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EDEXCEL A LEVEL MATHEMATICS: QUESTIONS BY TOPIC

Differentiation (7)

A Level Mathematics Topic 7

41 questions · 325 marks · Mark schemes included

Pearson Edexcel A Level Mathematics (9MA0), June 2018 - June 2025

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HOW TO USE THIS BOOKLET

Questions are grouped by specification sub-topic and run from the lowest mark tariff to the highest within each one.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme.

Diagrams, graphs, tables and data are printed exactly as they appeared in the original papers.

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QUESTIONS

41 questions · 325 marks · 6 sub-topics



SUB-TOPIC 7.1

DIFFERENTIATION FROM FIRST PRINCIPLES

Sub-topic 7.1 · 3 questions · 11 marks · lowest tariff first



(3)





4. Given that

$$y = 2x^2$$

use differentiation from first principles to show that

$$\frac{dy}{dx} = 4x$$

(3)





SUB-TOPIC 7.2

DIFFERENTIATING STANDARD FUNCTIONS

Sub-topic 7.2 · 1 question · 4 marks · lowest tariff first



14. Given that

$$y = \frac{x-4}{2+\sqrt{x}} \quad x > 0$$

show that

$$\frac{dy}{dx} = \frac{1}{A\sqrt{x}} \quad x > 0$$

where A is a constant to be found.

(4)







Question 14 continued

(Total for Question 14 is 4 marks)



SUB-TOPIC 7.3

TANGENTS, NORMALS AND STATIONARY POINTS

Sub-topic 7.3 · 18 questions · 133 marks · lowest tariff first



Question 2 continued

(Total for Question 2 is 4 marks)



(Total for Question 4 is 5 marks)



Question 7 continued

(Total for Question 7 is 5 marks)

1.

$$y = 4x^3 - 7x^2 + 5x - 10$$

(a) Find in simplest form

$$(i) \quad \frac{dy}{dx}$$

$$(ii) \quad \frac{d^2 y}{dx^2}$$

(3)

(b) Hence find the exact value of x when $\frac{d^2y}{dx^2} = 0$

(2)

(Total for Question 1 is 5 marks)

Question 5 continued

(Total for Question 5 is 5 marks)



Question 6 continued

(Total for Question 6 is 6 marks)



Question 2 continued

(Total for Question 2 is 7 marks)



5. The curve C has equation

$$y = 5x^4 - 24x^3 + 42x^2 - 32x + 11 \quad x \in \mathbb{R}$$

(a) Find

(i) $\frac{dy}{dx}$

$$(ii) \frac{d^2 y}{dx^2}$$

(3)

(b) (i) Verify that C has a stationary point at $x = 1$

(ii) Show that this stationary point is a point of inflection, giving reasons for your answer.

(4)

Question 5 continued

(Total for Question 5 is 7 marks)



Question 11 continued

Question 11 continued

Question 11 continued

(Total for Question 11 is 7 marks)



10.

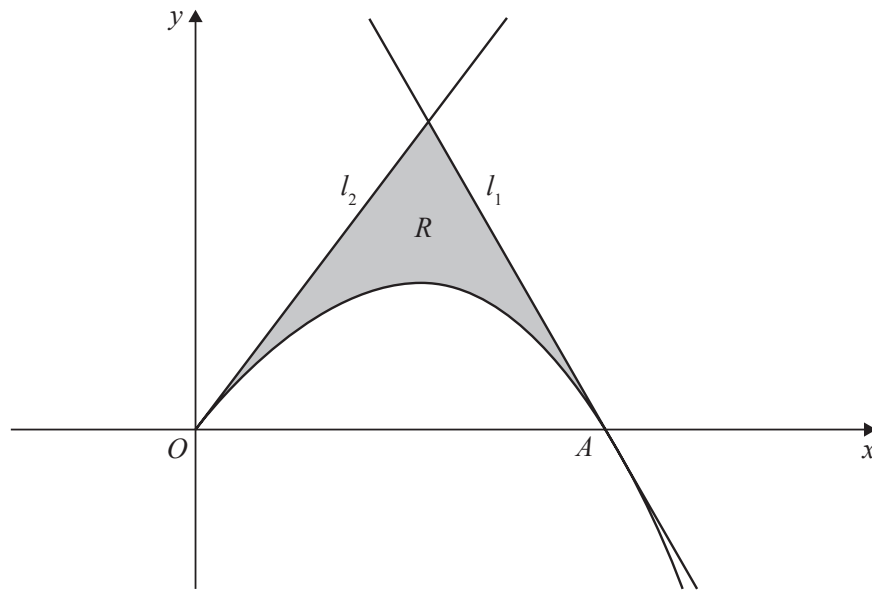


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve with equation

$$y = 8x - x^{\frac{5}{2}} \quad x \geq 0$$

The curve crosses the x -axis at the point A .

(a) Verify that the x coordinate of A is 4

(1)

The line l_1 is the tangent to the curve at A .

(b) Use calculus to show that an equation of line l_1 is

$$12x + y = 48$$

(3)

The line l_2 has equation $y = 8x$

The region R , shown shaded in Figure 3, is bounded by the curve, the line l_1 and the line l_2

(c) Use algebraic integration to find the exact area of R .

(5)



Question 10 continued

Question 10 continued

(Total for Question 10 is 9 marks)



7.

In this question you should show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

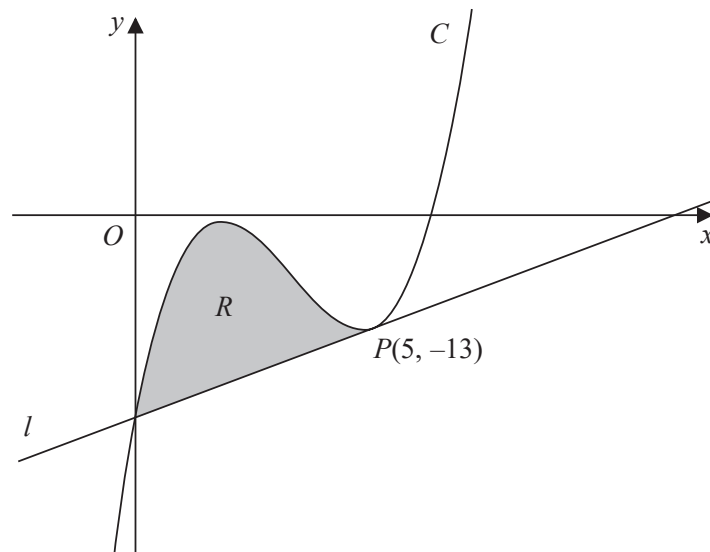


Figure 2

Figure 2 shows a sketch of part of the curve C with equation

$$y = x^3 - 10x^2 + 27x - 23$$

The point $P(5, -13)$ lies on C The line l is the tangent to C at P

- (a) Use differentiation to find the equation of l , giving your answer in the form $y = mx + c$ where m and c are integers to be found. (4)

- (b) Hence verify that l meets C again on the y -axis. (1)

The finite region R , shown shaded in Figure 2, is bounded by the curve C and the line l .

- (c) Use algebraic integration to find the exact area of R . (4)

Question 7 continued



(Total for Question 7 is 9 marks)



Question 5 continued



Question 5 continued

Question 5 continued

(Total for Question 5 is 10 marks)



15.

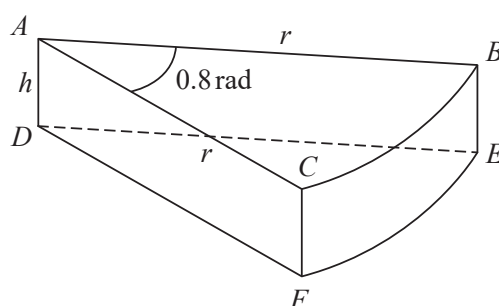


Figure 5

A company makes toys for children.

Figure 5 shows the design for a solid toy that looks like a piece of cheese.

The toy is modelled so that

- face ABC is a sector of a circle with radius r cm and centre A
- angle $BAC = 0.8$ radians
- faces ABC and DEF are congruent
- edges AD , CF and BE are perpendicular to faces ABC and DEF
- edges AD , CF and BE have length h cm

Given that the volume of the toy is 240 cm^3

(a) show that the surface area of the toy, $S \text{ cm}^2$, is given by

$$S = 0.8r^2 + \frac{1680}{r}$$

making your method clear.

(4)

Using algebraic differentiation,

(b) find the value of r for which S has a stationary point.

(4)

(c) Prove, by further differentiation, that this value of r gives the minimum surface area of the toy.

(2)

Question 15 continued



Question 15 continued

(Total for Question 15 is 10 marks)



7.

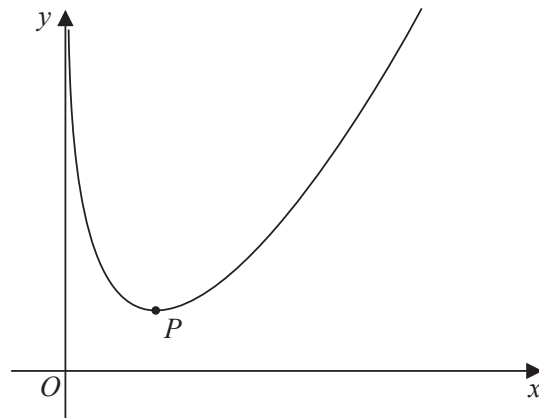


Figure 1

Figure 1 shows a sketch of the curve C with equation

$$y = \frac{4x^2 + x}{2\sqrt{x}} - 4\ln x \quad x > 0$$

(a) Show that

$$\frac{dy}{dx} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}} \quad (4)$$

The point P , shown in Figure 1, is the minimum turning point on C .

