{9}$ and achieves at least the first 3 terms in their expansion. May be seen separately e.g. $\frac{1}{9}$ and $(1 + \dots x + \dots x^2)$

M1: A correct method to find either the x or the x^2 term unsimplified.

Award for $(-2)(kx)$ or $\frac{(-2)(-2-1)}{2!}(kx)^2$ where $k \neq 1$. Condone invisible brackets.

A1: For a correct unsimplified or simplified expansion of $\left(1 + \frac{x}{3}\right)^{-2}$ e.g. $= 1 + (-2)\left(\frac{x}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2 - \dots$ or $1 - \frac{2x}{3} + \frac{x^2}{3} - \dots$ Do not condone missing brackets unless they are implied by subsequent work.

Condone $\left(-\frac{x}{3}\right)^2$ for $\left(\frac{x}{3}\right)^2$

Also allow this mark for 2 correct simplified terms from $\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}$ with both method marks scored.

A1: $\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}$ cao Allow terms to be listed. Ignore any extra terms. Isw once a correct **simplified** answer is seen.

Direct expansion, if seen, should be marked as follows:

$$\left((3+x)^{-2} = 3^{-2} - 2 \times 3^{-3} \times x + \frac{-2(-2-1)}{2!} \times 3^{-4} \times x^2\right)$$

$$\mathbf{M1:} \text{ For } (3+x)^{-2} = 3^{-2} + 3^{-3} \times \alpha x + 3^{-4} \times \beta x^2$$

M1: A correct method to find either the x or the x^2 term unsimplified.

Award for $(-2) \times 3^{-3} x$ or $\frac{(-2)(-2-1)}{2!} \times 3^{-4} x^2$. Condone invisible brackets.

A1: For a correct unsimplified or simplified expansion of $(3+x)^{-2}$ e.g. $3^{-2} - 2 \times 3^{-3} \times x + \frac{-2(-2-1)}{2!} \times 3^{-4} \times x^2$

Also award for at least 2 correct simplified terms from $\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}$ with both method marks scored.

A1: $\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}$ cao Allow terms to be listed. Ignore any extra terms. Isw once a correct **simplified** answer is seen.

Note that M0M1A1A0 is a possible mark trait in either method



Note regarding a possible misread in parts (b) and (c)

Some candidates are misreading $\int \frac{6x}{(3+x)^2} dx$ in parts (b) and (c) as $\int \frac{6}{(3+x)^2} dx$

If parts (b) and (c) are consistently attempted with $\int \frac{6}{(3+x)^2} dx$ then we will allow the M

marks in (b) **only**. **M1** for $x^n \rightarrow x^{n+1}$ applied to their expansion in part (a) or $6 \times$ (their expansion in part (a)) and **dM1** for substituting in 0.4 and 0.2 and subtracting either way round (may be implied).

No marks are available in part (c)

MARK PARTS (b) and (c) TOGETHER

(b)	$\int 6x \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27} \right) dx = \int \left(\frac{2x}{3} - \frac{4x^2}{9} + \frac{2x^3}{9} \right) dx = \dots$	M1	1.1b
	$\int \left(\frac{2x}{3} - \frac{4x^2}{9} + \frac{2x^3}{9} \right) dx = \frac{x^2}{3} - \frac{4x^3}{27} + \frac{x^4}{18} \text{ oe}$	A1	1.1b
	$\left[\frac{x^2}{3} - \frac{4x^3}{27} + \frac{x^4}{18} \right]_{0.2}^{0.4} = \left(\frac{(0.4)^2}{3} - \frac{4(0.4)^3}{27} + \frac{(0.4)^4}{18} \right) - \left(\frac{(0.2)^2}{3} - \frac{4(0.2)^3}{27} + \frac{(0.2)^4}{18} \right)$	dM1	3.1a
	$= \text{awrt } 0.03304 \text{ or } \frac{223}{6750}$	A1	1.1b
		(4)	

(b)

M1: Attempts to multiply their expansion from part (a) by $6x$ or just x and attempts to integrate. Condone copying slips and slips in expanding. Look for $x^n \rightarrow x^{n+1}$ at least once having multiplied by $6x$ or x . Ignore e.g. spurious integral signs.

A1: Correct integration, simplified or unsimplified.

$$\frac{x^2}{3} - \frac{4x^3}{27} + \frac{x^4}{18} \text{ oe e.g. } \frac{1}{9} \left(3x^2 - \frac{4x^3}{3} + \frac{x^4}{2} \right), 6 \left(\frac{x^2}{18} - \frac{2x^3}{81} + \frac{x^4}{108} \right)$$

If they have extra terms they can be ignored.

Ignore e.g. spurious integral signs.

dM1: An overall problem-solving mark for

- using part (a) by integrating $6x \times$ their binomial expansion and
- substituting in 0.4 and 0.2 and subtracting either way round (may be implied)

For evidence of using the correct limits we do not expect examiners to check so allow this mark if they obtain a value with (minimal) evidence of the use of the limits 0.4 and 0.2.

This could be e.g. $\left[f(x) \right]_{0.2}^{0.4} = \dots$ provided the first M was scored. If the integration was correct, evidence can be taken from answer of awrt 0.0330 if limits are not seen elsewhere. **Depends on the first M mark.**

A1: awrt 0.03304 (NB allow the exact value which is $\frac{223}{6750} = 0.033037037\dots$).

Is w following a correct answer.

Note answers which use additional terms in the expansion to give a different approximation score A0

Also note that the actual value is 0.032865...



Some may use integration by parts in (b) and the following scheme should be applied.

Integration by parts in (b):

Either by taking $u = 6x$ and $\frac{dv}{dx} = \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)$

$$\begin{aligned}\int 6x \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) dx &= 6x \times \left(\frac{1}{9}x - \frac{x^2}{27} + \frac{x^3}{81}\right) - 6 \int \left(\frac{1}{9}x - \frac{x^2}{27} + \frac{x^3}{81}\right) dx \\ &= 6x \left(\frac{1}{9}x - \frac{x^2}{27} + \frac{x^3}{81}\right) - \left(\frac{1}{3}x^2 - \frac{2x^3}{27} + \frac{6x^4}{324}\right)\end{aligned}$$

M1: A full attempt at integration by parts. This requires:

$$\int 6x \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) dx = kx \times f(x) - k \int f(x) dx = kx \times f(x) - kg(x)$$

Where $f(x)$ is an attempt to integrate their expansion from (a) with $x^n \rightarrow x^{n+1}$ at least once

and $g(x)$ is an attempt to integrate their $f(x)$ with $x^n \rightarrow x^{n+1}$ at least once

A1: Fully correct integration. Then **dM1A1** as in the main scheme

Or by taking $u = \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)$ and $\frac{dv}{dx} = 6x$

$$\begin{aligned}\int 6x \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) dx &= 3x^2 \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) - \int 3x^2 \times \left(-\frac{2}{27} + \frac{2x}{27}\right) dx \\ &= 3x^2 \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) - \int \left(\frac{6x^3}{27} - \frac{6x^2}{27}\right) dx = 3x^2 \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) - \left(\frac{6x^4}{108} - \frac{6x^3}{81}\right)\end{aligned}$$

M1: A full attempt at integration by parts. This requires:

$$\int 6x \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) dx = kx^2 \times f(x) - k \int x^2 g(x) dx = kx^2 \times f(x) - kh(x)$$

Where $f(x)$ is their expansion from (a) and $g(x)$ is an attempt to differentiate their $f(x)$ with

$x^n \rightarrow x^{n-1}$ at least once **and** $h(x)$ is an attempt to integrate their $x^2 g(x)$ with $x^n \rightarrow x^{n+1}$ at

least once

A1: Fully correct integration. Then **dM1A1** as in the main scheme



(c)	Overall problem-solving mark (see notes)	M1	3.1a
	$u = 3 + x \Rightarrow \int_{3.2}^{3.4} f(u) \, du \Rightarrow \int_{3.2}^{3.4} \frac{6(u-3)}{u^2} \, du = \int_{3.2}^{3.4} \frac{6}{u} - \frac{18}{u^2} \, du \Rightarrow \dots \ln u + \dots u^{-1}$	M1	1.1b
	$\int_{3.2}^{3.4} \frac{6(u-3)}{u^2} \, du = \int_{3.2}^{3.4} \frac{6}{u} - \frac{18}{u^2} \, du \Rightarrow 6 \ln u + 18u^{-1}$	A1	1.1b
	$\left[6 \ln u + 18u^{-1} \right]_{3.2}^{3.4} = \left(6 \ln 3.4 + \frac{18}{3.4} \right) - \left(6 \ln 3.2 + \frac{18}{3.2} \right) = \dots$	ddM1	1.1b
	$6 \ln \left(\frac{17}{16} \right) - \frac{45}{136} \text{ oe}$	A1	2.1
		(5)	
(c) Alt 1	Overall problem-solving mark (see notes)	M1	3.1a
	$\int 6x(3+x)^{-2} \, dx = \frac{\dots x}{3+x} \pm \dots \int (3+x)^{-1} \, dx = \frac{\dots x}{3+x} \pm \dots \ln(3+x) \text{ oe}$	M1	1.1b
	$= 6 \ln(3+x) - \frac{6x}{3+x} \text{ oe}$	A1	1.1b
	$\left(6 \ln(3+0.4) - \frac{6(0.4)}{3+0.4} \right) - \left(6 \ln(3+0.2) - \frac{6(0.2)}{3+0.2} \right) = \dots$	ddM1	1.1b
	$6 \ln \left(\frac{17}{16} \right) - \frac{45}{136} \text{ oe}$	A1	2.1

(c) Alt 2	Overall problem-solving mark (see notes)	M1	3.1a
	$\int 6x(3+x)^{-2} \, dx = \int \left(\frac{\dots}{(3+x)} + \frac{\dots}{(3+x)^2} \right) \, dx = \dots \ln(3+x) + \frac{\dots}{3+x} \text{ oe}$	M1	1.1b
	$= 6 \ln(3+x) + \frac{18}{3+x} \text{ oe}$	A1	1.1b
	$\left(6 \ln(3+0.4) + \frac{18}{3+0.4} \right) - \left(6 \ln(3+0.2) + \frac{18}{3+0.2} \right) = \dots$	ddM1	1.1b
	$6 \ln \left(\frac{17}{16} \right) - \frac{45}{136} \text{ oe}$	A1	2.1

(13 marks)**Notes****(c) There are various methods which can be used****M1:** An overall problem-solving mark for **all of**

- using an appropriate integration technique e.g. substitution, by parts or partial fractions – note that this may not be correct but mark positively if they have tried one of these approaches
- integrates one of their terms to a natural logarithm, e.g., $\frac{a}{3+x} \rightarrow b \ln(3+x)$ or $\frac{a}{u} \rightarrow b \ln u$
- substitutes in correct limits and subtracts either way round

M1: Integrates to achieve an expression of the required form for their chosen method

- substitution: $u = x + 3 \rightarrow \pm \frac{a}{u} \pm b \ln u$ or e.g. $u = (x+3)^2 \rightarrow \pm \frac{a}{\sqrt{u}} \pm b \ln u$
- parts: $\pm a \ln(3+x) \pm \frac{bx}{3+x}$ condone missing brackets e.g. $\dots \ln x + 3$ for $\dots \ln(3+x)$
- partial fractions: $\pm a \ln(3+x) \pm \frac{b}{3+x}$ condone missing brackets e.g. $\dots \ln 3 + x$ for $\dots \ln(3+x)$



A1: Correct integration for their method e.g.

- substitution: $u = x + 3 \rightarrow 6 \ln u + 18u^{-1}$ or e.g. $u = (x + 3)^2 \rightarrow 3 \ln u + \frac{18}{\sqrt{u}}$
- parts: $6 \ln(3 + x) - \frac{6x}{3 + x}$
- partial fractions: $6 \ln(3 + x) + \frac{18}{3 + x}$ or e.g. $3 \ln(9 + 6x + x^2) + \frac{18}{3 + x}$

Note that the above terms may appear “separated” but must be correct with the correct signs.
(ignore any reference to a constant of integration)

Do not condone missing brackets e.g. $6 \ln x + 3$ for $6 \ln(3 + x)$ unless they are implied by later work.

ddM1: Substitutes in the correct limits for their integral and subtracts either way round to find a value

Depends on both previous method marks.

For evidence of using the correct limits we do not expect examiners to check so allow this mark if they obtain a value with (minimal) evidence of the use of the appropriate limits.

This could be e.g. $[f(x)]_{0.2}^{0.4} = \dots$ provided both previous M marks were scored.

Note that for substitution they may revert back to $3 + x$ and so should be using 0.4 and 0.2

A1: A full and rigorous argument leading to $6 \ln\left(\frac{17}{16}\right) - \frac{45}{136}$ or exact equivalent e.g. $3 \ln\left(\frac{289}{256}\right) - \frac{45}{136}$ or

e.g. $-6 \ln\left(\frac{16}{17}\right) - \frac{45}{136}$

The brackets are not required around the $\frac{17}{16}$ and allow exact equivalents e.g. allow 1.0625 or $1\frac{1}{16}$

but not e.g. $\frac{3.4}{3.2}$. The $\frac{45}{136}$ must be exact or an exact equivalent. Also allow e.g. $6 \ln\left|\frac{17}{16}\right| - \frac{45}{136}$

Ignore spurious integral signs that may appear as part of their solution.