(b) Show that the x coordinate of P is a solution of

$$x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12} \right)^{\frac{2}{3}} \quad (3)$$

(c) Use the iteration formula

$$x_{n+1} = \left(\frac{4}{3} - \frac{\sqrt{x_n}}{12} \right)^{\frac{2}{3}} \quad \text{with } x_1 = 2$$

to find (i) the value of x_2 to 5 decimal places,

(ii) the x coordinate of P to 5 decimal places.

(3)

Question 7 continued

Question 7 continued

Question 7 continued

(Total for Question 7 is 10 marks)



7.

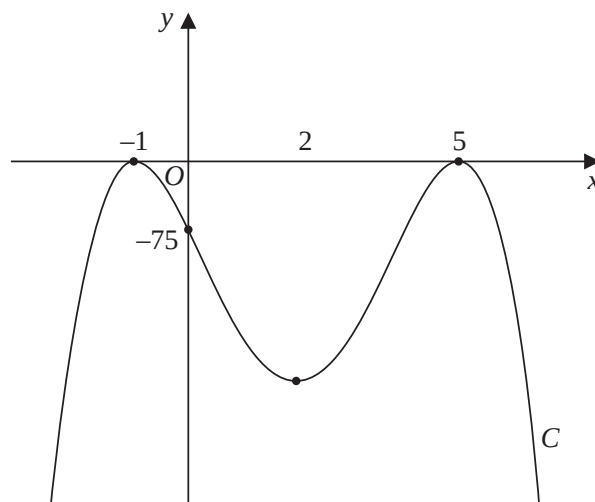


Figure 1

Figure 1 shows a sketch of a curve C with equation $y = f(x)$, where $f(x)$ is a quartic expression in x .

The curve

- has maximum turning points at $(-1, 0)$ and $(5, 0)$
- crosses the y -axis at $(0, -75)$
- has a minimum turning point at $x = 2$

(a) Find the set of values of x for which

$$f'(x) \geq 0$$

writing your answer in set notation.

(2)

(b) Find the equation of C . You may leave your answer in factorised form.

(3)

The curve C_1 has equation $y = f(x) + k$, where k is a constant.

Given that the graph of C_1 intersects the x -axis at exactly four places,

(c) find the range of possible values for k .

(2)





9.

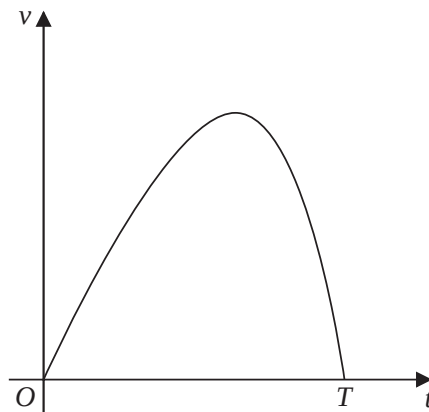


Figure 2

A racing car is driven along a straight road.

Figure 2 shows a graph of the speed of the car as it travels along the road.

The car starts from rest and is driven for T seconds before stopping.

The speed of the car is modelled by the equation

$$v = 15t - te^{0.2t} \quad 0 \leq t \leq T$$

where t seconds is the time after the car starts to move.

According to the model,

(a) find the value of T , giving your answer to one decimal place, (2)

(b) show that the maximum speed of the car occurs when

$$t = 5 \ln \left(\frac{75}{t + 5} \right) \quad (4)$$

Using the iteration formula

$$t_{n+1} = 5 \ln \left(\frac{75}{t_n + 5} \right) \quad \text{with } t_1 = 8$$

(c) (i) find the value of t_3 to 3 decimal places,
 (ii) find, by repeated iteration, the time taken for the car to reach maximum speed. (3)

Question 9 continued

(Total for Question 9 is 9 marks)



15.

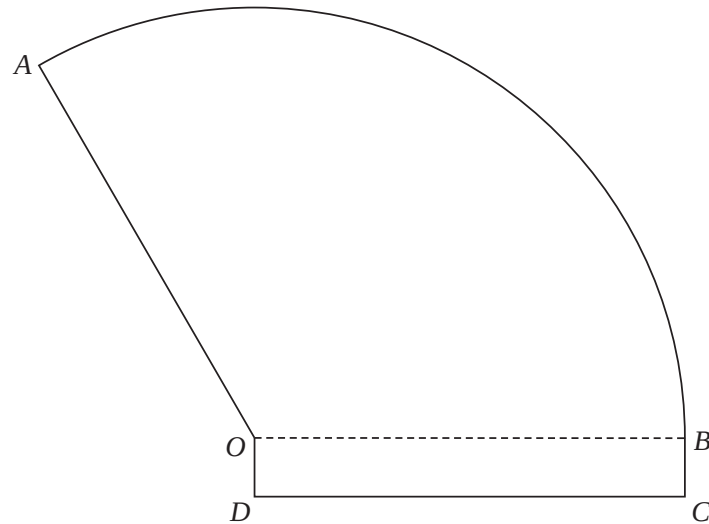


Figure 4

Figure 4 shows the plan view for the design of a stage.

The shape of this design consists of a sector of a circle AOB joined to a rectangle $OBCD$.

Given that

- the radius of the sector is r metres and angle AOB is θ radians
- the length and width of the rectangle are r metres and $\frac{1}{10}r$ metres respectively
- the total area of the stage is 240 m^2

(a) show that the perimeter of the stage, P metres, is given by

$$P = 2r + \frac{480}{r}$$

You must make your method clear.

(4)

Using algebraic differentiation,

(b) find the value of r for which P has a stationary value.

(3)

(c) Prove, by further differentiation, that this value of r gives the minimum perimeter of the stage.

(2)

Question 15 continued



Question 15 continued



Question 15 continued

(Total for Question 15 is 9 marks)



8.

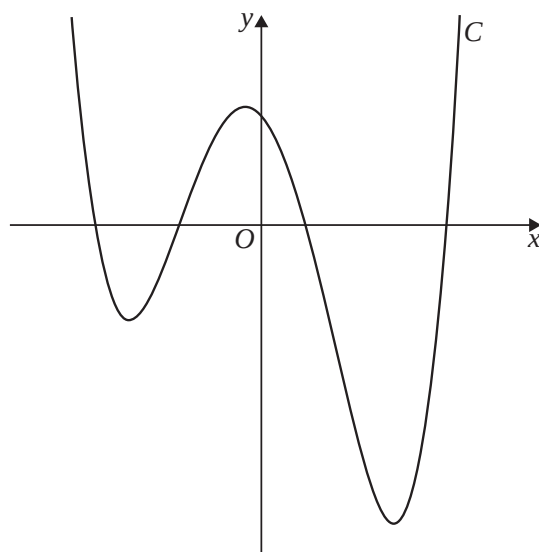


Figure 2

$$f(x) = x^4 + \frac{1}{3}x^3 - 8x^2 + ax + \frac{17}{3}$$

where a is a constant.

Figure 2 shows a sketch of the curve C with equation $y = f(x)$

Given that C has a local maximum at $x = -\frac{1}{4}$

(a) show that $a = -4$ (4)

(b) find the exact y coordinate of the local maximum. (1)

The equation $f(x) = k$, where k is a constant, has 4 distinct solutions.

(c) Using algebra and showing all stages of your working, find the range of values of k .
Give the answer using set notation.

(Solutions relying on calculator technology are not acceptable.)

(4)

Question 8 continued

(Total for Question 8 is 9 marks)



SUB-TOPIC 7.4

PRODUCT, QUOTIENT AND CHAIN RULES

Sub-topic 7.4 · 8 questions · 76 marks · lowest tariff first



3.

$$y = \frac{5x^2 + 10x}{(x+1)^2} \quad x \neq -1$$

- (a) Show that $\frac{dy}{dx} = \frac{A}{(x+1)^n}$ where A and n are constants to be found.

(4)

- (b) Hence deduce the range of values for x for which $\frac{dy}{dx} < 0$

(1)



Question 12 continued

(Total for Question 12 is 6 marks)



Question 4 continued



Question 4 continued



Question 4 continued

(Total for Question 4 is 9 marks)





Question 9 continued

Lined area for writing the answer to Question 9 continued.