Question	Scheme	Marks	AOs
3	$\log_3(12y+5) - \log_3(1-3y) = 2 \Rightarrow \log_3 \frac{12y+5}{1-3y} = 2$ or e.g. $2 = \log_3 9$	B1 M1 on EPEN	1.1b
	$\log_3 \frac{12y+5}{1-3y} = 2 \Rightarrow \frac{12y+5}{1-3y} = 3^2 \Rightarrow 9 - 27y = 12y + 5 \Rightarrow y = \dots$ or e.g. $\log_3(12y+5) = \log_3(3^2(1-3y)) \Rightarrow (12y+5) = 3^2(1-3y) \Rightarrow y = \dots$	M1	2.1
	$y = \frac{4}{39}$	A1	1.1b
		(3)	
(3 marks)			
Notes			
B1(M1 on EPEN): Applies at least one addition or subtraction law of logs correctly. Can also be awarded for using $2 = \log_3 9$. This may be implied by e.g. $\log_3 \dots = 2 \Rightarrow \dots = 9$ M1: A rigorous argument with no incorrect working to remove the log or logs correctly and obtain a <u>correct</u> equation in any form and solve for y. A1: Correct exact value. Allow equivalent fractions.			

Guidance on how to mark particular cases:

$$\log_3(12y+5) - \log_3(1-3y) = 2 \Rightarrow \frac{\log_3(12y+5)}{\log_3(1-3y)} = 2$$

$$\Rightarrow \frac{12y+5}{1-3y} = 3^2 \Rightarrow 9 - 27y = 12y + 5 \Rightarrow y = \frac{4}{39}$$

B1M0A0

$$\log_3(12y+5) - \log_3(1-3y) = 2 \Rightarrow \frac{\log_3(12y+5)}{\log_3(1-3y)} = 2 \Rightarrow \log_3 \frac{12y+5}{1-3y} = 2$$

$$\Rightarrow \frac{12y+5}{1-3y} = 3^2 \Rightarrow 9 - 27y = 12y + 5 \Rightarrow y = \frac{4}{39}$$

B1M0A0

$$\log_3(12y+5) - \log_3(1-3y) = 2 \Rightarrow \frac{12y+5}{1-3y} = 3^2 \Rightarrow 9 - 27y = 12y + 5 \Rightarrow y = \frac{4}{39}$$

B1M1A1

Question	Scheme	Marks	AOs
9 (a)	States $\log a - \log b = \log \frac{a}{b}$	B1	1.2
	Proceeds from $\frac{a}{b} = a - b \rightarrow \dots \rightarrow ab - a = b^2$	M1	1.1b
	$ab - a = b^2 \rightarrow a(b - 1) = b^2 \Rightarrow a = \frac{b^2}{b - 1} *$	A1*	2.1
		(3)	
(b)	States either $b > 1$ or $b \neq 1$ with reason $\frac{b^2}{b - 1}$ is not defined at $b = 1$ oe	B1	2.2a
	States $b > 1$ and explains that as $a > 0 \Rightarrow \frac{b^2}{b - 1} > 0 \Rightarrow b > 1$	B1	2.4
		(2)	
(5 marks)			

(a)

B1: States or uses $\log a - \log b = \log \frac{a}{b}$. This may be awarded anywhere in the question and may be implied

by a **starting line** of $\frac{a}{b} = a - b$ oe. Alternatively takes $\log b$ to the rhs and uses the addition law

$\log(a - b) + \log b = \log(a - b)b$. Watch out for $\log a - \log b = \frac{\log a}{\log b} = \log\left(\frac{a}{b}\right)$ which could score 010

M1: Attempts to make 'a' the subject. Awarded for proceeding from $\frac{a}{b} = a - b$ to a point where the two terms in a are on the same side of the equation and the term in b is on the other.

A1*: CSO. Shows clear reasoning and correct mathematics leading to $a = \frac{b^2}{b - 1}$. Bracketing must be correct.

Allow a candidate to proceed from $ab - a = b^2$ to $a = \frac{b^2}{b - 1}$ without the intermediate line.

(b)

B1: For deducing $b \neq 1$ as $a \rightarrow \infty$ oe such as "you cannot divide by 0" or correctly deducing that $b > 1$. They may state that b cannot be less than 1.

B1: For $b > 1$ and explaining that as $a > 0 \Rightarrow \frac{b^2}{b - 1} > 0 \Rightarrow b > 1$ (as b^2 is positive)

As a minimum accept that $b > 1$ as a cannot be negative.

Note that $a > b > 1$ is a correct statement but not sufficient on its own without an explanation.

Alt (a)

Note that it is possible to attempt part (a) by substituting $a = \frac{b^2}{b - 1}$ into both sides of the given identity.



$$\log a - \log b = \log(a - b) \Rightarrow \log\left(\frac{b^2}{b-1}\right) - \log b = \log\left(\frac{b^2}{b-1} - b\right)$$

B1: Score for $\log\left(\frac{b^2}{b-1}\right) - \log b = \log\left(\frac{b}{b-1}\right)$

M1: Attempts to write $\frac{b^2}{b-1} - b$ as a single fraction $\frac{b^2}{b-1} - b = \frac{b^2 - b(b-1)}{b-1}$

Allow as two separate fractions with the same common denominator

A1*: Achieves lhs and rhs as $\log\left(\frac{b}{b-1}\right)$ **and** makes a comment such as "hence true"

Question	Scheme	Marks	AOs	
11 (a) Way 1	$\{y = x^x \Rightarrow\} \ln y = x \ln x$	B1	1.1a	
	$\frac{1}{y} \frac{dy}{dx} = 1 + \ln x$	M1	1.1b	
		A1	2.1	
	$\left\{ \frac{dy}{dx} = 0 \Rightarrow \right\} \frac{x}{x} + \ln x = 0 \text{ or } 1 + \ln x = 0 \Rightarrow \ln x = k \Rightarrow x = \dots$	M1	1.1b	
	$x = e^{-1} \text{ or awrt } 0.368$	A1	1.1b	
	Note: $k \neq 0$	(5)		
(a) Way 2	$\{y = x^x \Rightarrow\} y = e^{x \ln x}$	B1	1.1a	
	$\frac{dy}{dx} = \left(\frac{x}{x} + \ln x \right) e^{x \ln x}$	M1	1.1b	
		A1	2.1	
	$\left\{ \frac{dy}{dx} = 0 \Rightarrow \right\} \frac{x}{x} + \ln x = 0 \text{ or } 1 + \ln x = 0 \Rightarrow \ln x = k \Rightarrow x = \dots$	M1	1.1b	
	$x = e^{-1} \text{ or awrt } 0.368$	A1	1.1b	
	Note: $k \neq 0$	(5)		
(b) Way 1	Attempts both $1.5^{1.5} = 1.8\dots$ and $1.6^{1.6} = 2.1\dots$ and at least one result is correct to awrt 1 dp	M1	1.1b	
	$1.8\dots < 2$ and $2.1\dots > 2$ and as C is continuous then $1.5 < \alpha < 1.6$	A1	2.1	
		(2)		
(c)	Attempts $x_{n+1} = 2x_n^{1-x_n}$ at least once with $x_1 = 1.5$	M1	1.1b	
	Can be implied by $2(1.5)^{1-1.5}$ or awrt 1.63			
	$\{x_4 = 1.67313\dots \Rightarrow\} x_4 = 1.673$ (3 dp) cao	A1	1.1b	
		(2)		
(d)	Give 1 st B1 for any of - oscillates - periodic - non-convergent - divergent - fluctuates - goes up and down 1, 2, 1, 2, 1, 2 - alternates (condone)	Give B1 B1 for any of - periodic {sequence} with period 2 - oscillates between 1 and 2	B1	2.5
		Condone B1 B1 for any of - fluctuates between 1 and 2 - keep getting 1, 2 - alternates between 1 and 2 - goes up and down between 1 and 2 1, 2, 1, 2, 1, 2, ...	B1	2.5
			(2)	
(11 marks)				
Note	A common solution A maximum of 3 marks (i.e. B1 1 st M1 and 2 nd M1) can be given for the solution $\log y = x \log x \Rightarrow \frac{1}{y} \frac{dy}{dx} = 1 + \log x$ $\left\{ \frac{dy}{dx} = 0 \Rightarrow \right\} 1 + \log x = 0 \Rightarrow x = 10^{-1}$ 1st B1 for $\log y = x \log x$ 1st M1 for $\log y \rightarrow \lambda \frac{1}{y} \frac{dy}{dx}; \lambda \neq 0 \text{ or } x \log x \rightarrow 1 + \log x \text{ or } \frac{x}{x} + \log x$ 2nd M1 can be given for $1 + \log x = 0 \Rightarrow \log x = k \Rightarrow x = \dots; k \neq 0$			

Question	Scheme	Marks	AOs
11 (b) Way 2	For $x^x - 2$, attempts both $1.5^{1.5} - 2 = -0.16...$ and $1.6^{1.6} - 2 = 0.12...$ and at least one result is correct to awrt 1 dp	M1	1.1b
	$-0.16... < 0$ and $0.12... > 0$ and as C is continuous then $1.5 < \alpha < 1.6$	A1	2.1
		(2)	
11 (b) Way 3	For $\ln y = x \ln x$, attempts both $1.5 \ln 1.5 = 0.608...$ and $1.6 \ln 1.6 = 0.752...$ and at least one result is correct to awrt 1 dp	M1	1.1b
	$0.608... < 0.69...$ and $0.752... > 0.69...$ and as C is continuous then $1.5 < \alpha < 1.6$	A1	2.1
		(2)	
11 (b) Way 4	For $\log y = x \log x$, attempts both $1.5 \log 1.5 = 0.264...$ and $1.6 \log 1.6 = 0.326...$ and at least one result is correct to awrt 2 dp	M1	1.1b
	$0.264... < 0.301...$ and $0.326... > 0.301...$ and as C is continuous then $1.5 < \alpha < 1.6$	A1	2.1
		(2)	
Notes for Question 11			
(a)	Way 1		
B1:	$\ln y = x \ln x$. Condone $\log_x y = x \log_x x$ or $\log_x y = x$		
M1:	For either $\ln y \rightarrow \frac{1}{y} \frac{dy}{dx}$ or $x \ln x \rightarrow 1 + \ln x$ or $\frac{x}{x} + \ln x$		
A1:	Correct differentiated equation. i.e. $\frac{1}{y} \frac{dy}{dx} = 1 + \ln x$ or $\frac{1}{y} \frac{dy}{dx} = \frac{x}{x} + \ln x$ or $\frac{dy}{dx} = y(1 + \ln x)$ or $\frac{dy}{dx} = x^x(1 + \ln x)$		
M1:	Sets $1 + \ln x = 0$ and rearranges to make $\ln x = k \Rightarrow x = ...$; k is a constant and $k \neq 0$		
A1:	$x = e^{-1}$ or awrt 0.368 only (with no other solutions for x)		
Note:	Give no marks for no working leading to 0.368		
Note:	Give M0 A0 M0 A0 for $\ln y = x \ln x \rightarrow x = 0.368$ with no intermediate working		
(a)	Way 2		
B1:	$y = e^{x \ln x}$		
M1:	For either $y = e^{x \ln x} \Rightarrow \frac{dy}{dx} = f(\ln x) e^{x \ln x}$ or $x \ln x \rightarrow 1 + \ln x$ or $\frac{x}{x} + \ln x$		
A1:	Correct differentiated equation. i.e. $\frac{dy}{dx} = \left(\frac{x}{x} + \ln x\right) e^{x \ln x}$ or $\frac{dy}{dx} = (1 + \ln x) e^{x \ln x}$ or $\frac{dy}{dx} = x^x(1 + \ln x)$		
M1:	Sets $1 + \ln x = 0$ and rearranges to make $\ln x = k \Rightarrow x = ...$; k is a constant and $k \neq 0$		
A1:	$x = e^{-1}$ or awrt 0.368 only (with no other solutions for x)		
Note:	Give B1 M1 A0 M1 A1 for the following solution: $\{y = x^x \Rightarrow \ln y = x \ln x \Rightarrow \frac{dy}{dx} = 1 + \ln x \Rightarrow 1 + \ln x = 0 \Rightarrow x = e^{-1} \text{ or awrt 0.368}$		

Notes for Question 11 Continued	
(b)	Way 1
M1:	Attempts both $1.5^{1.5} = 1.8...$ and $1.6^{1.6} = 2.1...$ and at least one result is correct to awrt 1 dp
A1:	Both $1.5^{1.5} = \text{awrt } 1.8...$ and $1.6^{1.6} = \text{awrt } 2.1...$, reason (e.g. $1.8... < 2$ and $2.1... > 2$ or states C cuts through $y = 2$), C continuous and conclusion
(b)	Way 2
M1:	Attempts both $1.5^{1.5} - 2 = -0.16...$ and $1.6^{1.6} - 2 = 0.12...$ and at least one result is correct to awrt 1 dp
A1:	Both $1.5^{1.5} - 2 = -0.16...$ and $1.6^{1.6} - 2 = 0.12...$ correct to awrt 1 dp, reason (e.g. $-0.16... < 0$ and $0.12... > 0$, sign change or states C cuts through $y = 0$), C continuous and conclusion
(b)	Way 3
M1:	Attempts both $1.5 \ln 1.5 = 0.608...$ and $1.6 \ln 1.6 = 0.752...$ and at least one result is correct to awrt 1 dp
A1:	Both $1.5 \ln 1.5 = 0.608...$ and $1.6 \ln 1.6 = 0.752...$ correct to awrt 1 dp, reason (e.g. $0.608... < 0.69...$ and $0.752... > 0.69...$ or states they are either side of $\ln 2$), C continuous and conclusion.
(b)	Way 4
M1:	Attempts both $1.5 \log 1.5 = 0.264...$ and $1.6 \log 1.6 = 0.326...$ and at least one result is correct to awrt 2 dp
A1:	Both $1.5 \log 1.5 = 0.264...$ and $1.6 \log 1.6 = 0.326...$ correct to awrt 2 dp, reason (e.g. $0.264... < 0.301...$ and $0.326... > 0.301...$ or states they are either side of $\log 2$), C continuous and conclusion.
(c)	
M1:	An attempt to use the given or their formula once. Can be implied by $2(1.5)^{1-1.5}$ or awrt 1.63
A1:	States $x_4 = 1.673$ cao (to 3 dp)
Note:	Give M1 A1 for stating $x_4 = 1.673$
Note:	M1 can be implied by stating their final answer $x_4 = \text{awrt } 1.673$
Note:	$x_2 = 1.63299...$, $x_3 = 1.46626...$, $x_4 = 1.67313...$
(d)	
B1:	see scheme
B1:	see scheme
Note:	Only marks of B1B0 or B1B1 are possible in (d)
Note:	Give B0 B0 for “Converges in a cob-web pattern” or “Converges up and down to α ”

Question	Scheme	Marks	AOs
2	$4^{3p-1} = 5^{210} \Rightarrow (3p-1)\log 4 = 210\log 5$	M1	1.1b
	$\Rightarrow 3p = \frac{210\log 5}{\log 4} + 1 \Rightarrow p = \dots$	dM1	2.1
	$p = \text{awrt } 81.6$	A1	1.1b
		(3)	
(3 marks)			
Notes:			

M1: Takes logs of both sides and uses the power law on **each** side.

Condone a missing bracket on lhs and slips.

Award for any base including ln but the logs must be the same base.

dM1: A full method leading to a value for p .

It is dependent upon the previous M mark and there must be an attempt to change the subject of the equation in the correct order.

Look for $(3p-1)\log 4 = 210\log 5 \Rightarrow 3p = \frac{210\log 5}{\log 4} + 1 \Rightarrow p = \dots$ condoning slips.

You may see numerical versions E.g. $(3p-1) \times 0.60 = 210 \times 0.7 \Rightarrow 1.8p - 0.6 = 147 \Rightarrow p = 82$

Use of incorrect log laws would be dM0. E.g. $(3p-1)\log 4 = 210\log 5 \Rightarrow 3p = 210\log \frac{5}{4} + 1$

A1: awrt 81.6 following a correct method. Bracketing errors can be recovered for full marks

A correct answer with no working scores 0 marks. The demand in the question is clear.

There are alternatives:

E.g. A starting point could be $4^{3p-1} = 5^{210} \Rightarrow \frac{4^{3p}}{4} = 5^{210}$

but the first M mark is still for using the power law correctly on each side

In such a method the dM1 mark is for using **all** log rules correctly and proceeding to a value for p .

Using base 4 or 5

E.g. $4^{3p-1} = 5^{210} \Rightarrow (3p-1) = \log_4 5^{210}$

The M mark is not scored until $(3p-1) = 210\log_4 5$

Question	Scheme	Marks	AOs
3	$3^x = 7^y \rightarrow x \log 3 = y \log 7$	M1	3.1a
	e.g. $\left(\frac{x}{y} = \right) \frac{\log 7}{\log 3}$	A1	1.1b
		(2)	
(2 marks)			
Notes			
Note: Condone absence of reference to the value of $\frac{x}{y}$ being undefined when $x = y = 0$ M1: For the key step in attempting to take logarithms with the same base of both sides and apply the power law to both sides, e.g., $x = y \log_3 7$ or $y = x \log_7 3$ or $x \ln 3 = y \ln 7$ Condone e.g. $x = \log_3 7y$ but not $x = \log_3 (7y)$ Alternatively, takes a root of both sides (either x or y), takes logs with the same base of both sides and applies the power law e.g., $3^x = 7^y \rightarrow 3^{\frac{x}{y}} = 7 \rightarrow \frac{x}{y} \log 3 = \log 7$ May be implied by a correct answer. A1: $\left(\frac{x}{y} = \right) \frac{\log 7}{\log 3}$ or equivalent, e.g., $\frac{\ln 7}{\ln 3}$ or $\log_3 7$ or $\frac{1}{\log_7 3}$ Condone e.g. $\frac{\ln 7 }{\ln 3 }$ Correct answer only scores both marks provided there is no incorrect log work. Any base k may be used for e.g. $\frac{\log_k 7}{\log_k 3}$ provided it is the same base in both logs, although watch out for e.g. $\left(\frac{\log_3 7}{\log_7 3}\right)^{\frac{1}{2}}$ or $\frac{\log_3 k}{\log_7 k}$ for constant $k > 0$ which are exceptions and correct. There is no need to see $\frac{x}{y} =$ but it should be clear what their answer is. Do not ISW if they incorrectly apply log laws e.g. $\frac{\log 7}{\log 3} = \log \frac{7}{3}$ or $= \log 7 - \log 3$ or $= \log (7 - 3)$ all of which score M1A0. You may ISW after a correct answer if they go on to provide a decimal approximation. A decimal approximation with no correct log work seen (1.7712...) scores no marks.			

Question	Scheme	Marks	AOs
13(a)	$\log_{10} b = 0.0054 \Rightarrow b = 10^{0.0054}$ or $\log_{10} a = 0.81 \Rightarrow a = 10^{0.81}$	M1	3.1a
	$b = 1.01$ or $a = 6.46$	A1	1.1b
	$\log_{10} b = 0.0054 \Rightarrow b = 10^{0.0054}$ and $\log_{10} a = 0.81 \Rightarrow a = 10^{0.81}$	M1	2.1
	$b = 1.013$ and $a = 6.457$	A1	1.1b
		(4)	
(b)(i)	e.g. The world population in billions in 2004	B1ft	3.2a
(ii)	$b = 1.013$ represents the scale factor of the <u>yearly increase</u> in the world population	B1ft	3.2a
		(2)	
(c)	$P = 6.457 \dots (1.013 \dots)^{26}$ or e.g. $\log P = 0.81 + 26 \times 0.0054 \Rightarrow P = \dots$	M1	3.4
	awrt 9 billion	A1	2.2b
		(2)	
(d)	Not reliable since the data used for the model covered the years 2004 – 2007 and it would not be sensible to assume that the model still holds in 2030	B1	3.2b
		(1)	
(9 marks)			
Notes			
(a) Must be using base 10 in (a). Ignore any units associated with a and b in part (a). M1: Correct strategy to get a numerical expression or value for a or b e.g. $a = 10^{0.81}$ or $b = 10^{0.0054}$. This may be implied by $a = \text{awrt } 6.46$ or $b = \text{awrt } 1.01$ if no incorrect work is seen. A1: Correct value for a or b. Allow 3 sf for this mark so allow $a = \text{awrt } 6.46$ or $b = \text{awrt } 1.01$. May be seen embedded in their formula. M1: Correct strategy to get a numerical expression or value for a and b e.g. $a = 10^{0.81}$ and $b = 10^{0.0054}$. This may be implied by $a = \text{awrt } 6.46$ and $b = \text{awrt } 1.01$ if no incorrect work is seen. A1: Correct values. This requires $a = \text{awrt } 6.457$ and $b = \text{awrt } 1.013$ for this mark. May be seen embedded in their formula. Is w once correct answers are seen. Special case: Constants the wrong way round: $a = 1.013$ and $b = 6.457$ with or without working scores M1A1M1A0 unless the equation is formed correctly in which case the final A mark can be recovered. Note that having found the value of a, it is possible to find b by substituting e.g. $t = 1$ as follows: $a = 10^{0.81} = 6.457 \quad t = 1 \Rightarrow P = ab \Rightarrow b = \frac{P}{a}$ $t = 1 \Rightarrow \log_{10} P = 0.0054 + 0.81 = 0.8154 \Rightarrow P = 10^{0.8154} \Rightarrow b = \frac{P}{a} = \frac{10^{0.8154}}{6.457} = 1.013$			

Note that a misread of 0.0054 as 0.054 is quite common and may score 1110 as it does not simplify the question.

(b)(i) Follow through their a .

B1ft: Correct interpretation for a but must reference “billions”.

Allow equivalent alternatives e.g.

- The original/initial population **in billions**
- The population in 2004 was “6.46” **billion**

(b)(ii) Follow through their b .

B1ft: Correct interpretation for b but must reference “each year” or e.g. “yearly” or

Allow equivalent alternatives e.g.

- The proportional increase/change in each year.
- The population will rise by “1.3%” each year. Must follow their value for b .
- The rate/factor at which the population is rising/increasing/changing per annum.
- “1.013” is the multiplier representing the year on year increase.

Do **not** accept

- The amount it is rising
- How much it is rising
- The rate the population increases
- The percentage increase each year
- The rate of increase in billions annually

If they are not labelled (b)(i) and (b)(ii) mark in the order given but accept any way round as long as clearly labelled “ a is.....” and “ b is”

(c)

M1: Substitutes $t = 25$ or 26 or 27 into their model to find a value for P

Must be using **their** a and b correctly in $P = ab^t$

May be implied by sight of “9” or 9 billion if no incorrect working is seen.

A1: Correct value including units (allow awrt 9 billion) **from a correct model but condone incorrect/premature rounding or truncating in an otherwise correct model that leads to the correct value of awrt 9 billion.**

Allow e.g. awrt 9 000 000 000 or e.g. awrt 9×10^9

Just awrt 9 without the “billions” is A0

(d)

B1: The response must refer to the fact that the answer is unreliable together with a reference to the fact that the data used for the model is a long way from 2030

Examples:

- Not good as 2030 is a long way from 2004 – 2007
- Unreliable as based on old data
- Questionable as it has been extrapolated over a long time
- Not reliable due to how far out we have extrapolated
- By the time 2030 arrives it will be unreliable

But not e.g.

- Unreliable, extrapolation
- Not good as outside the range
- Not good as the population rises 101.3% each year
- Disease may happen
- Reliable as based on old data

Question	Scheme	Marks	AOs
4(a)	$(A =) 55$	B1	3.4
		(1)	
(b)	$\left\{ \frac{dH}{dt} = \right\} -A\text{Be}^{-Bt}$ or $\left\{ \frac{dH}{dt} = \right\} -"55" \text{Be}^{-Bt}$	M1	3.1b
	$-B \times "55" = -7.5 \Rightarrow B = \dots \left(\frac{3}{22} = \text{awrt } 0.136 \right)$	M1	1.1b
	$H = 55\text{e}^{-0.136t} + 30$	A1cso	3.3
		(3)	
(4 marks)			
Notes			
(a) B1: 55 only. Just look for this value e.g. “A =” is not required. Ignore any “units” if given e.g. 55 °C (b) M1: Differentiates to obtain an expression of the form $\pm A\text{Be}^{-Bt}$ which may have their A already substituted in so allow for $\pm A\text{Be}^{-Bt}$ or $\pm "55" \text{Be}^{-Bt}$ M1: Substitutes $t = 0$ and their A into their $\frac{dH}{dt}$, sets $= \pm 7.5$ and proceeds to find a value for B which may be implied by $\frac{3}{22}$ or awrt 0.136 Their $\frac{dH}{dt}$ must not be H. i.e. it must be a “changed” function. A1cso: Correct equation which follows fully correct work $H = 55\text{e}^{-0.136t} + 30$ but condone $H = 55\text{e}^{-\frac{3}{22}t} + 30$ The final equation must be correct but you can ignore spurious notation within their solution such as integral signs and “+ c” which do not affect their solution. <u>Marking guidance is as follows for particular cases in (b)</u> Case 1: $\left\{ \frac{dH}{dt} = \right\} -"55" \text{Be}^{-Bt}$, $- "55" \text{Be}^{-Bt} = 7.5 \Rightarrow B = -0.136 \Rightarrow H = 55\text{e}^{-0.136t} + 30$ scores M1M1A0 Error: it should be – 7.5 Case 2: $\left\{ \frac{dH}{dt} = \right\} "55" \text{Be}^{-Bt}$, $"55" \text{Be}^{-Bt} = -7.5 \Rightarrow B = -0.136 \Rightarrow H = 55\text{e}^{-0.136t} + 30$ scores M1M1A0 Error: incorrect derivative Case 3: $\left\{ \frac{dH}{dt} = \right\} "55" \text{Be}^{-Bt}$, $"55" \text{Be}^{-Bt} = 7.5 \Rightarrow B = 0.136 \Rightarrow H = 55\text{e}^{-0.136t} + 30$ scores M1M1A0 Error: incorrect derivative Case 4: $\left\{ \frac{dH}{dt} = \right\} -"55" \text{Be}^{-Bt}$, $"55" B = 7.5 \Rightarrow B = 0.136 \Rightarrow H = 55\text{e}^{-0.136t} + 30$ scores M1M1A1 No errors			



Question	Scheme	Marks	AOs
9(a)	Deduces that $A = \pm 50$ or $b = \frac{1}{4}$	B1	3.4
	Deduces that $A = \pm 50$ and $b = \frac{1}{4}$	B1	3.4
	Uses $t = 0, H = 1 \Rightarrow \alpha = \dots$ E.g. $1 = 50 \sin(\alpha)^\circ \Rightarrow \alpha = \dots$	M1	3.4
	$H = \left \pm 50 \sin\left(\frac{1}{4}t + 1.15\right)^\circ \right $	A1	3.3
		(4)	
(b)	E.g. the minimum height above the ground of the passenger on the original model was 0 m or Adding “d” means the passenger does not touch the ground.	B1	3.5b
		(1)	
(5 marks)			
Notes:			

(a) Note that B0B1 is not possible

B1: Uses the equation of the given model to deduce that $A = \pm 50$ **or** $b = \frac{1}{4}$ o.e.