Question 9 continued

(Total for Question 9 is 9 marks)



12.

$$f(x) = 10e^{-0.25x} \sin x, \quad x \geq 0$$

- (a) Show that the x coordinates of the turning points of the curve with equation $y = f(x)$ satisfy the equation $\tan x = 4$

(4)

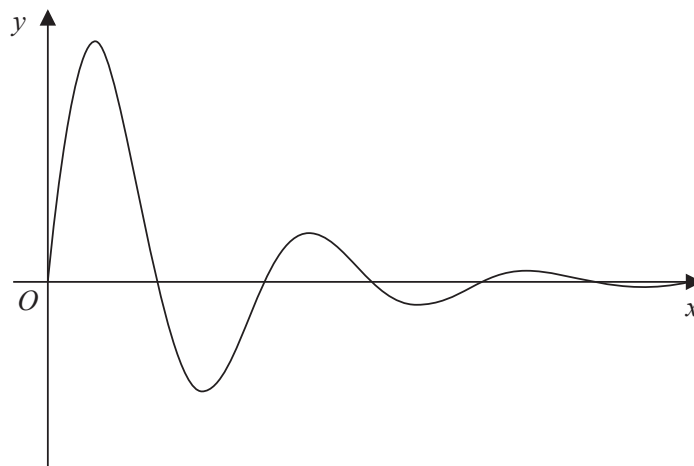


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = f(x)$.

- (b) Sketch the graph of H against t where

$$H(t) = |10e^{-0.25t} \sin t| \quad t \geq 0$$

showing the long-term behaviour of this curve.

(2)

The function $H(t)$ is used to model the height, in metres, of a ball above the ground t seconds after it has been kicked.

Using this model, find

- (c) the maximum height of the ball above the ground between the first and second bounce.

(3)

- (d) Explain why this model should not be used to predict the time of each bounce.

(1)

Question 12 continued



Question 12 continued



Question 12 continued

(Total for Question 12 is 10 marks)



Question 11 continued





Question 11 continued

(Total for Question 11 is 11 marks)



15. A curve has equation $y = f(x)$, where

$$f(x) = \frac{7xe^x}{\sqrt{e^{3x} - 2}} \quad x > \ln \sqrt[3]{2}$$

(a) Show that

$$f'(x) = \frac{7e^x(e^{3x}(2-x) + Ax + B)}{2(e^{3x} - 2)^{\frac{3}{2}}}$$

where A and B are constants to be found.

(5)

(b) Hence show that the x coordinates of the turning points of the curve are solutions of the equation

$$x = \frac{2e^{3x} - 4}{e^{3x} + 4}$$

(2)

The equation $x = \frac{2e^{3x} - 4}{e^{3x} + 4}$ has two positive roots α and β where $\beta > \alpha$

A student uses the iteration formula

$$x_{n+1} = \frac{2e^{3x_n} - 4}{e^{3x_n} + 4}$$

in an attempt to find approximations for α and β

Diagram 1 shows a plot of part of the curve with equation $y = \frac{2e^{3x} - 4}{e^{3x} + 4}$ and part of the line with equation $y = x$

Using Diagram 1 on page 42

(c) draw a staircase diagram to show that the iteration formula starting with $x_1 = 1$ can be used to find an approximation for β

(1)

Use the iteration formula with $x_1 = 1$, to find, to 3 decimal places,

(d) (i) the value of x_2

(ii) the value of β

(3)

Using a suitable interval and a suitable function that should be stated

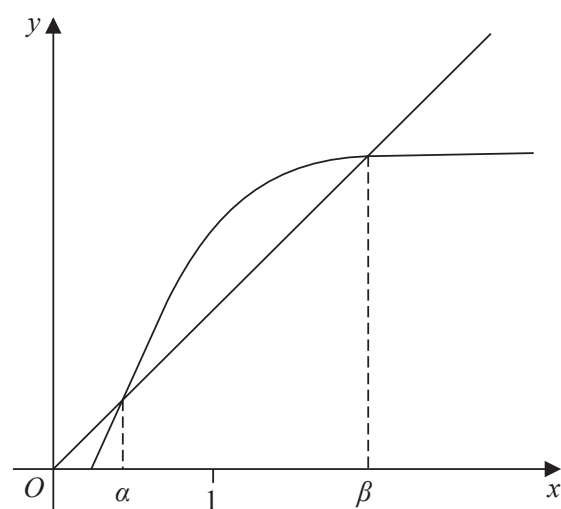
(e) show that $\alpha = 0.432$ to 3 decimal places.

(2)





Only use the copy of Diagram 1 if you need to redraw your answer to part (c).



copy of Diagram 1

Question 15 continued





15.

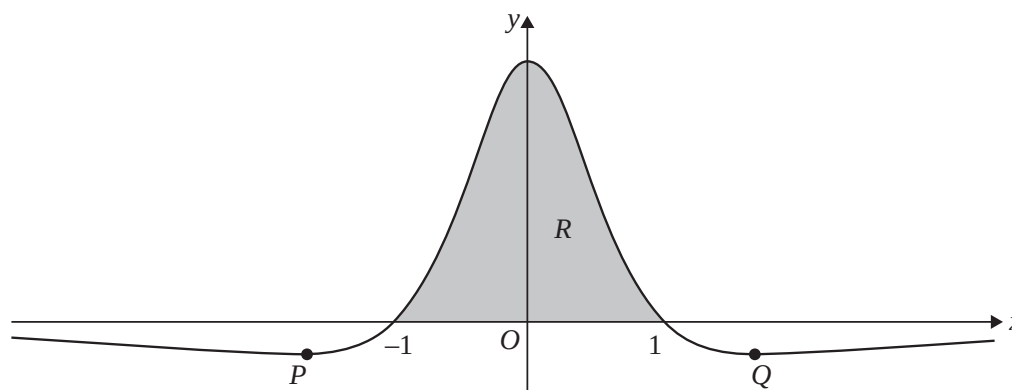


Figure 5

**In this question you must show all stages of your working.
Solutions relying on calculator technology are not acceptable.**

Figure 5 shows a sketch of part of the curve with equation $y = f(x)$, where

$$f(x) = \frac{1 - x^2}{(1 + x^2)^2}$$

The curve

- intersects the x -axis at -1 and 1
- has minimum turning points at P and Q

as shown in Figure 5.

(a) Use calculus to find the exact coordinates of P .

(5)

(b) Using the substitution $x = \tan \theta$ show that

$$\int_{-1}^1 f(x) dx = \int_{\alpha}^{\beta} \cos 2\theta d\theta$$

where α and β are constants to be found.

(5)

The finite region R , shown shaded in Figure 5, is bounded by the x -axis and the curve.

(c) Use algebraic integration to find the area of R .

(3)





(Total for Question 15 is 10 marks)



SUB-TOPIC 7.5

IMPLICIT AND PARAMETRIC DIFFERENTIATION

Sub-topic 7.5 · 8 questions · 67 marks · lowest tariff first



13. The curve C has parametric equations

$$x = \sin 2\theta \quad y = \operatorname{cosec}^3 \theta \quad 0 < \theta < \frac{\pi}{2}$$

- (a) Find an expression for $\frac{dy}{dx}$ in terms of θ

(3)

- (b) Hence find the exact value of the gradient of the tangent to C at the point where $y = 8$

(3)



(Total for Question 13 is 6 marks)





Question 15 continued

(Total for Question 15 is 7 marks)









Question 9 continued

(Total for Question 9 is 10 marks)



Question 15 continued





Question 15 continued

(Total for Question 15 is 12 marks)



16.

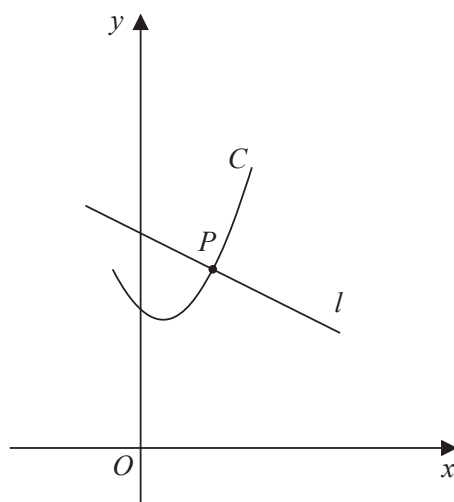


Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 2 \tan t + 1 \quad y = 2 \sec^2 t + 3 \quad -\frac{\pi}{4} \leq t \leq \frac{\pi}{3}$$

The line l is the normal to C at the point P where $t = \frac{\pi}{4}$

(a) Using parametric differentiation, show that an equation for l is

$$y = -\frac{1}{2}x + \frac{17}{2} \quad (5)$$

(b) Show that all points on C satisfy the equation

$$y = \frac{1}{2}(x-1)^2 + 5 \quad (2)$$

The straight line with equation

$$y = -\frac{1}{2}x + k \quad \text{where } k \text{ is a constant}$$

intersects C at two distinct points.

(c) Find the range of possible values for k .

(5)







Question 16 continued

(Total for Question 16 is 12 marks)



(Total for Question 5 is 7 marks)



SUB-TOPIC 7.6

RATES OF CHANGE AND CONSTRUCTING DIFFERENTIAL EQUATIONS

Sub-topic 7.6 · 3 questions · 34 marks · lowest tariff first



11.

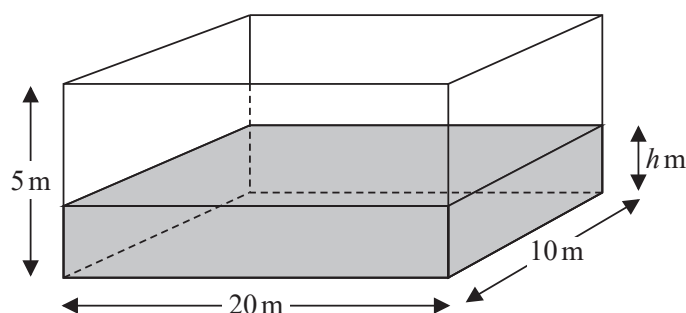


Figure 1

A tank in the shape of a cuboid is being filled with water.