May be seen embedded within their equation.

B1: Uses the equation of the given model to deduce that $A = \pm 50$ **and** $b = \frac{1}{4}$ o.e.

May be seen embedded within their equation.

M1: Uses $t = 0$ and $H = 1$ in the equation of the model to find a value for α .

Follow through on their value for A . Allow for $\pm 1 = 50 \sin(\alpha)^\circ \Rightarrow \alpha = \dots$ where α is in degrees or radians.

Note that in radians $\sin^{-1}\left(\frac{1}{50}\right) \approx \frac{1}{50}$ (0.0200...) which may appear incorrect but is in fact ok.

Also in degrees a value of e.g. 1.14 (truncated) would indicate the method.

A1: Writes down the correct full equation of the model: $H = \left| \pm 50 \sin\left(\frac{1}{4}t + 1.15\right)^\circ \right|$ o.e.

Condone omission of degrees symbol and allow awrt 1.15 for α .

Allow if a correct equation is seen anywhere in their solution.

(b)

B1: Gives a suitable explanation with no contradictory statements.

Condone “so that pod/capsule/seat/passenger/ferris wheel/it etc. will not hit/touch the ground”

Responses that focus on the starting point of the model are likely to score B0

Question	Scheme	Marks	AOs
5 (a)	Attempts to use $h^2 = at + b$ with either $t = 2, h = 2.6$ or $t = 10, h = 5.1$	M1	3.1b
	Correct equations $2a + b = 6.76$ $10a + b = 26.01$	A1	1.1b
	Solves simultaneously to find values for a and b	dM1	1.1b
	$h^2 = 2.41t + 1.95$ cao	A1	3.3
		(4)	
(b)	Substitutes $t = 20$ into their $h^2 = 2.41t + 1.95$ and finds h or h^2 Or substitutes $h = 7$ into their $h^2 = 2.41t + 1.95$ and finds t	M1	3.4
	Compares the model with the true values and concludes "good model" with a minimal reason E.g. I Finds $h = 7.08$ (m) and states that it is a good model as 7.08 (m) is close to 7 (m) E.g II Finds $t = 19.5$ years and states that the model is accurate as 19.5 (years) ≈ 20 (years)	A1	3.5a
		(2)	
(6 marks)			
Notes:			



(a)

M1: For translating the problem into mathematics. Attempts to use the given equation o.e. with either of the pieces of information to form one correct equation.

Award for unsimplified equations as well, such as $2.6^2 = 2a + b$ or $2.6 = \sqrt{2a + b}$

A1: Two correct (and different) equations which may be unsimplified

dM1: Solves simultaneously to find values for a and b . It is dependent upon the previous M

Don't be too concerned with the process here as calculators may be used.

Score if values of a and b are reached from a pair of simultaneous equations

A1: Establishes **the full equation of the model** with values of a and b given to **exactly** 3sf. Award if seen in either (a) or (b). It is not scored for the values of a and b .

Allow either $h^2 = 2.41t + 1.95$ or $h = \sqrt{2.41t + 1.95}$

If they go on to square root each term from $h^2 = 2.41t + 1.95$ then it is A0. E.g. $h = 1.55t + 1.40$

Special case for candidates who mistakenly use $h = at + b$

For $2.6 = 2a + b$, $5.1 = 10a + b \Rightarrow h = 0.3125t + 1.975$ or $h = 0.313t + 1.98$

can score M1 correct equations with attempt to solve and A1 for either correct answer shown above. These are the only marks available to them for a maximum mark of 1100 00

(b)

M1: A full and valid attempt to

either substitute $t = 20$ into their $h^2 = 2.41t + 1.95$ o.e. and find a value for h or h^2

or substitute $h = 7$ into their $h^2 = 2.41t + 1.95$ o.e. and find a value for t

(to enable the candidate to compare real life data with that of the model.)

The equation of the model must be of the correct form, either $h^2 = at + b$ or $h = \sqrt{at + b}$

Do not be too concerned with the mechanics of the solution but the square or $\sqrt{\quad}$ must have been used appropriately to enable the comparison to be made.

In cases with no working you will need to check the calculation

A1: Compares their $h=7.08\text{m}$ to 7m o.e using h^2 or their $t=19.5$ years to 20 years and makes valid conclusion with reason.

For this mark you require

- a statement that it is a "good" or "accurate" model or similar wording
- a reason such as "the values are close", "the values are similar" or "the predicted values are within 5% of the true values."
- a model with equation $h^2 = at + b$ o.e. where $a = \text{awrt } 2.4$ and $b \in [1.9, 2.0]$
- correct calculations

Condone a statement like 'the model is pretty accurate as it predicted 7.08m and the actual value is 7m'

Do not allow incorrect statements such as the model is incorrect as it does not give 7 metres.

Do not allow just "the model gives an underestimate of the true value."

Do not allow 'bad' or 'poor' model



Question	Scheme	Marks	AOs
9(a)	$t = 0, \theta = 18 \Rightarrow 18 = A - B$ or $t = 10, \theta = 44 \Rightarrow 44 = A - Be^{-0.7}$	M1	3.1b



Question	Scheme	Marks	AOs
6 (a)	$R = \sqrt{5}$	B1	1.1b
	$\tan \alpha = 2 \Rightarrow \alpha = \dots$	M1	1.1b
	$\alpha = 1.107$	A1	1.1b
		(3)	
	$\theta = 5 + \sqrt{5} \sin\left(\frac{\pi t}{12} + 1.107 - 3\right)$		
(b)	$(5 + \sqrt{5})^\circ\text{C}$ or awrt 7.24°C	B1ft	2.2a
		(1)	
(c)	$\frac{\pi t}{12} + 1.107 - 3 = \frac{\pi}{2} \Rightarrow t =$	M1	3.1b
	$t = \text{awrt } 13.2$	A1	1.1b
	Either 13:14 or 1:14 pm or 13 hours 14 minutes after midnight.	A1	3.2a
		(3)	
(7 marks)			
Notes:			

(a)**B1:** $R = \sqrt{5}$ only.**M1:** Proceeds to a value of α from $\tan \alpha = \pm 2$, $\tan \alpha = \pm \frac{1}{2}$, $\sin \alpha = \pm \frac{2}{\text{"R"}}$ OR $\cos \alpha = \pm \frac{1}{\text{"R"}}$

It is implied by either awrt 1.11 (radians) or 63.4 (degrees)

A1: $\alpha = \text{awrt } 1.107$ **(b)****B1ft:** Deduces that the maximum temperature is $(5 + \sqrt{5})^\circ\text{C}$ or awrt 7.24°C Remember to isw
Condone a lack of units. Follow through on their value of R so allow $(5 + \text{"R"})^\circ\text{C}$ **(c)****M1:** An complete strategy to find t from $\frac{\pi t}{12} \pm 1.107 - 3 = \frac{\pi}{2}$.

Follow through on their 1.107 but the angle must be in radians.

It is possible via degrees but only using $15t \pm 63.4 - 171.9 = 90$ **A1:** awrt $t = 13.2$ **A1:** The question asks for the time of day so accept either 13:14, 1:14 pm, 13 hours 14 minutes after midnight, 13h 14, or 1 hour 14 minutes after midday. If in doubt use review.....
It is possible to attempt parts (b) and (c) via differentiation but it is unlikely to yield correct results.

$$\frac{d\theta}{dt} = \frac{\pi}{12} \cos\left(\frac{\pi t}{12} - 3\right) - \frac{2\pi}{12} \sin\left(\frac{\pi t}{12} - 3\right) = 0 \Rightarrow \tan\left(\frac{\pi t}{12} - 3\right) = \frac{1}{2} \Rightarrow t = 13.23 = 13:14 \text{ scores M1 A1 A1}$$

$$\frac{d\theta}{dt} = \cos\left(\frac{\pi t}{12} - 3\right) - 2 \sin\left(\frac{\pi t}{12} - 3\right) = 0 \Rightarrow \tan\left(\frac{\pi t}{12} - 3\right) = \frac{1}{2} \Rightarrow t = 13.23 = 13:14 \text{ they can score M1 A0 A1 (SC)}$$

A value of $t = 1.23$ implies the minimum value has been found and therefore incorrect method M0.
.....

Question	Scheme	Marks	AOs
7 (a)	Uses a model $V = Ae^{\pm kt}$ oe (See next page for other suitable models)	M1	3.3
	Eg. Substitutes $t = 0, V = 20\,000 \Rightarrow A = 20\,000$	M1	1.1b
	Eg. Substitutes $t = 1, V = 16\,000 \Rightarrow 16\,000 = 20\,000e^{-1k} \Rightarrow k = ..$	dM1	3.1b
	$V = 20\,000e^{-0.223t}$	A1	1.1b
		(4)	
(b)	Substitutes $t = 10$ in their $V = 20\,000e^{-0.223t} \Rightarrow V = (£\,2150)$	M1	3.4
	Eg. The model is reliable as $£2150 \approx £2000$	A1	3.5a
		(2)	
(c)	Make the "-0.223" less negative. Alt: Adapt model to for example $V = 18\,000e^{-0.223t} + 2000$	B1ft	3.3
		(1)	
(7 marks)			

(a) Option 1

M1: For $V = Ae^{\pm kt}$ Do not allow if k is fixed, eg $k = -0.5$

Condone different variables $V \leftrightarrow y$ $t \leftrightarrow x$ for this mark, but for A1 V and t must be used.

M1: Substitutes $t = 0 \Rightarrow A = 20\,000$ into their exponential model

Candidates may start by simply writing $V = 20\,000e^{kt}$ which would be M1 M1

dM1: Substitutes $t = 1 \Rightarrow 16\,000 = 20\,000e^{-1k} \Rightarrow k = ..$ via the correct use of logs.

It is dependent upon both previous M's.

A1: $V = 20\,000e^{-0.223t}$ (with accuracy to at least 3sf) or $V = 20\,000e^{t \ln 0.8}$

A correct linking formula with correct constants must be seen somewhere in the question

(b)

M1: Uses a model of the form $V = Ae^{\pm kt}$ to find the value of V when $t = 10$.

Alternatively substitutes $V = 2000$ into their model and finds t

A1: This can only be scored from an acceptable model with correct constants with accuracy to at least 2sf.

Compares $V = (£)\,2150$ with $(£)\,2\,000$ and states "reliable as $2150 \approx 2000$ " or "reasonably good as they are close" or "OK but a little high".

Allow a candidate to argue that it is unreliable as long as they state a suitable reason. Eg. "It is too far away from £2000" or "It is over £100 away, so it is not good"

Do not allow "it is not a good model because it is not the same"

In the alternative it is for comparing their value of t with 10 and making a suitable comment as to the reliability of their model with a reason.

$V = 20\,000e^{-0.223t} \Rightarrow 2000 = 20\,000e^{-0.223t} \Rightarrow t = 10.3$ years.

Deduction Reliable model as the time is approximately the same as 10 years. A candidate can argue that the model is unreliable if they can give a suitable reason.

(c)

B1ft: For a correct statement. Eg states that the value of their '-0.223' should become less negative.

Alt states that the value of their '0.223' should become smaller. If they refer to k then refer to the model and apply the same principles.

Condone the fact that they don't state their -0.223 doesn't lie in the range $(-0.223, 0)$



(a) Option 2
M1: For $V = Ar^t$ or equivalent such as $V = kr^{t-1}$

 Condone different variables $V \leftrightarrow y$ $t \leftrightarrow x$ for this mark, but for A1 V and t must be used.

M1: Uses $t = 0 \Rightarrow A = 20\,000$ in their model. Alternatively uses $(0, 20\,000)$ and $(1, 16\,000)$ to give $r = \frac{4}{5}$ oe

 You may award if one of the number pair $(0, 20\,000)$ or $(1, 16\,000)$ works in an allowable model

dM1: $t = 1 \Rightarrow 16\,000 = 20\,000r^1 \Rightarrow r = \dots$ Dependent upon both previous M's

 In the alternative it would be for using $r = \frac{4}{5}$ with one of the points to find $A = 20\,000$

 You may award if both number pairs $(0, 20\,000)$ or $(1, 16\,000)$ work in an allowable model

A1: $V = 20\,000 \times 0.8^t$ Note that $V = 20\,000 \times 1.25^{-t}$ $V = 16\,000 \times 0.8^{t-1}$ and is also correct

(b)
M1: Uses a model of the form $V = Ar^t$ oe to find the value of V when $t = 10$. Eg. $20\,000 \times 0.8^{10}$

 Alternatively substitutes $V = 2000$ into their model and finds t
A1: This can only be scored from an acceptable model with correct constants also allowing an accuracy to 2sf. Compares (£) 2147 with (£) 2 000 and states "reliable as $2147 \approx 2000$ " or "reasonably good as they are close" or "OK but a little high".

Allow a candidate to argue that it is unreliable as long as they state a suitable reason. Eg. "It is too far away from £2000" or "It is over £100 away, so it is not good"

Do not allow "it is not a good model because it is not the same"

(c)
B1ft: States a value of r in the range $(0.8, 1)$ or states would increase the value of "0.8"

 They do not need to state that "0.8" must lie in the range $(0.8, 1)$

 Condone increase the 0.8. Also allow decrease the "1.25" for $V = 20\,000 \times 1.25^{-t}$
(a) Option 3
M1: They may suggest an exponential model with a lower bound. For example, for $V = Ae^{\pm kt} + 2000$ The bound must be stated but do not allow k to be fixed. Allow as long as the bound $< 10\,000$
M1: $t = 0, V = 20\,000 \Rightarrow A = 18\,000$
dM1: $t = 1, V = 16\,000 \Rightarrow 16\,000 = 2\,000 + 18\,000e^k \Rightarrow k = \dots$ Dependent upon both previous M's

A1: $V = 18\,000 \times e^{-0.251t} + 2000$
(b)
M1: Uses their model to find the value of V when $t = 10$.

 Alternatively substitutes $V = 2000$ into their model and finds t
A1: For $V = 18\,000 \times e^{-0.251 \times 10} + 2000 = £3462.83$ Deduction: Unreliable model as £3462.83

is not close to £2 000 This can only be scored from an acceptable model with correct constants

(c)
B1: States make the value of k or the -0.251 greater (or less negative) so that it lies in the range $(-0.251, 0)$

 Condone 'make the value of k or the -0.251 greater (or less negative)'

It is entirely possible that they start part (a) from a differential equation.

M1: $\frac{dV}{dt} = kV \Rightarrow \int \frac{dV}{V} = \int k dt \Rightarrow \ln V = kt + c$ **M1:** $\ln 20\,000 = c$
dM1: Using $t = 1, V = 16\,000 \Rightarrow k = \dots$ **A1:** $\ln V = -\ln\left(\frac{5}{4}\right)t + \ln 20\,000$


Question	Scheme	Marks	AOs														
13(a)	$a = 60$	B1	3.1b														
	$2 = "60"-b(-20)^2 \Rightarrow b = \dots$	M1	3.4														
	$H = 60 - 0.145(t - 20)^2$	A1	3.3														
		(3)															
(b)	Height = 2 m	B1	3.4														
		(1)															
(c)	$\alpha = 180$ or $\beta = 31$	M1	3.4														
	$H = 29 \cos(9t + 180)^\circ + 31$	A1	3.3														
		(2)															
(d)	e.g. "The model allows for more than one circuit"	B1	3.5a														
		(1)															
(7 marks)																	
Notes																	
(a)																	
B1:	$a = 60$ (may be seen in their final equation of the model or implied by 60 substituted for a in the model)																
M1:	Attempts to find b by substituting in $t = 0$, $H = 2$ and their a and proceeding to a value for b . May be seen as two simultaneous equations formed: $2 = a - b(-20)^2$ and $60 = a - b(20 - 20)^2$ proceeding to a value for b																
A1:	$H = 60 - 0.145(t - 20)^2$ or equivalent such as $H = -\frac{29}{200}t^2 + 5.8t + 2$ or $H = 60 - \frac{29}{200}(t - 20)^2$ isw once a correct equation for the model is seen. Must be in terms of H and t . If they just state $a = 60$, $b = 0.145$ then A0 A correct answer with no working seen scores full marks.																
(b)																	
B1:	2 cao (condone lack of units) This can be scored even if their model in (a) is incorrect (they may have used symmetry to determine this value)																
(c)																	
M1:	$(\alpha =) 180$ or $(\beta =) 31$ Condone $(\alpha =) \pi$																
A1:	$H = 29 \cos(9t + 180)^\circ + 31$ or equivalent e.g. $H = -29 \cos(9t) + 31$ isw once a correct equation for the model is seen. Must be in terms of H and t . If they just state $\alpha = 180$, $\beta = 31$ then A0. A correct equation with no working seen scores both marks. Does not require the degree symbol.																
(d)																	
B1:	Score for a reason which makes reference to any of - the alternative model allows repetition (allow phrases e.g. "multiple cycles", "repeated circuits", "cyclical", "periodic", "loops around", "the original model can only go up and down once") - the alternative model after 2 minutes the carriage will be back at the start (e.g. "at 2 mins, $H = 2$") - the original/quadratic model after 40 seconds (or any time after this) will be negative (e.g. "the height will be negative which cannot happen") - the original model after 2 minutes would not be back at the start Do not allow vague responses on their own e.g. "the original model is a parabola" If calculations are used then they must be correct using a correct model (allow rounded or truncated) Look for a valid reason and ignore reference to anything else as long as it does not contradict																
	t	0	5	10	15	20	25	30	35	40	45	50	55	60	80	100	120
	h	2	27	46	56	60	56	46	27	2	-31	-71	-118	-172	-462	-868	-1390

Question	Scheme	Marks	AOs
8 (a)	25	B1	3.4
		(1)	
(b)	Attempts to differentiate using the product rule $\frac{dv}{dt} = \ln(t+1) \times -0.4 + \frac{(10-0.4t)}{t+1}$	M1 A1	3.1b 1.1b
	Sets their $\frac{dv}{dt} = 0 \Rightarrow \frac{(10-0.4t)}{(t+1)} = 0.4 \ln(t+1)$ and then makes progress towards making "t" the subject (See notes for this)	dM1	1.1b
	$t = \frac{25 - \ln(t+1)}{1 + \ln(t+1)}$ $t = \frac{26}{1 + \ln(t+1)} - 1$ *	A1*	2.1
		(4)	
(c)	(i) Attempts $t_2 = \frac{26}{1 + \ln 8} - 1$	M1	1.1b
	awrt 7.298	A1	1.1b
	(ii) awrt 7.33 seconds	A1	3.2a
		(3)	
(8 marks)			
Notes:			

(a)

B1: 25 but condone 25 seconds. If another value is given (apart from 0) it is B0

(b)

 M1: Attempts to use the product rule in an attempt to differentiate $v = (10 - 0.4t) \ln(t+1)$

 Look for $(10 - 0.4t) \times \frac{1}{(t+1)} \pm k \ln(t+1)$, where k is a constant, condoning slips.

 If you see direct evidence of an incorrect rule used e.g. $vu' - uv'$ it is M0

 You will see attempts from $v = 10 \ln(t+1) - 0.4t \ln(t+1)$ which can be similarly marked.

 In this case look for $\frac{a}{t+1} \pm \frac{bt}{t+1} \pm c \ln(t+1)$

 A1: Correct differentiation. Condone a missing left hand or it seen as v' , $\frac{dy}{dx}$ or even $= 0$
 $\left(\frac{dv}{dt}\right) = \ln(t+1) \times -0.4 + \frac{(10-0.4t)}{t+1}$ or equivalent such as $\left(\frac{dv}{dt}\right) = \frac{10}{t+1} - \frac{0.4t}{(t+1)} - 0.4 \ln(t+1)$


dM1: Score for setting their $dV/dt = 0$ (which must be in an appropriate form) and proceeding to an equation where the variable t occurs only once – ignoring $\ln(t+1)$.

See two examples of how this can be achieved below. It is dependent upon the previous M.

Look for the following steps

- An allowable derivative set (or implied) $= 0$ E.g. $\ln(t+1) \times 0.4 = \frac{(10-0.4t)}{t+1}$
- Cross multiplication (or division) and rearrangement to form an equation where the variable t only occurs once.

E.g.1.

$$\begin{aligned}\ln(t+1) \times 0.4 &= \frac{(10-0.4t)}{t+1} \\ \Rightarrow \ln(t+1) &= \frac{25-t}{t+1} \\ \Rightarrow \ln(t+1) &= -1 + \frac{26}{t+1}\end{aligned}$$

E.g.2

$$\begin{aligned}\ln(t+1) \times 0.4 &= \frac{(10-0.4t)}{t+1} \\ \Rightarrow 0.4t \ln(t+1) + 0.4 \ln(t+1) &= 10 - 0.4t \\ \Rightarrow 0.4t(1 + \ln(t+1)) &= 10 - 0.4 \ln(t+1)\end{aligned}$$

A1*: Correctly proceeds to the given answer of $t = \frac{26}{1 + \ln(t+1)} - 1$ showing all key steps.

The key steps must include

- use of $\frac{dv}{dt}$ or v' which must be correct
- a correct line preceding the given answer, usually $t = \frac{25 - \ln(t+1)}{1 + \ln(t+1)}$ or $\frac{26}{t+1} - 1 = \ln(t+1)$

(c) (i)

M1: Attempts to use the iteration formula at least once.

Usually to find $t_2 = \frac{26}{1 + \ln 8} - 1$ which may be implied by awrt 7.44

A1: awrt 7.298. This alone will score both marks as iteration is implied. ISW after sight of this value. As t_3 is the only value that rounds to 7.298 just score the rhs, it does not need to be labelled t_3

(c) (ii)

A1: Uses repeated iteration until value established as awrt 7.33 **seconds**. Allow awrt 7.33 s

Requires units. It also requires some evidence of iteration which will be usually be awarded from the award of the M



Question	Scheme	Marks	AOs
10 (a)	265 thousand	B1	3.4
		(1)	
(b)	Attempts $\frac{dN_b}{dt} = 11e^{0.05t}$	M1	1.1b
	Substitutes $t = 10$ into their $\frac{dN_b}{dt}$	M1	3.4
	$\frac{dN_b}{dt} = \text{awrt } 18.1$ which is approximately 18 thousand per year *	A1*	2.1
		(3)	
(c)	Sets $45 + 220e^{0.05t} = 10 + 800e^{-0.05t} \Rightarrow 220e^{0.05t} + 35 - 800e^{-0.05t} = 0$	M1	3.1b
	Correct quadratic equation $\Rightarrow 220(e^{0.05t})^2 + 35e^{0.05t} - 800 = 0$	A1	1.1b
	$e^{0.05t} = 1.829, (-1.988) \Rightarrow 0.05t = \ln(1.829)$	M1	2.1
	$T = 12.08$	A1	1.1b
		(4)	
(8 marks)			
Notes:			

(a) May be seen in the question so watch out.

B1: Accept 265 thousand or 265 000 or equivalent such as 265 k but not just 265.

(b)

M1: Differentiates to a form $ke^{0.05t}$, $k > 0, k \neq 220$. Do not be too concerned about the lhs.

M1: Substitutes $t = 10$ into a changed function that was formed from an attempt at differentiation.

The left hand side must have implied differentiation. E.g. Rate = , N' , $\frac{dN_b}{dt}$, $\frac{dN}{dt}$ or even $\frac{dy}{dx}$

A1*: Full and complete proof that requires

- some correct lhs seen at some point. E.g. "Rate = , " $\frac{dN_b}{dt}$, $\frac{dN}{dt}$ but condone N' .
- an intermediate line/answer of either $11e^{0.05 \times 10}$ or awrt 18.1 before a minimal conclusion which must be referencing the 18 000 or 18 thousand



(c)

M1: Attempts to set both equations equal to each other and simplify the constant terms.

Look for $220e^{0.05t} + 35 = 800e^{-0.05t}$ o.e but condone slipsIt is also possible to set $\frac{N-45}{220} = \left(e^{0.05t} = \right) \frac{800}{N-10}$ and form an equation in N

A1: Correct quadratic form.

Look for $220(e^{0.05t})^2 + 35e^{0.05t} - 800 = 0$ or $220e^{0.1t} + 35e^{0.05t} - 800 = 0$ but allow with terms in different order such as $220e^{0.1t} + 35e^{0.05t} = 800$ FYI the equation in N is $N^2 - 55N - 175550 = 0$ M1: Full attempt to find the value of t (or a constant multiple of t)This involves the key step of recognising and solving a 3TQ in $e^{0.05t}$ followed by the use of lns.

If the answers to the quadratic just appear (from a calculator) you will need to check.

Accuracy should be to 3sf.

You may see different variables used such as x

$$x = e^{0.05t}, 220e^{0.1t} + 35e^{0.05t} = 800 \Rightarrow 220x^2 + 35x = 800 \Rightarrow x = 1.82... \Rightarrow t = 20 \ln 1.82...$$

Allow use of calculator for solving the quadratic and for $e^{0.05t} = 1.82.. \Rightarrow t = 12.08$ Via the N route it will involve substituting the positive solution to their quadratic into either equation to find a value for t/T using same rules as above.

A1: AWRT 12.08

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Answers with limited or no working in (b) and (c)

(b) A derivative in the correct form must be seen

(c) Candidates who state $45 + 220e^{0.05t} = 10 + 800e^{-0.05t}$ followed by awrt 12.08 (presumably from using num-solv on their calculators) can score SC 1100. Rubric on the front of the paper states that "Answers without working may not gain full credit" so we demand a method in this part.

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Question	Scheme	Marks	AOs
8(a)	$A = 1000$	B1	3.4
	$2000 = 1000e^{5k}$ or $e^{5k} = 2$	M1	1.1b
	$e^{5k} = 2 \Rightarrow 5k = \ln 2 \Rightarrow k = \dots$	M1	2.1
	$N = 1000e^{\left(\frac{1}{5}\ln 2\right)t}$ or $N = 1000e^{0.139t}$	A1	3.3
		(4)	
(b)	$\frac{dN}{dt} = 1000 \times \left(\frac{1}{5}\ln 2\right)e^{\left(\frac{1}{5}\ln 2\right)t}$ or $\frac{dN}{dt} = 1000 \times 0.139e^{0.139t}$ $\left(\frac{dN}{dt}\right)_{t=8} = 1000 \times \left(\frac{1}{5}\ln 2\right)e^{8 \times \frac{1}{5}\ln 2}$ or $\left(\frac{dN}{dt}\right)_{t=8} = 1000 \times 0.139e^{0.139 \times 8}$	M1	3.1b
	= awrt 420	A1	1.1b
		(2)	
(c)	$500e^{1.4 \times \left(\frac{1}{5}\ln 2\right)T} = 1000e^{\left(\frac{1}{5}\ln 2\right)T}$ or $500e^{1.4 \times "0.139" T} = 1000e^{0.139 T}$	M1	3.4
	Correct method of getting a linear equation in T E.g. $0.08T \ln 2 = \ln 2$ or $1.4 \times "0.339" T = \ln 2 + "0.339" t$	M1	2.1
	$T = 12.5$ hours	A1	1.1b
		(3)	
(9 marks)			
Notes			

Mark as one complete question. Marks in (a) can be awarded from (b)

(a)

B1: Correct value of A for the model. Award if equation for model is of the form $N = 1000e^{kt}$

M1: Uses the model to set up a correct equation in k . Award for substituting $N = 2000, t = 5$

following through on their value for A .