The base of the tank measures 20 m by 10 m and the height of the tank is 5 m, as shown in Figure 1.

At time t minutes after water started flowing into the tank the height of the water was h m and the volume of water in the tank was V m³

In a model of this situation

- the sides of the tank have negligible thickness
- the rate of change of V is inversely proportional to the square root of h

(a) Show that

$$\frac{dh}{dt} = \frac{\lambda}{\sqrt{h}}$$

where λ is a constant.

(3)

Given that

- initially the height of the water in the tank was 1.44 m
- exactly 8 minutes after water started flowing into the tank the height of the water was 3.24 m

(b) use the model to find an equation linking h with t , giving your answer in the form

$$h^{\frac{3}{2}} = At + B$$

where A and B are constants to be found.

(5)

(c) Hence find the time taken, from when water started flowing into the tank, for the tank to be completely full.

(2)



Question 11 continued



Question 11 continued



Question 11 continued

(Total for Question 11 is 10 marks)









(Total for Question 14 is 15 marks)



10. Water flows at a constant rate into a large container.

There is a tap at the bottom of the container.

At time t hours after the tap was opened

- the volume of water in the container is $V\text{m}^3$
- water is flowing into the container at a constant rate of 0.45m^3 per hour
- water is leaving the container through the tap at a rate of $0.3V\text{m}^3$ per hour

(a) Show that

$$20 \frac{dV}{dt} = 9 - 6V \quad (2)$$

Given that when the tap was opened, there was 0.25 m^3 of water in the container,

(b) solve the differential equation to show that

$$V = P - Qe^{-kt}$$

where P , Q and k are positive constants to be found.

(5)

Given that

- the capacity of the container is 2 m^3
- the tap remains open
- the water continues to flow into the tank at the same rate

(c) determine whether the container will ever become full, giving a reason for your answer.

(2)

[illegible]

Question 10 continued



Question 10 continued



(Total for Question 10 is 9 marks)



MARK SCHEMES

official mark schemes



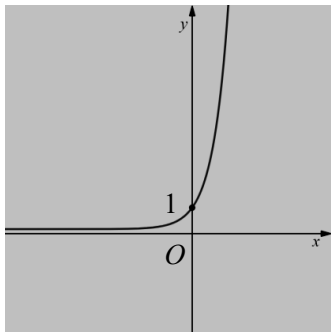
Question	Scheme	Marks	AOs
4	$\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h}$	M1	2.1
	$= \frac{2xh + h^2}{h}$	A1	1.1b
	$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x^*$	A1*	2.5
		(3)	
(3 marks)			
Notes			
Note: Throughout the question allow use of δx for h or any other letter e.g. α if used consistently. If δx is used then you can condone e.g. $\delta^2 x$ for δx^2 as well as condoning e.g. poorly formed δ's M1: Begins the process by writing down the gradient of the chord and attempts to expand the correct squared bracket – you can condone “poor” squaring e.g. $(x+h)^2 = x^2 + h^2$ but the $-x^2$ must be present. A1: Reaches a correct fraction o.e. with the x^2 terms cancelled out and with no algebraic errors, e.g. $\frac{x^{\cancel{2}} + 2xh + h^2 - x^{\cancel{2}}}{h}$, $2x + h$ is correct. A1*: Completes the process by applying a limiting argument and deduces that $\frac{dy}{dx} = 2x$ with no errors seen. They must have $= 2x$ and not just $\lim_{h \rightarrow 0} 2x$ to complete the proof. $\frac{dy}{dx} =$ or an equivalent e.g. $f'(x) =$ or “Gradient =” must be evident somewhere in their working or final line. If $f'(x)$ is used then there is no requirement to see $f(x)$ defined first. Condone e.g. $\frac{dy}{dx} \rightarrow 2x$ or $f'(x) \rightarrow 2x$. Condone missing brackets to allow e.g. $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \lim_{h \rightarrow 0} 2x + h = 2x$ Do not allow $h = 0$ if there is never a reference to $h \rightarrow 0$. e.g. $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \lim_{h \rightarrow 0} 2x + 0 = 2x$ is acceptable but e.g. $\frac{dy}{dx} = \frac{2xh + h^2}{h} = 2x + 0 = 2x$ is not unless $h \rightarrow 0$ is seen. The $h \rightarrow 0$ does not need to be present throughout the proof e.g. appear on every line but must appear at least once. They must reach $2x + h$ at the end and not $\frac{2xh + h^2}{h}$ (without the h's cancelled) to complete the limiting argument.			

for $-\frac{4}{3}$ e.g. $-1\frac{1}{3}$ or $-1.\dot{3}$ but not e.g. -1.33

Is w if necessary e.g. ignore subsequent attempts to put the values in an inequality e.g. $-\frac{4}{3} < x < 10$

But if e.g. $x = -\frac{4}{3}$ is obtained and a candidate states $x = \left| -\frac{4}{3} \right|$ then score A0



Question	Scheme	Marks	AOs	
2(a)		Correct shape or correct intercept – see notes	B1	1.2
		Fully correct – see notes	B1	1.1b
			(2)	
(b)	$4^x = 100 \Rightarrow x = \log_4 100$ or e.g. $x \log 4 = \log 100 \Rightarrow x = \frac{\log 100}{\log 4}$	M1	1.1b	
	$\Rightarrow (x =) \text{awrt } 3.32$	A1	1.1b	
		(2)		
(4 marks)				
Notes:				

Note that B0B1 is not possible in part (a)

(a) Axes do not need to be labelled. No sketch is no marks.

B1: Correct shape or correct intercept.

Shape: A positive exponential curve in quadrants 1 and 2 only, passing through a point on the positive y-axis. Must “level out” in quadrant 2 but not necessarily asymptotic to the x-axis and allow if the curve bends up slightly for $x < 0$ but do not allow a clear “U” shape. It must not clearly “stop” on the x-axis to the left of the y-axis.

OR

Intercept: The intercept can be marked as 1 or (0, 1) or $y = 1$ or (1, 0) as long as it is in the correct place. May also be seen away from the sketch but must be seen as (0, 1) or possibly these coordinates in a table but it must correspond to the sketch. If there is any ambiguity, the sketch takes precedence.

B1: Fully correct.

Shape: A positive exponential curve in quadrants 1 and 2 only, passing through a point on the positive y-axis. The curve must appear to be asymptotic to the x-axis **and it must level out at least half way below the intercept**. Allow if the curve bends up slightly for $x < 0$ but do not allow a clear “U” shape. The curve must not bend back on itself on the rhs of the y-axis. There must be no suggestion that the curve approaches another horizontal asymptote other than the x-axis e.g. a horizontal dotted line that the curve approaches.

AND

Intercept: As above

See practice items and below for some examples:

(b)

M1: Uses logs in an attempt to solve the equation. E.g. takes $\log$ base 4 and obtains $x = \log_4 100$ Alternatively takes logs (any base) to obtain $x \log 4 = \log 100$ and proceeds to $x = \frac{\log 100}{\log 4}$ Allow if this subsequently becomes e.g. $\log 25$ as long as $\frac{\log 100}{\log 4}$ is seen **but** $x \log 4 = \log 100 \Rightarrow x = \log 25$ or $x \log 4 = \log 100 \Rightarrow x = \log 100 - \log 4$ scores M0**A1:** awrt 3.32 . A correct answer only of awrt 3.32 scores M1A1Note that a common incorrect answer is $x = 3.218875\dots$ and comes from $\ln 25$ or $\ln 100 - \ln 4$ and unless $x = \frac{\ln 100}{\ln 4}$ is seen previously, this scores M0A0

Question	Scheme	Marks	AOs
3(a)(i) (ii)	$a_1 = 3, a_2 = 5, a_3 = 3 \dots$	B1	1.1b
	2	B1	1.1b
		(2)	
(b)	$\sum_{n=1}^{85} a_n = 42 \times (3+5) + 3 \text{ o.e.}$	M1	3.1a
	$= 339$	A1	1.1b
		(2)	
(4 marks)			
Notes:			

(a)(i) Mark (a)(i) and (a)(ii) together.

B1: States the values of at least $a_2 = 5$ and $a_3 = 3$. This is sufficient but if more terms are given they must be correct. There is no need to see e.g. $a_2 = \dots, a_3 = \dots$ just look for values.Allow an algebraic approach e.g. $a_{n+1} = 8 - a_n, a_{n+2} = 8 - (8 - a_n) = a_n$ A conclusion is **not** needed.

(a)(ii)

B1: States that the order of the periodic sequence is 2

Allow “second order”, “it repeats every 2 numbers” or equivalent statements that convey the idea of the period being 2.

Note that ± 2 is B0

(b)

M1: Attempts a **correct** method to find $\sum_{n=1}^{85} a_n$ For example $\sum_{n=1}^{85} a_n = 42 \times (3+5) + 3, \sum_{n=1}^{85} a_n = \frac{84}{2} \times 3 + 42 \times 5 + 3$ or $\sum_{n=1}^{85} a_n = 43 \times (3+5) - 5$ or $\sum_{n=1}^{85} a_n = 43 \times 3 + 42 \times 5$ or $\sum_{n=1}^{85} a_n = \frac{85}{2} \times 8 - 1$ There may be other methods e.g. “Chunking”: $5 \times (3+5) = 40, 40 \times 8 = 320, 320 + 3 \times 3 + 2 \times 5 = 339$ **A1:** 339. Correct answer only scores both marks.