M1: Uses correct $\ln$ work to solve an equation of the form $ae^{5k} = b$ and obtain a value for k

A1: Correct equation of model. Condone an ambiguous $N = 1000e^{\frac{1}{5}\ln 2t}$ unless followed by something incorrect. Watch for $N = 1000 \times 2^{\frac{1}{5}t}$ which is also correct

(b)

M1: Differentiates ae^{kt} to βe^{kt} and substitutes $t = 8$ (Condone $\alpha = \beta$ so long as you can see an attempt to differentiate)

A1: For awrt 420 (2sf).

(c)

M1: Uses both models to set up an equation in T using their value for k , but also allow in terms of k

M1: Uses correct processing using $\ln$ s to obtain a linear equation in T (or t)

A1: Awrt 12.5

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 Answers to (b) and (c) appearing without working (i.e. from a calculator).

It is important that candidates show sufficient working to make their methods clear.

(b) If candidate has for example $N = 1000e^{0.139t}$, and then writes at $t = 8$ $\frac{dN}{dt} = \text{awrt } 420$ award both marks. Just the answer from a correct model equation score SC 1,0.

(c) The first M1 should be seen E.g. $500e^{1.4 \times "0.139" T} = 1000e^{0.139 T}$

If the answer $T = 12.5$ appears without any further working score SC M1 M1 A0

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Question	Scheme	Marks	AOs
12 (a)	(i) Method to find p Eg. Divides $32000 = Ap^4$ by $50000 = Ap^{11}$ $p^7 = \frac{50000}{32000} \Rightarrow p = \sqrt[7]{\frac{50000}{32000}} = \dots$	M1	3.1a
	$p = 1.0658$	A1	1.1b
	(ii) Substitutes their $p = 1.0658$ into either equation and finds A $A = \frac{32000}{1.0658^4} \text{ or } A = \frac{50000}{1.0658^{11}}$	M1	1.1b
	$A = 24795 \rightarrow 24805 \approx 24800^*$	A1*	1.1b
		(4)	
(b)	A / (£) 24 800 is the value of the car on 1st January 2001	B1	3.4
	$p / 1.0658$ is the factor by which the value rises each year. Accept that the value rises by 6.6 % a year (ft on their p)	B1	3.4
		(2)	
(c)	Attempts $100000 = '24800' \times '1.0658'^t$		
	$'1.0658'^t = \frac{100000}{24800}$	M1	3.4
	$t = \log_{1.0658} \left(\frac{100000}{24800} \right)$	dM1	1.1b
	$t = 21.8 \text{ or } 21.9$	A1	1.1b
	cso 2022	A1	3.2a
		(4)	
(10 marks)			
(a) (i) M1: Attempts to use both pieces of information within $V = Ap^t$, eliminates A correctly and solves an equation of the form $p^n = k$ to reach a value for p. Allow for slips on the 32 000 and 50 000 and the values of t. A1: $p = \text{awrt } 1.0658$ Both marks can be awarded from incorrect but consistent interpretations of t. Eg. $32000 = Ap^5$, $50000 = Ap^{12}$ (a)(ii) M1: Substitutes their $p = 1.0658$ into either of their equations and finds A Eg $A = \frac{32000}{1.0658^4}$ or $A = \frac{50000}{1.0658^7}$ but you may follow through on incorrect equations from part (i) A1*: Shows that A is between 24 795 and 24 805 before you see '≈ 24800' or '≈ 24800'. Accept with or without units. An alternative to (ii) is to start with the given answer.			

A1: 24800×1.0658^4 , achieves a value between 31095 and 32005 followed by $\approx 32\,000$ hence A must be $\approx 24\,800$

(b)

B1: States that A is the value of the car on 1st January 2001.

The statement must reference **the car**, its **cost/value**, and **"0" time**

Allow 'it is the initial value of the car' 'it is the cost of the car at $t = 0$ ' 'it is the cars starting value'

B1: States that p is the rate at which the value of the car rises each year.

The statement must reference **a yearly rate** and **an increase in value or multiplier**.

They could reference the 1.0658 Eg "The cars value rises by 6.5 % each year."

Allow " p is the rate the cars value is rising each year" "it is the proportional increase in value of the car each year" "the factor by which the value of the car is rising each year" 'its value appreciates by 6.5% per year' Allow 'the value of the car multiplies by p each year'

Do not allow "by how much the value of the car rises each year" or "it is the rate of inflation"

(c)

M1: Uses the model $100000 = 24800 \times 1.0658^t$ and proceeds to their $1.0658^t = k$

Allow use of any inequality here.

dM1: For the complete method of (i) using the information given with their equation of the model **and** (ii) translating the situation into a correct method to find ' t '

A1: $(t) = \text{awrt } 21.8 \text{ or } 21.9 \text{ or } \log_{1.0658} \left(\frac{100000}{24800} \right) \text{ oe}$

A1: States in the year 2022. A candidate using a GP formula can be awarded full marks

Allow different methods in part (c).

Eg Via GP a formula

M1: $24800 \times 1.0658^{n-1} = 100000 \Rightarrow 1.0658^{n-1} = K$

dM1: Uses a correct method to find n .

A2: 2022

Via (trial and improvement)

M1: Uses the model by substituting integer values of t into their $V = Ap^t$ so that for $t = n, V < 100\,000$ or $t = n+1, V > 100\,000$

(So for the correct A and p this would be scored for $t = 21, V \approx £95\,000$ or $t = 21, V \approx £101\,000$)

dM1: For a complete method showing that this is the least value. So both of the above values

A1: Allow for 22 following correct and accurate results (awrt nearest £1000 is sufficient accuracy)

A1: As before



Question	Scheme	Marks	AOs
14	$N = \frac{900}{3 + 7e^{-0.25t}} = 900(3 + 7e^{-0.25t})^{-1}, t \in \mathbb{R}, t \geq 0; \quad \frac{dN}{dt} = \frac{N(300 - N)}{1200}$		
(a)	90	B1	3.4
		(1)	
(b) Way 1	$\frac{dN}{dt} = -900(3 + 7e^{-0.25t})^{-2} (7(-0.25)e^{-0.25t}) \quad \left\{ = \frac{900(0.25)(7)e^{-0.25t}}{(3 + 7e^{-0.25t})^2} \right\}$	M1	2.1
		A1	1.1b
	$\Rightarrow \frac{dN}{dt} = \frac{900(0.25) \left(\left(\frac{900}{N} - 3 \right) \right)}{\left(\frac{900}{N} \right)^2}$	dM1	2.1
	correct algebra leading to $\frac{dN}{dt} = \frac{N(300 - N)}{1200}$ *	A1*	1.1b
		(4)	
(b) Way 2	$\frac{dN}{dt} = -900(3 + 7e^{-0.25t})^{-2} (7(-0.25)e^{-0.25t}) \quad \left\{ = \frac{900(0.25)(7)e^{-0.25t}}{(3 + 7e^{-0.25t})^2} \right\}$	M1	2.1
		A1	1.1b
	$\frac{N(300 - N)}{1200} = \frac{\left(\frac{900}{3 + 7e^{-0.25t}} \right) \left(300 - \frac{900}{3 + 7e^{-0.25t}} \right)}{1200}$	dM1	2.1
	$\text{LHS} = \frac{1575e^{-0.25t}}{(3 + 7e^{-0.25t})^2} \text{ o.e.,}$ $\text{RHS} = \frac{900(300(3 + 7e^{-0.25t}) - 900)}{1200(3 + 7e^{-0.25t})^2} = \frac{1575e^{-0.25t}}{(3 + 7e^{-0.25t})^2} \text{ o.e.}$ and states hence $\frac{dN}{dt} = \frac{N(300 - N)}{1200}$ (or LHS = RHS) *	A1*	1.1b
		(4)	
(c)	Deduces $N = 150$ (can be implied)	B1	2.2a
	so $150 = \frac{900}{3 + 7e^{-0.25T}} \Rightarrow e^{-0.25T} = \frac{3}{7}$	M1	3.4
	$T = -4 \ln \left(\frac{3}{7} \right)$ or $T = \text{awrt } 3.4$ (months)	dM1	1.1b
		A1	1.1b
(d)		(4)	
	either one of 299 or 300	B1	3.4
		(1)	
(10 marks)			

Notes for Question 14	
14 (b)	
M1:	Attempts to differentiate using - the chain rule to give $\frac{dN}{dt} = \pm Ae^{-0.25t}(3+7e^{-0.25t})^{-2}$ or $\frac{\pm Ae^{-0.25t}}{(3+7e^{-0.25t})^2}$ o.e. - the quotient rule to give $\frac{dN}{dt} = \frac{(3+7e^{-0.25t})(0) \pm Ae^{-0.25t}}{(3+7e^{-0.25t})^2}$ - implicit differentiation to give $N(3+7e^{-0.25t}) = 900 \Rightarrow (3+7e^{-0.25t})\frac{dN}{dt} \pm ANe^{-0.25t} = 0$, o.e. where $A \neq 0$
Note:	Condone a slip in copying $(3+7e^{-0.25t})$ for the M mark
A1:	A correct differentiation statement
Note:	Implicit differentiation gives $(3+7e^{-0.25t})\frac{dN}{dt} - 1.75Ne^{-0.25t} = 0$
dM1:	Way 1: Complete attempt, by eliminating t, to form an equation linking $\frac{dN}{dt}$ and N only Way 2: Complete substitution of $N = \frac{900}{3+7e^{-0.25t}}$ into $\frac{dN}{dt} = \frac{N(300-N)}{1200}$
Note:	Way 1: e.g. substitutes $3+7e^{-0.25t} = \frac{900}{N}$ and $e^{-0.25t} = \frac{900}{N} - 3$ or substitutes $e^{-0.25t} = \frac{\frac{900}{N} - 3}{7}$ into their $\frac{dN}{dt} = \dots$ to form an equation linking $\frac{dN}{dt}$ and N
A1*:	Way 1: Correct algebra leading to $\frac{dN}{dt} = \frac{N(300-N)}{1200}$ * Way 2: See scheme
(c)	
B1:	Deduces or shows that $\frac{dN}{dt}$ is maximised when $N = 150$
M1:	Uses the model $N = \frac{900}{3+7e^{-0.25t}}$ with their $N = 150$ and proceeds as far as $e^{-0.25T} = k, k > 0$ or $e^{0.25T} = k, k > 0$. Condone $t \equiv T$
dM1:	Correct method of using logarithms to find a value for T . Condone $t \equiv T$
A1:	see scheme
Note:	$\frac{d^2N}{dt^2} = \frac{dN}{dt} \left(\frac{300}{1200} - \frac{2N}{1200} \right) = 0 \Rightarrow N = 150$ is acceptable for B1
Note:	Ignore units for T
Note:	Applying $300 = \frac{900}{3+7e^{-0.25t}} \Rightarrow t = \dots$ or $0 = \frac{900}{3+7e^{-0.25t}} \Rightarrow t = \dots$ is M0 dM0 A0
Note:	M1 dM1 can only be gained in (c) by using an N value in the range $90 < N < 300$
(d)	
B1:	300 (or accept 299)



Question	Scheme	Marks	AOs
14	$N = \frac{900}{3 + 7e^{-0.25t}} = 900(3 + 7e^{-0.25t})^{-1}, t \in \mathbb{R}, t \geq 0; \quad \frac{dN}{dt} = \frac{N(300 - N)}{1200}$		
(b) Way 3	$\int \frac{1}{N(300 - N)} dN = \int \frac{1}{1200} dt$	M1	2.1
	$\int \frac{1}{300} \left(\frac{1}{N} + \frac{1}{300 - N} \right) dN = \int \frac{1}{1200} dt$	A1	1.1b
	$\frac{1}{300} \ln N - \frac{1}{300} \ln(300 - N) = \frac{1}{1200} t \quad \{+c\}$		
	$\{t = 0, N = 90 \Rightarrow\} \quad c = \frac{1}{300} \ln(90) - \frac{1}{300} \ln(210) \Rightarrow c = \frac{1}{300} \ln\left(\frac{3}{7}\right)$ $\frac{1}{300} \ln N - \frac{1}{300} \ln(300 - N) = \frac{1}{1200} t + \frac{1}{300} \ln\left(\frac{3}{7}\right)$ $\ln N - \ln(300 - N) = \frac{1}{4} t + \ln\left(\frac{3}{7}\right)$ $\ln\left(\frac{N}{300 - N}\right) = \frac{1}{4} t + \ln\left(\frac{3}{7}\right) \Rightarrow \frac{N}{300 - N} = \frac{3}{7} e^{\frac{1}{4}t}$	dM1	2.1
	$7N = 3e^{\frac{1}{4}t} (300 - N) \Rightarrow 7N + 3Ne^{\frac{1}{4}t} = 900e^{\frac{1}{4}t}$ $N(7 + 3e^{\frac{1}{4}t}) = 900e^{\frac{1}{4}t} \Rightarrow N = \frac{900e^{\frac{1}{4}t}}{7 + 3e^{\frac{1}{4}t}} \Rightarrow N = \frac{900}{3 + 7e^{-0.25t}} *$	A1*	1.1b
		(4)	
(b) Way 4	$N(3 + 7e^{-0.25t}) = 900 \Rightarrow e^{-0.25t} = \frac{1}{7} \left(\frac{900}{N} - 3 \right) \Rightarrow e^{-0.25t} = \frac{900 - 3N}{7N}$	M1	2.1
	$\Rightarrow t = -4(\ln(900 - 3N) - \ln(7N))$ $\Rightarrow \frac{dt}{dN} = -4 \left(\frac{-3}{900 - 3N} - \frac{7}{7N} \right)$	A1	1.1b
	$\frac{dt}{dN} = 4 \left(\frac{1}{300 - N} + \frac{1}{N} \right) \Rightarrow \frac{dt}{dN} = 4 \left(\frac{N + 300 - N}{N(300 - N)} \right)$	dM1	2.1
	$\frac{dt}{dN} = \left(\frac{1200}{N(300 - N)} \right) \Rightarrow \frac{dN}{dt} = \frac{N(300 - N)}{1200} *$	A1*	1.1b
		(4)	