Attempts to use an AP formula score M0



Question	Scheme	Marks	AOs
4	$\frac{2(x+h)^2 - 2x^2}{h} = \dots$	M1	2.1
	$\frac{2(x+h)^2 - 2x^2}{h} = \frac{4xh + 2h^2}{h}$	A1	1.1b
	$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{4xh + 2h^2}{h} = \lim_{h \rightarrow 0} (4x + 2h) = 4x^*$	A1*	2.5
		(3)	
(3 marks)			
Notes:			

Throughout the question allow the use of δx for h or any other letter e.g. α if used consistently. If δx is used then you can condone e.g. $\delta^2 x$ for δx^2 as well as condoning e.g. poorly formed δ 's

M1: Begins the process by writing down the gradient of the chord and attempts to expand the correct bracket – you can condone “poor” squaring e.g. $(x+h)^2 = x^2 + h^2$.

Note that $\frac{2(x-h)^2 - 2x^2}{-h} = \dots$ is also a possible approach.

A1: Reaches a correct fraction or with the x^2 terms cancelled out.

E.g. $\frac{4xh + 2h^2}{h}$, $\frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{2x^2}}{h}$, $4x + 2h$

A1*: Completes the process by applying a limiting argument and deduces that $\frac{dy}{dx} = 4x$ with no

errors seen. The “ $\frac{dy}{dx} =$ ” doesn’t have to appear but there must be something equivalent e.g.

“ $f'(x) =$ ” or “Gradient =” which can appear anywhere in their working. If $f'(x)$ is used then

there is no requirement to see $f(x)$ defined first. Condone e.g. $\frac{dy}{dx} \rightarrow 4x$ or $f'(x) \rightarrow 4x$.

Condone missing brackets so allow e.g. $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{4xh + 2h^2}{h} = \lim_{h \rightarrow 0} 4x + 2h = 4x$

Do not allow $h = 0$ if there is never a reference to $h \rightarrow 0$

e.g. $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{4xh + 2h^2}{h} = \lim_{h \rightarrow 0} 4x + 2(0) = 4x$ is acceptable

but e.g. $\frac{dy}{dx} = \frac{4xh + 2h^2}{h} = 4x + 2h = 4x + 2(0) = 4x$ is not if there is no $h \rightarrow 0$ seen.

The $h \rightarrow 0$ does not need to be present throughout the proof e.g. on every line.

They must reach $4x + 2h$ at the end and not $\frac{4xh + 2h^2}{h}$ (without the h 's cancelled) to complete the limiting argument.



Question	Scheme	Marks	AOs
12	$\frac{\sin(x+h) - \sin x}{h}$	B1	2.1
	$\frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$	M1	1.1b
		A1	1.1b
	$(\text{As } h \rightarrow 0), \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \rightarrow 0 \times \sin x + 1 \times \cos x$	dM1	2.1
	$\text{so } \frac{dy}{dx} = \cos x \quad *$	A1*	2.5

(5 marks)

Notes

Throughout the question allow the use of $h = \delta x$ if used consistently

There is no requirement to see "gradient of chord" written down.

B1: Gives the correct fraction such as $\frac{\sin(x+h) - \sin x}{x+h-x}$ or $\frac{\sin x - \sin(x+h)}{-h}$ or $\frac{\sin(x+h) - \sin(x-h)}{2h}$ or $\frac{\sin(x-h) - \sin x}{x-h-x}$. Condone invisible brackets. May be implied by $\frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$

M1: Uses the compound angle formula for $\sin(x \pm h)$ to give $\sin x \cos h \pm \cos x \sin h$

A1: Achieves $\frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$ or equivalent (may be implied by further work).
Allow invisible brackets to be recovered.

dM1: **It is dependent on both the B and the M marks being awarded.**

Complete attempt to apply the given limits to the gradient of their chord. They must isolate

$\left(\frac{\cos h - 1}{h} \right)$ and replace with 0 and isolate $\left(\frac{\sin h}{h} \right)$ and replace with 1.

e.g. $\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) = \sin x \times 0 + \cos x \times 1$

Accept as a minimum $\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) = \cos x$ (implying the application of the limits)

If they do not fully show $\left(\frac{\cos h - 1}{h} \right)$ and $\left(\frac{\sin h}{h} \right)$ being isolated but proceed from

e.g. $\frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$ to $0 \times \sin x + \cos x$ (or e.g. $0 + \cos x$) then this can be implied and

score dM1

$\frac{\sin x(\cos h - 1) + \cos x \sin h}{h} = \cos x$ is dM0

Condone if limit notation remains within their expression after the limits have been applied.

e.g. $\lim_{h \rightarrow 0} (\sin x \times 0 + \cos x \times 1)$

Alternatively, condone use of the small angle approximations such that

$$\frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \rightarrow \frac{-\frac{h^2}{2} \sin x + h \cos x}{h} = -\frac{h}{2} \sin x + \cos x$$
 and replaces $\frac{h}{2}$ with 0



A1*: Uses correct mathematical language of limiting arguments to show that $\frac{dy}{dx} = \cos x$ with no errors seen. (cso)

We need to see $h \rightarrow 0$ at some point in their solution and linking $\frac{dy}{dx}$ with $\cos x$ e.g.

- $\frac{dy}{dx} = \dots = \lim_{h \rightarrow 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right) = \cos x$
- $\frac{dy}{dx} = \dots = \lim_{h \rightarrow 0} \left(-\frac{h}{2} \sin x + \cos x \right) = 0 \times \sin x + \cos x = \cos x$ (using small angle approximations)
- $\frac{dy}{dx} = \dots = \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} = \sin x \times 0 + 1 \times \cos x = \cos x$ as $h \rightarrow 0$

Condone $f'(x)$ or y' in place of $\frac{dy}{dx}$

Give final A0 for no evidence of limiting arguments:

e.g. when $h = 0$ $\frac{dy}{dx} = \dots = \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) = \sin x \times 0 + \cos x \times 1 = \cos x$ is A0

Do not allow the final A1 for just stating $\frac{\sin h}{h} = 1$ and $\frac{\cos h - 1}{h} = 0$ and attempting to apply these (without seeing e.g. $h \rightarrow 0$ at some point in their solution)

If they work in another variable (e.g. θ) then withhold the final mark. If they have mixed variables within some of their statements, then allow recovery but withhold the final mark.

Withhold this mark if there has been incorrect bracketing or invisible brackets when isolating $\sin x (\cos h - 1)$ e.g. $\frac{\sin x \cos h - 1 + \cos x \sin h}{h}$ but accept terms written as e.g. $\sin x \frac{\cos h - 1}{h}$ which do not require brackets. Condone a missing trailing bracket if the intention is clear.

Question	Scheme	Marks	AOs
14	$y = \frac{x-4}{2+\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{2+\sqrt{x}-(x-4)\frac{1}{2}x^{-\frac{1}{2}}}{(2+\sqrt{x})^2}$	M1 A1	2.1 1.1b
	$= \frac{2+\sqrt{x}-(x-4)\frac{1}{2}x^{-\frac{1}{2}}}{(2+\sqrt{x})^2} = \frac{2+\sqrt{x}-\frac{1}{2}\sqrt{x}+2x^{-\frac{1}{2}}}{(2+\sqrt{x})^2} = \frac{2\sqrt{x}+\frac{1}{2}x+2}{\sqrt{x}(2+\sqrt{x})^2}$	M1	1.1b
	$= \frac{x+4\sqrt{x}+4}{2\sqrt{x}(2+\sqrt{x})^2} = \frac{(2+\sqrt{x})^2}{2\sqrt{x}(2+\sqrt{x})^2} = \frac{1}{2\sqrt{x}}$	A1	2.1
		(4)	
(4 marks)			
Notes			

M1: Attempts to use a correct rule e.g. quotient or product (& chain) rule to achieve the following forms

Quotient : $\frac{\alpha(2+\sqrt{x})-\beta(x-4)x^{-\frac{1}{2}}}{(2+\sqrt{x})^2}$ but be tolerant of attempts where the $(2+\sqrt{x})^2$ has been

incorrectly expanded

Product: $\alpha(2+\sqrt{x})^{-1} + \beta x^{-\frac{1}{2}}(x-4)(2+\sqrt{x})^{-2}$

Alternatively with $t = \sqrt{x}$, $y = \frac{t^2-4}{2+t} \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2t(2+t)-(t^2-4)}{(2+t)^2} \times \frac{1}{2}x^{-\frac{1}{2}}$ with same rules

A1: Correct derivative in any form. Must be in terms of a single variable (which could be t)

M1: Following a correct attempt at differentiation, it is scored for multiplying both numerator and denominator by $\sqrt{x}$ and collecting terms to form a single fraction. It can also be scored from $\frac{uv'-vu'}{v^2}$

For the $t = \sqrt{x}$, look for an attempt to simplify $\frac{t^2+4t+4}{(2+t)^2} \times \frac{1}{2t}$