Notes for Question 14 Continued	
(b) Way 3	
M1:	Separates the variables, an attempt to form and apply partial fractions and integrates to give $\ln \text{ terms} = kt \{+c\}$, $k \neq 0$, with or without a constant of integration c
A1:	$\frac{1}{300} \ln N - \frac{1}{300} \ln(300 - N) = \frac{1}{1200} t \{+c\}$ or equivalent with or without a constant of integration c
dM1:	Uses $t = 0$, $N = 90$ to find their constant of integration and obtains an expression of the form $\lambda e^{\frac{1}{4}t} = f(N)$; $\lambda \neq 0$ or $\lambda e^{-\frac{1}{4}t} = f(N)$; $\lambda \neq 0$
A1*:	Correct manipulation leading to $N = \frac{900}{3 + 7e^{-0.25t}}$ *
(b) Way 4	
M1:	Valid attempt to make t the subject, followed by an attempt to find two $\ln$ derivatives, condoning sign errors and constant errors.
A1:	$\frac{dt}{dN} = -4 \left(\frac{-3}{900 - 3N} - \frac{7}{7N} \right)$ or equivalent
dM1:	Forms a common denominator to combine their fractions
A1*:	Correct algebra leading to $\frac{dN}{dt} = \frac{N(300 - N)}{1200}$ *



Question	Scheme	Marks	AOs
12(a)	$K = 500$	B1	1.1b
	$\tan \alpha = \frac{480}{140} \Rightarrow \alpha = \dots$	M1	1.1b
	$\alpha = \text{awrt } 73.74^\circ \text{ or } 500 \cos(\theta + 73.74)^\circ$	A1	1.1b
		(3)	
(b)(i)	$R = 1000 + 500 \cos(30t + 73.74)^\circ$ or $R = 1000 + 140 \cos(30t)^\circ - 480 \sin(30t)^\circ$	B1ft	3.3
(b)(ii)	$\{R_{\min} =\} 500$	B1ft	3.4
		(2)	
(c)	$t = 3.5 \Rightarrow R = "1000" + "500" \cos(30(3.5) + "73.74")^\circ = \dots$	M1	3.4
	$R = \text{awrt } 500.1 \dots$ so the model is reliable	A1	3.5a
		(2)	
(d)	$\sin(30t + 70)^\circ = -1 \Rightarrow 30t + 70 = 270 \Rightarrow 30t = \dots \text{ (or } t = \dots)$	M1	3.4
	$30t = 200 \left(\text{or } t = \frac{20}{3} \right)$	A1	1.1b
	$R = "1000" + "500" \cos\left(30\left(\frac{20}{3}\right) + "73.74" \right)^\circ$ or $R = "1000" + 140 \cos("200")^\circ - 480 \sin("200")^\circ$	dM1	3.4
	$R = 1032 \text{ (or } 1033)$	A1	1.1b
		(4)	

(11 marks)**Notes**

Note: Candidates working in radians are able to score all the M and B marks in this question. Condone the absence of the degrees symbol throughout the whole question.

(a)**B1:** Correct value for K . Condone $R = 500$

M1: Award for $\tan \alpha = \pm \frac{480}{140} \Rightarrow \alpha = \dots$, $\tan \alpha = \pm \frac{140}{480} \Rightarrow \alpha = \dots$, $\sin \alpha = \pm \frac{480}{"500"} \Rightarrow \alpha = \dots$ or $\cos \alpha = \pm \frac{140}{"500"} \Rightarrow \alpha = \dots$

Note $\alpha = \text{awrt } 1.3 \text{ (rad)}$ implies this mark.**A1:** $\alpha = \text{awrt } 73.74\{\circ\}$ or correct expression $500 \cos(\theta + 73.74)\{\circ\}$ **(b)(i)** Note: mark parts (b)(i) and (b)(ii) together.**B1ft:** Correct equation of the model in either form including the $R =$ following through on their numerical K ($0 < K \leq 750$) and their numerical α .Allow for e.g. $R = 1500 - "500" + "500" \cos(30t + "73.74")\{\circ\}$ or for $R = 1500 - "500" + 140 \cos(30t)^\circ - 480 \sin(30t)^\circ$ but not e.g. $R = 1500 - K + K \cos(30t + \alpha)^\circ$ $R = 1000 + 140 \cos 30t - 480 \sin 30t$ (without the brackets) is correct.Allow this mark if they have truncated or rounded an otherwise correct α (to 3s.f.)

(b)(ii)

B1ft: 500 or follow through on (their A – their K) or $(1500 - 2 \times \text{their } K)$ provided it is non-negative and less than 1500. It must be clear this is their answer to (b)(ii) so expect to see e.g. (b) or $R_{\min} =$ or an indication it is the minimum.

(c) Note: if θ is used in place of $30t$ then they must revert back to $30t$ correctly to access the marks.

M1: Substitutes $t = 3.5$ into their model for the number of rabbits (you may need to check if no method is shown)

or substitutes $t = 3.5$ into their $\cos(30t + \alpha)^\circ$

Condone substitution of a value of t in the range $3 \leq t \leq 4.5$ for this mark.

A1: $R = \text{awrt } 500.1 \dots$ **or** 500 (not awrt) following substitution of $t = 3.5$, suggesting that the model is valid/reliable/appropriate/good.

or $\cos(30(3.5) + 73.74)^\circ \approx -1$ suggesting that the model is valid/reliable/appropriate/good.

Allow this mark if they have truncated or rounded an otherwise correct α (to 3s.f.)

Alt:

M1: Minimum occurs when $A + K \cos(30t + \alpha)^\circ = R_{\min} \Rightarrow \cos(30t + \alpha)^\circ = \lambda$ with $|\lambda| \leq 1$ leading to $t = \dots$

May just see $\cos(30t + 73.74)^\circ = -1 \Rightarrow t = \dots$ (or their $\cos(30t + \alpha)^\circ = -1 \Rightarrow t = \dots$)

$30t + \alpha = 180 \Rightarrow t = \dots$ implies this mark. Condone $30t + \alpha = \pi \Rightarrow t = \dots$ for this mark.

A1: $t = 3.54 \dots$ (i.e. the middle of April) so the model is valid/reliable/appropriate/good.

Do not condone incorrect statements, e.g., $t = 3.54 \dots$ i.e. the middle of March so close to middle of April. If using $A + K \cos(30t + \alpha)^\circ = R_{\min}$ then $R_{\min}$ must be = their A – their K

Allow this mark if they have truncated or rounded an otherwise correct α (to 3s.f.)

Alt 2 using differentiation (Condoned)

M1: Condone finding the minimum using $\dots \sin(30t + 73.74)^\circ = 0 \Rightarrow t = \dots$ (or their $\sin(30t + \alpha)^\circ = 0 \Rightarrow t = \dots$)

$30t + \alpha = 180 \Rightarrow t = \dots$ implies this mark. Condone $30t + \alpha = \pi \Rightarrow t = \dots$ for this mark.

A1: $t = 3.54 \dots$ (i.e. the middle of April) suggesting that the model is valid/reliable/appropriate.

Do not condone incorrect statements, e.g., $t = 3.54 \dots$ i.e. the middle of March so close to middle of April.

The complete derivative for $\frac{dR}{dt}$ does not need to be seen.

Allow this mark if they have truncated or rounded an otherwise correct α (to 3s.f.)

(d) Note: if θ is used in place of $30t$ then they must revert back to $30t$ correctly to access the marks.

M1: Realises that $\sin(30t + 70)^\circ = -1$, reaches $30t + 70 = 270$ or -90 and attempts to find t (or $30t$)

Condone attempts using differentiation. The minimum occurs when $\cos(30t + 70)^\circ = 0 \Rightarrow$

$30t + 70 = 270 \Rightarrow 30t = \dots$ (or $t = \dots$). They must use 270 or -90 and **not** 90 to achieve the

minimum. Condone $30t + \alpha = \frac{3\pi}{2} \Rightarrow t = \dots$ for this mark but not $30t + \alpha = \frac{\pi}{2} \Rightarrow t = \dots$

A1: Correct value for $30t$ (or t) Accept rounded or truncated values to at least 3s.f. e.g. 6.66 or 6.67

dm1: Substitutes their value of $t > 0$ (or $30t > 0$) coming from $30t + 70 = 270$ into their model for R

A1: Correct number of rabbits. Allow 1032 or 1033 but must be whole numbers and not just 1030.

Allow this mark if they have truncated or rounded an otherwise correct α (to 3s.f.)



Question	Scheme	Marks	AOs
15(a)	$R = \sqrt{5}$	B1	1.1b
	$\tan \alpha = \frac{1}{2}$ or $\sin \alpha = \frac{1}{\sqrt{5}}$ or $\cos \alpha = \frac{2}{\sqrt{5}} \Rightarrow \alpha = \dots$	M1	1.1b
	$\alpha = 0.464$	A1	1.1b
		(3)	
(b)(i)	$3 + 2\sqrt{5}$	B1ft	3.4
(ii)	$\cos(0.5t + 0.464) = 1 \Rightarrow 0.5t + 0.464 = 2\pi$ $\Rightarrow t = \dots$	M1	3.4
	$t = 11.6$	A1	1.1b
		(3)	
(c)	$3 + 2\sqrt{5} \cos(0.5t + 0.464) = 0$ $\cos(0.5t + 0.464) = -\frac{3}{2\sqrt{5}}$	M1	3.4
	$\cos(0.5t + 0.464) = -\frac{3}{2\sqrt{5}} \Rightarrow 0.5t + 0.464 = \cos^{-1}\left(-\frac{3}{2\sqrt{5}}\right)$ $\Rightarrow t = 2\left(\cos^{-1}\left(-\frac{3}{2\sqrt{5}}\right) - 0.464\right)$	dM1	1.1b
	So the time required is e.g.: $2(3.977\dots - 0.464) - 2(2.306\dots - 0.464)$	dM1	3.1b
	$= 3.34$	A1	1.1b
		(4)	
(d)	e.g. the "3" would need to vary	B1	3.5c
		(1)	
(11 marks)			
Notes			
(a) B1: $R = \sqrt{5}$ only. M1: Proceeds to a value for α from $\tan \alpha = \pm \frac{1}{2}$ or $\sin \alpha = \pm \frac{1}{\sqrt{5}}$ or $\cos \alpha = \pm \frac{2}{\sqrt{5}}$ It is implied by either awrt 0.464 (radians) or awrt 26.6 (degrees) A1: $\alpha =$ awrt 0.464 (b)(i) B1ft: For $(3 + 2\sqrt{5})$ m or awrt 7.47 m and remember to isw. Condone lack of units. Follow through on their R value so allow $3 + 2 \times$ Their R . (Allow in decimals with at least 3sf accuracy) (b)(ii) M1: Uses $0.5t \pm "0.464" = 2\pi$ to obtain a value for t Follow through on their 0.464 but this angle must be in radians. It is possible in degrees but only using $0.5t \pm "26.6" = 360$ A1: Awrt 11.6			

Alternative for (b):

$$H = 3 + 4 \cos(0.5t) - 2 \sin(0.5t) \Rightarrow \frac{dH}{dt} = -2 \sin(0.5t) - \cos(0.5t) = 0$$

$$\Rightarrow \tan(0.5t) = -\frac{1}{2} \Rightarrow 0.5t = 2.677..., 5.819... \Rightarrow t = 5.36, 11.6$$

$$t = 11.6 \Rightarrow H = 7.47$$

Score as follows:

M1: For a complete method:

Attempts $\frac{dH}{dt}$ and attempts to solve $\frac{dH}{dt} = 0$ for t

A1: For $t = \text{awrt } 11.6$

B1ft: For awrt 7.47 or $3 + 2 \times \text{Their } R$

(c)

M1: Uses the model and sets $3 + 2\sqrt{5} \cos(\dots) = 0$ and proceeds to $\cos(\dots) = k$ where $|k| < 1$.

Allow e.g. $3 + 2\sqrt{5} \cos(\dots) < 0$

dM1: Solves $\cos(0.5t \pm 0.464) = k$ where $|k| < 1$ to obtain at least one value for t

This requires e.g. $2\left(\pi + \cos^{-1}(k) \pm \tan^{-1}\left(\frac{1}{2}\right)\right)$ or e.g. $2\left(\pi - \cos^{-1}(k) \pm \tan^{-1}\left(\frac{1}{2}\right)\right)$

Depends on the previous method mark.

dM1: A fully correct strategy to find the required duration. E.g. finds 2 consecutive values of t when $H = 0$ and subtracts. Alternatively finds t when H is minimum and uses the times found correctly to find the required duration.

Depends on the previous method mark.