A1: Correct expression showing all key steps with no errors or omissions. $\frac{dy}{dx}$ must be seen at least once

Question	Scheme	Marks	AOs
14	$y = \frac{x-4}{2+\sqrt{x}} \Rightarrow y = \frac{(\sqrt{x}+2)(\sqrt{x}-2)}{2+\sqrt{x}} = \sqrt{x}-2$	M1 A1	2.1 1.1b
	$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$	M1 A1	1.1b 2.1
		(4)	
(4 marks)			
Notes			

M1: Attempts to use difference of two squares. Can also be scored using

$$t = \sqrt{x} \Rightarrow y = \frac{t^2-4}{t+2} \Rightarrow y = \frac{(t+2)(t-2)}{t+2}$$

A1: $y = \sqrt{x}-2$ or $y = t-2$

M1: Attempts to differentiate an expression of the form $y = \sqrt{x} + b$

A1: Correct expression showing all key steps with no errors or omissions. $\frac{dy}{dx}$ must be seen at least once



Question	Scheme	Marks	AOs
2(a)	$f(x) = (x-2)^2 \pm \dots$	M1	1.2
	$f(x) = (x-2)^2 + 1$	A1	1.1b
		(2)	
(b)(i)	$P = (0, 5)$	B1	1.1b
(b)(ii)	$Q = (2, 1)$	B1ft	1.1b
		(2)	
(4 marks)			
Notes			

(a)

M1: Achieves $(x-2)^2 \pm \dots$ or states $a = -2$

A1: Correct expression $(x-2)^2 + 1$ ISW after sight of this

Condone $a = -2$ and $b = 1$. Condone $(x-2)^2 + 1 = 0$

(b)

(i) B1: Correct coordinates for P . Allow to be expressed $x = 0, y = 5$

(ii) B1ft: Correct coordinates for Q . Allow to be expressed $x = 2, y = 1$ (Score for the correct answer or follow through their part (a) so allow $(-a, b)$ where a and b are numeric)

Score in any order if they state $P = (0, 5)$ and $Q = (2, 1)$

.....
Allow part (b) to be awarded from a sketch. So award

First B1 from a sketch crossing the y -axis at 5

Second B1 from a sketch with minimum at $(2, 1)$

.....



A1: $\frac{4}{15}x^{\frac{5}{2}} - \frac{10}{9}x^{\frac{3}{2}} + c$ and including the constant or simplified exact equivalent such as

$\frac{4}{15}\sqrt{x^5} - \frac{10}{9}\sqrt{x^3} + c$ or $\frac{4}{15}x^{2.5} - \frac{10}{9}x^{1.5} + c$ or $\frac{1}{45}\left(12x^{\frac{5}{2}} - 50x^{\frac{3}{2}}\right) + c$ or $\frac{x^{\frac{3}{2}}}{45}(12x - 50) + c$. Fractions must be in their lowest terms and indices processed.

Do not accept e.g. $0.267x^{\frac{5}{2}} - 1.11x^{\frac{3}{2}} + c$ but allow $0.2\dot{6}x^{\frac{5}{2}} - 1.\dot{1}x^{\frac{3}{2}} + c$

Is w once a correct answer is seen but withhold this mark if there is spurious notation around

their final answer e.g. $\int \frac{4}{15}x^{\frac{5}{2}} - \frac{10}{9}x^{\frac{3}{2}} + c \, dx$ is M1A1dM1A0

Alternative method using integration by parts example

M1: e.g. $\int x^{\frac{1}{2}}(2x-5) \, dx = \dots x^{\frac{3}{2}}(2x-5) - \int \dots x^{\frac{3}{2}} \, dx$ (applies integration by parts correctly to typically achieve this form – the $(2x-5)$ may also be split up as well – send to review if unsure how to mark)

This may also be done the other way round e.g. $\int x^{\frac{1}{2}}(2x-5) \, dx = \dots x^{\frac{1}{2}}(x^2-5x) - \int \dots x^{\frac{3}{2}} \pm \dots x^{\frac{1}{2}} \, dx$

The $\frac{1}{3}$ does not need to be considered for this mark.

A1: A correct intermediate stage applying integration by parts with correct coefficients.

e.g. $\int \frac{x^{\frac{1}{2}}(2x-5)}{3} \, dx = \frac{2}{3}x^{\frac{3}{2}}\left(\frac{2x-5}{3}\right) - \int \frac{4}{9}x^{\frac{3}{2}} \, dx$ (or unsimplified equivalent).

Coefficients must be exact. (See main scheme notes above) The other way round this could appear as

e.g. $\int \frac{x^{\frac{1}{2}}(2x-5)}{3} \, dx = \frac{1}{3}x^{\frac{1}{2}}(x^2-5x) - \frac{1}{6}\int x^{\frac{3}{2}} - 5x^{\frac{1}{2}} \, dx$. Condone a missing dx . May be implied.

dM1: Increases the power by one on an x^n term where n is a fraction e.g. $\int \dots x^{\frac{3}{2}} \, dx \rightarrow \dots x^{\frac{5}{2}}$ The index does not need to be processed. It is dependent on the previous method mark.

A1: $\frac{4}{15}x^{\frac{5}{2}} - \frac{10}{9}x^{\frac{3}{2}} + c$ or exact simplified equivalent. (See main scheme notes above)

Alternative method using the substitution method

M1: e.g. let $u = x^{\frac{1}{2}} \Rightarrow \int \dots u^4 + \dots u^2 \, du$ (uses a substitution to express the integral in terms of another variable. Allow slips with the coefficients, but the indices should be correct for their substitution)

The $\frac{1}{3}$ does not need to be considered for this mark.

A1: e.g. $\int \frac{4u^4}{3} - \frac{10u^2}{3} \, du$ or unsimplified equivalent. Coefficients must be exact. See main scheme notes above). May be implied by further work. Condone a missing dx .

dM1: $\int \dots u^4 + \dots u^2 \, du \rightarrow \dots u^5 + \dots u^3$ (increases the power by one on at least one of their indices – does not need to be processed. It is dependent on the previous method mark.

A1: $\frac{4}{15}x^{\frac{5}{2}} - \frac{10}{9}x^{\frac{3}{2}} + c$ or exact simplified equivalent. (See main scheme notes above)

There may be alternative substitutions, but the same marking principles apply.

Question	Scheme	Marks	AOs
2(a)	$(f(a) =) 4a^3 + 5a^2 - 10a + 4a = 0 \Rightarrow a(\dots) = 0$	M1	3.1a
	$a(4a^2 + 5a - 6) = 0$ *	A1*	1.1b
		(2)	
(b)(i)	$a = \frac{3}{4}$	B1	2.2a
(ii)	$4x^3 + 5x^2 - 10x + 4 \times \frac{3}{4} = 3 \Rightarrow 4x^3 + 5x^2 - 10x (= 0)$	M1	1.1b
	$x = 0$	B1	1.1b
	$x = \frac{-5 \pm \sqrt{185}}{8}$	A1	1.1b
		(4)	
(6 marks)			
Notes			
(a) M1: Attempts $f(a) = 0$ leading to an equation in a only and attempts to take a factor of a out. Condone slips. The $= 0$ may appear on a later line which is fine, but must be seen at some point in their solution in part (a). A1*: Achieves the given answer with no errors including brackets. Minimum acceptable is $4a^3 + 5a^2 - 10a + 4a = 0 \Rightarrow a(4a^2 + 5a - 6) = 0$ If the $= 0$ is absent at the start of their solution, it must appear before achieving the given answer. Do not allow attempts to find the value of a and substitute that into $f(x)$ More difficult alternative methods may be seen: Alt 1: You may see attempts via division / inspection $ \begin{array}{r} 4x^2 \quad + (5+4a)x \quad + (-10+5a+4a^2) \\ x-a \overline{) 4x^3 \quad + 5x^2 \quad - 10x \quad + 4a} \\ \underline{4x^3 \quad - 4ax^2} \\ (5+4a)x^2 \quad - 10x \\ \underline{(5+4a)x^2 \quad - a(5+4a)x} \\ (-10+5a+4a^2)x \quad + 4a \\ \underline{(-10+5a+4a^2)x - a(-10+5a+4a^2)} \\ -6a+5a^2+4a^3 \end{array} $ Then sets remainder $-6a + 5a^2 + 4a^3 = 0$ M1: For dividing the cubic by $(x - a)$ leading to a quadratic quotient in x and a cubic remainder in a which is then set $= 0$ and attempts to take a factor of a out. A1*: Completely correct with $a(4a^2 + 5a - 6) = 0$			

Alt 2: You may also see a grid or an attempt at factorisation via inspection

	$4x^2$	$+(5+4a)x$	$+(-10+5a+4a^2)$
x	$4x^3$	$+(5+4a)x^2$	$(-10+5a+4a^2)x$
$-a$	$-4ax^2$	$-a(5+4a)x$	$-a(-10+5a+4a^2)$

OR $4x^3 + 5x^2 - 10x + 4a \equiv (x-a)(4x^2 + px - 4)$ which should be followed by equating the x terms and x^2 terms to form two equations which can be solved simultaneously.

$$-10 = -ap - 4 \quad \text{and} \quad 5 = -4a + p \Rightarrow p = 5 + 4a$$

$$\Rightarrow -10 = -a(5+4a) - 4 \Rightarrow 4a^2 + 5a - 6 = 0 \Rightarrow a(4a^2 + 5a - 6) = 0$$

The above are examples. There may be other correct attempts so look at what is done.

M1: For an attempt to set up two simultaneous equations by equating coefficients for x and equating coefficients for x^2 . Condone slips.

A1*: $4a^2 + 5a - 6 = 0 \Rightarrow a(4a^2 + 5a - 6) = 0$ Completely correct with no errors.

(b) Mark (i) and (ii) together**(i)**

B1: Deduces that $a = \frac{3}{4}$ only. May be implied by their resultant cubic. If they do (b)(ii) multiple times using other roots for which $a \neq \frac{3}{4}$, then the solutions arising from using the other roots $a \neq \frac{3}{4}$ must be rejected

(ii)

M1: Attempts to substitute their $a = \frac{3}{4}$ (which must be positive) into $f(x)$, sets their $f(x) = 3$ and collects terms on one side ($= 0$ may be implied). Condone arithmetical and sign slips. Condone if they repeat this step using their other root(s).

B1: $(x =) 0$

A1: $(x =) \frac{-5 \pm \sqrt{185}}{8}$ (**and these values only**) or exact equivalent (ignore 0 for this mark). Withhold this mark if the fraction line was clearly not intended to be under both terms. This mark cannot be scored if they proceed directly to the roots from $4x^3 + 5x^2 - 10x$ without taking a factor of or dividing by x first to **see the quadratic factor**. Isw once the correct answers are seen if they proceed to provide rounded answers after.

e.g. $4x^3 + 5x^2 - 10x + 4 \times \frac{3}{4} = 3 \Rightarrow 0, \frac{-5 \pm \sqrt{185}}{8}$ is M0B1A0

e.g. $4x^3 + 5x^2 - 10x = 0 \Rightarrow 0, \frac{-5 \pm \sqrt{185}}{8}$ is M1B1A0

e.g. $4x^3 + 5x^2 - 10x = 0 \Rightarrow 4x^2 + 5x - 10 = 0 \Rightarrow \frac{-5 \pm \sqrt{185}}{8}$ is M1B0A1

e.g. $4x^3 + 5x^2 - 10x = 0 \Rightarrow x(4x^2 + 5x - 10) = 0 \Rightarrow 0, \frac{-5 \pm \sqrt{185}}{8}$ is M1B1A1



Question	Scheme	Marks	AOs
3(a)	$(OA =) \sqrt{5^2 + 3^2 + 2^2} = \sqrt{38} *$	B1*	1.1b
(b)		(1)	
	$ OB = \sqrt{2^2 + 4^2 + a^2} = \sqrt{20 + a^2}$ so when $a = 5$ $ OB = \sqrt{20 + 25} = \sqrt{45}$	M1	1.1b
	$= 5$	A1 cso	2.3
		(2)	
(3 marks)			
Notes			
(a)			
B1*: Shows the magnitude of OA is $\sqrt{38}$. Must see $\sqrt{5^2 + 3^2 + 2^2}$ or e.g. $\sqrt{25 + 9 + 4}$. We need to see how the value 38 or $\sqrt{38}$ is formed using the three components. Withhold this mark for incorrect working such as $OA = 5^2 + 3^2 + 2^2 = 38 \Rightarrow OA = \sqrt{38}$ but do not penalise poor notation to denote vectors or the magnitude as long as the intention is clear as to what they are finding (OA instead of OA is fine). Do not penalise if their square root does not go fully over all three terms as long as the intention is clear. May find AO instead which is acceptable. $(OA ^2 =) 25 + 9 + 4 = 38 \Rightarrow (OA =) \sqrt{38}$ scores B1 (we see how 38 is found) $(OA ^2 =) 25 + 9 + 4 \Rightarrow \sqrt{38}$ scores B1 (we see how 38 is found) $25 + 9 + 4 = \sqrt{38}$ scores B0 (they are not equal) $(OA ^2 =) 38 \Rightarrow (OA =) \sqrt{38}$ scores B0 (no method seen to show how 38 is found)			
(b)			
M1: Attempts to find OB (or $OB ^2$) in terms of a and substitutes in a positive integer for a to find a value for OB (or $OB ^2$). e.g. $OB = \sqrt{20 + a^2} \Rightarrow$ when $a = 2 \Rightarrow \sqrt{24}$. Also accept e.g. BO (or $BO ^2$). Alternatively sets up an equation or an inequality e.g. $\sqrt{20 + a^2} > \sqrt{38}$ and proceeds to $a^2 > \dots$ (or $a^2 = \dots$). Condone sign slips in their rearrangement only. Allow the use of $=$, $<$ or $>$ for this mark. "$20 + a^2 \dots 38 \Rightarrow a^2 \dots 18$" (may be implied by sight of $\sqrt{18} = 4.24\dots$) A1: 5 cso (answer on its own with no incorrect working seen scores M1A1). Withhold this mark if OB (or $OB ^2$) is incorrect (or BO or $BO ^2$). Do not be concerned with the notation as long as the intention is clear or implied as to what they are trying to calculate (the calculations must be correct) Withhold this mark if at any point they set $OB < OA$ but accept an argument leading to an answer of 5 from either $OB > OA$ or $OB = OA$			



Question	Scheme	Marks	AOs
4a	$2\alpha + \frac{1}{2}\left(1 - \frac{\alpha^2}{2}\right)$	M1	1.2
	$2\alpha + \frac{1}{2}\left(1 - \frac{\alpha^2}{2}\right) = 0 \Rightarrow 2\alpha + \frac{1}{2} - \frac{\alpha^2}{4} = 0 \Rightarrow \alpha = \dots$	dM1	1.1b
	$\alpha = -0.243$ (3dp) only	A1	2.3
		(3)	
b	$f'(0) = \frac{1}{2} \cos 0 \Rightarrow \dots \Rightarrow y = \dots x + 3$	M1	1.1b
	$y = \frac{1}{2}x + 3$	A1	1.1b
		(2)	
(5 marks)			
Notes			
(a) Accept to be in terms of α or another variable e.g. x Note: -0.243 with no working is 0 marks M1: Fully substitutes $\cos x = 1 - \frac{x^2}{2}$ into the derivative. dM1: Attempts to multiply out to achieve a 3TQ (= 0) and attempts to find a value for α . Condone slips. Allow solving the quadratic via any method (usual rules apply). If they use a calculator then you may need to check this. A1: ($\alpha =$) -0.243 only cao Can only be scored provided a correct 3TQ is seen. If both roots found then the other one must be rejected (or a choice made of -0.243 e.g. underlining it or a tick) Condone $x = -0.243$			
(b) M1: Attempts to find the gradient of the curve when $x = 0$ and achieves an equation of the form $y = "f'(0)"x + 3$. $x = 0$ must be fully substituted in and a value must be found for the gradient. Do not allow this mark if they attempt to use a changed gradient e.g. the gradient of the normal. Also allow attempts using the small angle approximation: $f'(x) \approx 2x + \frac{1}{2}\left(1 - \frac{x^2}{2}\right)$ when $x = 0$, $f'(0) = \frac{1}{2} \Rightarrow y = "f'(0)"x + 3$ A1: $y = \frac{1}{2}x + 3$ or equivalent in the form $y = mx + c$ isw Stating just the values $m = 0.5$, $c = 3$ without the correct equation is A0			

Question	Scheme	Marks	AOs
7	Attempts equation of line Eg Substitutes $(-2,13)$ into $y = mx + 25$ and finds m	M1	1.1b
	Equation of l is $y = 6x + 25$	A1	1.1b
	Attempts equation of C Eg Attempts to use the intercept $(0,25)$ within the equation $y = a(x \pm 2)^2 + 13$, in order to find a	M1	3.1a
	Equation of C is $y = 3(x+2)^2 + 13$ or $y = 3x^2 + 12x + 25$	A1	1.1b
	Region R is defined by $3(x+2)^2 + 13 < y < 6x + 25$ o.e.	B1ft	2.5
		(5)	
(5 marks)			
Notes:			

The first two marks are awarded for finding the equation of the line

M1: Uses the information in an attempt to find an equation for the line l .

E.g. Attempt using two points: Finds $m = \pm \frac{25-13}{2}$ and uses of one of the points in their $y = mx + c$ or equivalent to find c . Alternatively uses the intercept as shown in main scheme.

A1: $y = 6x + 25$ seen or implied. This alone scores the first two marks. Do not accept $l = 6x + 25$

It must be in the form $y = \dots$ but the correct equation can be implied from an inequality. E.g. $\dots < y < 6x + 25$

The next two marks are awarded for finding the equation of the curve

M1: A complete method to find the constant a in $y = a(x \pm 2)^2 + 13$ or the constants a, b in $y = ax^2 + bx + 25$.

An alternative to the main scheme is deducing equation is of the form $y = ax^2 + bx + 25$ and setting and solving a pair of simultaneous equations in a and b using the point $(-2, 13)$ the gradient being 0 at $x = -2$. Condone slips. Implied by $C = 3x^2 + 12x + 25$ or $3x^2 + 12x + 25$

FYI the correct equations are $13 = 4a - 2b + 25$ ($2a - b = -6$) and $-4a + b = 0$

A1: $y = 3(x+2)^2 + 13$ or equivalent such as $y = 3x^2 + 12x + 25$, $f(x) = 3(x+2)^2 + 13$.