Examples:

Second time at water level – first time at water level:

$$2\left(\pi + \cos^{-1}\left(\frac{3}{2\sqrt{5}}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right) - 2\left(\pi - \cos^{-1}\left(\frac{3}{2\sqrt{5}}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right) = 7.02685... - 3.68492...$$

$2 \times (\text{first time at minimum point} - \text{first time at water level}):$

$$2\left(2\left(\pi - \tan^{-1}\left(\frac{1}{2}\right)\right) - 2\left(\pi - \cos^{-1}\left(\frac{3}{2\sqrt{5}}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right)\right) = 2(5.35589... - 3.68492...)$$

Note that both of these examples equate to $4\cos^{-1}\left(\frac{3}{2\sqrt{5}}\right)$ which is not immediately obvious

but may be seen as an overall method.

There may be other methods – if you are not sure if they deserve credit send to review.

A1: Correct value. Must be 3.34 (not awrt).

Special Cases in (c):

Note that if candidates have an incorrect α and have e.g. $3 + 2\sqrt{5} \cos(0.5t - 0.464)$, this has no impact on the final answer. So for candidates using $3 + 2\sqrt{5} \cos(0.5t \pm \alpha)$ in (c) allow all the marks including the A mark as a correct method should always lead to 3.34

Some values to look for:

$$0.5t \pm 0.464 = \pm 2.306, \pm 3.977, \pm 8.598, \pm 10.26$$

(d)

B1: Correct refinement e.g. As in scheme. If they suggest a specific function to replace the “3” then it must be sensible e.g. a trigonometric function rather than e.g. a quadratic/linear one.



Question	Scheme	Marks	AOs
11 (a)	Uses $t = 0, V = 20\,000 \Rightarrow 20\,000 = 1500 + Ae^0$	M1	3.4
	$A = 18500$	A1	1.1b
	$t = 2.5, V = 12\,000 \Rightarrow 12\,000 = 1500 + 18\,500e^{-k \times 2.5}$ $\Rightarrow 10500 = 18500e^{-k \times 2.5} \Rightarrow k = \dots \left(= -\frac{2}{5} \ln \frac{21}{37} = \text{awrt } 0.227 \right)$	dM1	3.1b
	$V = 1500 + 18500e^{-0.227t}$	A1	3.3
		(4)	
(b)	Achieves $\left(\frac{dV}{dt} = \right) -kAe^{-kt}$ or $"-0.227" \times "18500" e^{-0.227t}$	B1	1.1b
	Substitutes $Ae^{-kt} = V - 1500$ into $\left(\frac{dV}{dt} = \right) -kAe^{-kt}$ or $"18500" e^{-0.227t} = V - 1500$ into $"-0.227" \times "18500" e^{-0.227t}$	M1	3.4
	Rate of change in value of car is $\left(\frac{dV}{dt} = \right) -k(V - 1500)$ or $"-0.227"(V - 1500) *$	A1*	2.1
		(3)	
(c)	Suggests a suitable limitation of the model (see notes)	B1	3.5b
		(1)	
(8 marks)			
Notes:			
Mark (a) and (b) together (a) M1: Uses the equation of the model with $t = 0, V = 20\,000 \Rightarrow 20000 = 1500 + Ae^0$ o.e. May be implied by 18500 A1: $A = 18500$ (18500 with no working seen scores M1A1) Ignore £ if present. dM1: Attempts to use the equation of the model $t = 2.5, V = 12\,000$ $\Rightarrow 12000 = 1500 + "18500" e^{-2.5k}$ and proceeds to $Ce^{\pm k \times 2.5} = D$ (where $C \times D > 0$ and allow $C = 1$) before proceeding to find a value for k. Allow them to have a non-numerical C for this mark. Note it cannot be implied by their awrt ± 0.227 so $12000 = 1500 + "18500" e^{-2.5k} \Rightarrow k = 0.227$ scores dM0A0 as we need to see the intermediate stage $Ce^{\pm k \times 2.5} = D$. (typically look for $18\,500e^{-k \times 2.5} = 10\,500 \Rightarrow k = \dots$ or condone to be implied by a correct expression involving logarithms for their A) It is dependent on the previous method mark. A1: $V = 1500 + 18500e^{\text{awrt } -0.227t}$ o.e. e.g. $t = \frac{\ln\left(\frac{V-1500}{18500}\right)}{\text{awrt } -0.227}$ Allow k to be exact.			

(b)

- B1: Differentiates to a form $\left(\frac{dV}{dt} = \right) -kAe^{-kt}$ where k and A may be their values from (a)
 e.g. $-0.227 \times 18500 e^{-0.227t}$ May just see e.g. (using an exact k) $-4191.3... e^{-0.227t}$ or
 e.g. (using k to 3sf) $-4199.5 e^{-0.227t}$ but do not be too concerned by over rounding
 provided the intention is clear that it is their $-kA$
 Do not be too concerned by the left hand side / poor labelling of the derivative.
- M1: Substitutes $Ae^{-kt} = V - 1500$ into their $\left(\frac{dV}{dt} = \right) \pm kAe^{-kt}$ to form an expression for $\frac{dV}{dt}$ in
 terms of V . May see e.g. $-0.227 \times 18500 e^{-0.227t} \Rightarrow -0.227(V - 1500)$.
 Condone recovery of a sign slip on the index on $e^{-kt} \rightarrow e^{kt}$ provided it is before they
 substitute $Ae^{-kt} = V - 1500$ for this mark.
- Using the given answer: substitutes $V - 1500 = Ae^{-kt}$ and shows that
 $-k(V - 1500) = -kAe^{-kt}$ (may be in terms of their A and k)
- A1*: Full and complete proof with sight of $\frac{dV}{dt}$ oe seen somewhere in their solution and no errors
 seen. Must see $-kAe^{-kt}$ (or using their values) before proceeding to the given answer which
 may be written using their numerical value for k
 Using the given answer they must conclude that $\frac{dV}{dt} = -k(V - 1500)$ which may be written
 using their numerical value for k

Alt (b) Separating the variables – may be in terms of their numerical value for k

- B1ft: Separates the variables correctly and integrates to $\ln(V - 1500) = -kt$ with or without $+c$
- $$\int \frac{1}{(V - 1500)} dV = \int -k dt \Rightarrow \ln(V - 1500) = -kt (+c) \text{ oe}$$
- M1: Proceeds from $\dots \ln(V - 1500) = \dots kt + c$ o.e. and rearranges to make V the subject. Condone
 slips.
 e.g. $\ln(V - 1500) = -kt + c \Rightarrow V - 1500 = Ae^{-kt} \Rightarrow V = 1500 + Ae^{-kt}$ or $V = 1500 + e^{-kt+c}$
 (just look for proceeding to $V = \dots$ for this mark though.) May be in terms of their numerical
 values for k and A
- A1*: Achieves $V = 1500 + Ae^{-kt}$ with no errors seen and concludes that $\frac{dV}{dt} = -k(V - 1500)$
 May be in terms of their numerical values for k and A



Alt (b) Rearranging to make t the subject

B1ft: For a correct rearrangement to $t = -\frac{1}{k} \ln\left(\frac{V-1500}{18500}\right)$ ft on their k

M1: Differentiates $\dots \ln(V-1500)$ to $\frac{\dots}{V-1500}$ and then finding $\frac{dV}{dt} = \frac{1}{\left(\frac{dt}{dV}\right)}$

Typically look for $t = -\frac{1}{k} \ln\left(\frac{V-1500}{18500}\right) \Rightarrow \frac{dt}{dV} = -\frac{1}{k} \left(\frac{1}{V-1500}\right)$ o.e. so they may have an unsimplified version of this e.g.

$$t = -\frac{1}{k} \ln\left(\frac{V-1500}{18500}\right) \Rightarrow \frac{dt}{dV} = -\frac{1}{k} \left(\frac{\frac{1}{18500}}{\frac{V-1500}{18500}}\right) \Rightarrow \frac{dV}{dt} = -k \left(\frac{V-1500}{18500}\right)$$

May be in terms of their numerical k . Condone slips.

A1*: Achieves the given answer with no errors and $\frac{dV}{dt}$ seen somewhere in their solution

(c) **Note the question asks for a limitation of the model. If there is ambiguity over whether the response is referring to the model then try putting “the model suggests” or “the model” in front of their comment to see if this is a valid limitation.**

Ignore comments which do not contradict a valid limitation.

If values are given then they must be correct and if it is the value of the car then it must have units (£ or pounds)

B1: **Suitable limitations in context referring to the limitation of the model which score B1**

e.g. (the model suggests) the value/price of the car will never go below £1500

e.g. (the model suggests) the rate of decrease of the value of the car is proportionally the same each year

e.g. (the model suggests) after a certain period of time the car will no longer lose value (condone this property of the model that it will tend to a limit)

e.g. (the model) only takes account of age

e.g. (the model) does not take into account damage / alteration to the car/ mileage

e.g. (the model) predicts that the car's value will always go down

e.g. after many years the car may become worthless whereas the model does not allow for this (valid limitation comparing the car to the model)

e.g. the value of the car may go up where as it only decreases according to the model

Do not accept vague/incorrect/irrelevant or non-contextual comments which score B0

e.g. after many years the car may become worthless

e.g. the value of the car may increase

e.g. damage or alterations to the car may impact the value

e.g. the car will still have value when it is very old (should refer to £1500)

e.g. (the model suggests) the minimum value is 1500 (no units for money)

e.g. (the model suggests) the (value of the) car cannot be negative

e.g. car value will fluctuate which the model will not show (a model is not for this purpose)

e.g. the rate of decrease is proportionally the same each year (no context)



Question	Scheme	Marks	AOs
9(a)	$5000 - 5000e^{-0.075 \times 3} = \dots$	M1	3.4
	1007	A1	1.1b
		(2)	
(b)	$5000 - 5000e^{-0.075T} = 3000 \rightarrow 5000e^{-0.075T} = 2000$	M1	1.1b
	$\Rightarrow T = \frac{1}{-0.075} \ln \frac{2000}{5000}$	dM1	1.1b
	$\Rightarrow T = 12.22$	A1	1.1b
		(3)	
(c)	$\left(\frac{dN}{dt} =\right) -5000 \times -0.075e^{-0.075 \times 3} = \dots$	M1	3.4
	299 (car sales per month)	A1	1.1b
		(2)	
(d)	Change the constant 5000 to 6500.	B1	3.5c
		(1)	
(8 marks)			
Notes			
(a) M1: Substitutes $t = 3$ into the given model and proceeds to a value. Implied by awrt 1007, 1008 or 1010 Condone copying slips e.g. -0.75 or -0.0075 in place of -0.075 or 500 in place of 5000 – in such cases you may need to check their calculation if the substitution isn't shown. A1: 1007 but allow 1008 or 1010 (3sf). Must be a whole number. Correct answer only scores both marks. A0 for e.g. 1007.4 Do not ISW if they go on to sum their values of N from $t = 1$ to 3			
(b) M1: Sets $5000 - 5000e^{-0.075T}$ equal to 3000 and proceeds to either $Ae^{-0.075T} = B$ or $e^{-0.075T} = k$ where A, B, k are constants with no restrictions for this mark. May use t instead of T. Condone slips. Condone copying slips e.g. -0.75 or -0.0075 in place of -0.075 or 500 in place of 5000. dM1: Uses the correct order of operations and correct log work from an equation of the form $Ae^{-0.075T} = B$ with $AB > 0$ or $e^{-0.075T} = k$ with $k > 0$ to proceed to a value for T or t which may be a numerical expression. e.g. $Ae^{-0.075T} = B \Rightarrow \ln A - 0.075T = \ln B \Rightarrow T = \frac{\ln A - \ln B}{0.075}$ or $= \frac{\ln(A/B)}{0.075}$ Condone log being used in place of $\ln$ unless there is clear evidence that they have used an inconsistent base. May not be scored if “recovered” from e.g. $e^{-0.075T} = -\frac{2}{5}$ Condone copying slips e.g. -0.75 or -0.0075 in place of -0.075 or 500 in place of 5000. A1: 12.22 cao but must see a correct equation following use of logs, e.g., $-0.075T = \ln \frac{2}{5}$ or $T = \frac{1}{-0.075} \times -0.916$ which may have intermediate rounding. Ignore any units given. This mark is available following the occasional slip in writing -0.075 as -0.75 or -0.0075 provided it is not consistently used.			

(c)

- M1: Differentiates once to the form $\lambda e^{-0.075t}$, λ a constant (even ± 5000), and substitutes in $t = 3$
 Award for an expression of the form $ke^{-0.075 \times 3}$ with constant k if no incorrect working is seen.
 The substitution of $t = 3$ may be implied by a correct value for their derivative of the required form, provided the derivative is seen. Do not be concerned with what they call their derivative.
- A1: awrt 299 (car sales per month). Answer only scores no marks. Condone 300 following 299.4
 Requires a correct derivative to be seen, e.g., $375e^{-0.075t}$ o.e., with or without the substitution of $t = 3$ present.
 Units may be omitted, but score A0 if an incorrect time frame is given e.g. cars per week

(d)

- B1: Acceptable refinement. Some examples:
- Change the 5000 to 6500 (condone ambiguity about which 5000)
 - Change the first 5000 to 6500
 - Change both 5000s to 6500s
 - Multiply the model by 1.3 (or e.g. $\frac{65}{50}$ or $\frac{13}{10}$)
 - Add 1500 to the model
 - Change the constant to 6500
- The statement of an acceptable refined model e.g. $(N =) 6500 - 6500e^{-0.075t}$
 or e.g. $(N =) 6500 - 5000e^{-0.075t}$ scores B1.
 Ignore any extra unnecessary refinements such as increase/decrease the 0.075.
- The following score B0:
- Change 5000 in the model (not specific enough – they haven't said to 6500)
 - Change the second 5000 to 6500 (incorrect)