Do not accept $C = 3x^2 + 12x + 25$ or just $3x^2 + 12x + 25$ for the A1 but may be implied from an inequality or from an attempt at the area, E.g. $\int 3x^2 + 12x + 25 \, dx$

B1ft: Fully defines the region R . Follow through on their equations for l and C .

Allow strict or non-strict inequalities as long as they are used consistently.

E.g. Allow for example " $3(x+2)^2 + 13 < y < 6x + 25$ " " $3(x+2)^2 + 13 \leq y \leq 6x + 25$ "

Allow the inequalities to be given separately, e.g. $y < 6x + 25, y > 3(x+2)^2 + 13$. Set notation may be used so

$\{(x, y) : y > 3(x+2)^2 + 13\} \cap \{(x, y) : y < 6x + 25\}$ is fine but condone with or without any of $(x, y) \leftrightarrow y \leftrightarrow x$

Incorrect examples include " $y < 6x + 25$ or $y > 3(x+2)^2 + 13$ ", $\{(x, y) : y > 3(x+2)^2 + 13\} \cup \{(x, y) : y < 6x + 25\}$

Values of x could be included but they must be correct. So $3(x+2)^2 + 13 < y < 6x + 25, x < 0$ is fine

If there are multiple solutions mark the final one.



Question	Scheme	Marks	AOs
1(a)(i)	$y = 4x^3 - 7x^2 + 5x - 10 \Rightarrow \left(\frac{dy}{dx} = \right) 12x^2 - 14x + 5$	M1 A1	1.1b 1.1b
(ii)	$\left(\frac{d^2y}{dx^2} = \right) 24x - 14$	A1ft	1.1b
		(3)	
(b)	$24x - 14 = 0 \Rightarrow x = \dots$	M1	1.1b
	$x = \frac{7}{12}$ oe e.g. $x = \frac{14}{24}$	A1	1.1b
		(2)	
(5 marks)			
Notes			
(a)(i) If “+ c” is included with either derivative penalise it only once on the first occurrence. M1: Award for $x^3 \rightarrow x^2$ or $x^2 \rightarrow x$ or $5x \rightarrow 5$ or $-10 \rightarrow 0$ Indices may be unprocessed e.g. $x^3 \rightarrow x^{3-1}$ or $x^2 \rightarrow x^{2-1}$ or $5x \rightarrow 5x^0$ A1: Correct <u>simplified</u> expression with indices processed $12x^2 - 14x + 5$. Do not allow x^1 for x or $5x^0$ for 5. Apply isw if necessary once a correct answer is seen. The “$\frac{dy}{dx} =$” is not required. (ii) A1ft: Correct simplified second derivative $24x - 14$ or follow through their first derivative. Must be <u>simplified</u> so do not allow e.g. x^1 for x or x^0 for 1 as above. Apply isw if necessary once a correct answer is seen. The “$\frac{d^2y}{dx^2} =$” is not required. (b) M1: Sets their second derivative of the form $ax + b$, $a, b \neq 0$ equal to 0 and proceeds to a value for x. Condone slips in rearranging as long as a value for x is obtained. This may be implied by their value of x or may be implied by their working e.g. $\left(\frac{d^2y}{dx^2} = \right) 24x - 14 \rightarrow 24x = 14 \Rightarrow x = \dots$ Condone one slip in copying their second derivative. Also condone if they “cancel” e.g. $\left(\frac{d^2y}{dx^2} = \right) 24x - 14 \rightarrow 12x - 7 = 0 \Rightarrow x = \dots$ A1: Correct value from correct work and a correct second derivative but allow recovery if they “cancel” their second derivative to obtain e.g. $12x - 7$. Allow exact equivalents e.g. $\frac{14}{24}$ but not rounded decimals e.g. 0.583 Allow recurring decimal if clearly indicated e.g. $0.58\dot{3}$ Correct answer only from a correct second derivative (or correctly cancelled second derivative) scores both marks. Isw after a correct answer is seen.			

Question	Scheme	Marks	AOs
5(a)	$2(3)^3 - 9(3)^2 + 5(3) + k = 0 \Rightarrow k = \dots$	M1	1.1b
	$54 - 81 + 15 + k = 0 \Rightarrow k = 12^*$ or $-12 + k = 0 \Rightarrow k = 12^*$	A1*	1.1b
		(2)	
	(a) Alternative by verification:		
	$2(3)^3 - 9(3)^2 + 5(3) + 12 = 0$	M1	1.1b
	$54 - 81 + 15 + 12 = 0$ Hence $k = 12^*$	A1*	1.1b
		(2)	
(b)	$\int (2x^3 - 9x^2 + 5x + 12) dx \dots x^4 \pm \dots x^3 \pm \dots x^2 \pm \dots x \pm \dots$	M1	3.1a
	$\frac{1}{2}(3)^4 - 3(3)^3 + \frac{5}{2}(3)^2 + 12(3) + c = -10 \Rightarrow c = \dots$	dM1	1.1b
	$(0, -28)$	A1	2.2a
		(3)	
(5 marks)			
Notes			
(a) Mark (a) and (b) together M1: Substitutes $x = 3$ completely into the given derivative, sets = 0 and solves for k. e.g., $2(3)^3 - 9(3)^2 + 5(3) + k = 0 \Rightarrow k = \dots$ May be implied by e.g. $54 - 81 + 15 + k = 0 \Rightarrow k = \dots$ with at least 2 correctly evaluated powers. A1*: Obtains $k = 12$ with no errors seen and sufficient working shown. As a minimum you would need to see e.g., $2(3)^3 - 9(3)^2 + 5(3) + k = 0 \Rightarrow -12 + k = 0 \Rightarrow k = 12$ or $54 - 81 + 15 + k = 0 \Rightarrow k = 12$ or $2 \times 27 - 9 \times 9 + 5(3) + k = 0 \Rightarrow k = 12$ But $2(3)^3 - 9(3)^2 + 5(3) + k = 0 \Rightarrow k = 12$ scores M1A0 for lack of working Note that some are just writing the expression for $\frac{dy}{dx}$, then write “sub in $x = 3$” but don't actually show 3 substituted in and then go on to write $-12 + k = 0$ leading to $k = 12$ scores M0A0*. Alternative: M1: Substitutes $x = 3$ and $k = 12$ into the given derivative and attempts to evaluate A1*: Correct work to obtain an answer of 0 with a (minimal) conclusion e.g., tick, hence proven etc. As a minimum you would need to see e.g., $2(3)^3 - 9(3)^2 + 5(3) + 12 = 54 - 81 + 15 + 12 = 0 \checkmark$ (b) M1: Attempts to integrate. Evidence can be taken for integrating to obtain at least 2 from: $2x^3 \rightarrow \dots x^4$ or $-9x^2 \rightarrow \dots x^3$ or $5x \rightarrow \dots x^2$ or $12 \rightarrow \dots x$ where ... are constants dM1: Substitutes $x = 3$ into their integrated expression that includes a constant of integration, sets this equal to ± 10 and proceeds to find their constant. Depends on the previous mark. If the substitution is not shown this mark may be implied by their value for c or by their equation e.g., $18 + c = \pm 10$ A1: $(0, -28)$ Condone -28 or $y = -28$ but not just $c = -28$. There must be no other values or points. Condone $(-28, 0)$ following $y = -28$ <u>Beware of circular arguments which avoid doing part (a) e.g.</u> Integration is used on the given derivative to give y in terms of x, k and c $(3, -10)$ is substituted to give $3k + c = 8$ Part (b) is then done first using $k = 12$ to find $c = -28$			

This is then substituted into $3k + c = 8$ to give $k = 12$
 This scores (a) M0A0 (b) M1dM1A1 (if -28 is identified as the intercept)

Alternative for part (a) using algebraic division:

$$\begin{array}{r}
 2x^2 - 3x - 4 \\
 x - 3 \overline{) 2x^3 - 9x^2 + 5x + k} \\
 \underline{2x^3 - 6x^2} \\
 -3x^2 + 5x \\
 \underline{-3x^2 + 9x} \\
 -4x + k \\
 \underline{-4x + 12} \\
 k - 12 \text{ (or 0)}
 \end{array}$$

leading to $k - 12 = 0$ and then $k = 12$.

M1: Attempts to divide the given cubic by $(x - 3)$ and proceeds as far as a remainder set $= 0$.

Requires at least $2x^2 \pm 3x$.

A1*: Obtains $k = 12$ with no errors seen and sufficient working. Their algebraic division needs to be correct but allow them to have either $k - 12$ or 0 as their “remainder”. If their remainder is given in their working as 0 they may proceed directly to $k = 12$.

Question	Scheme	Marks	AOs
6 (a)	$2 < x < 6$	B1	1.1b
		(1)	
(b)	States either $k > 8$ or $k < 0$	M1	3.1a
	States e.g. $\{k : k > 8\} \cup \{k : k < 0\}$	A1	2.5
		(2)	
(c)	Please see notes for alternatives		
	States $y = ax(x-6)^2$ or $f(x) = ax(x-6)^2$	M1	1.1b
	Substitutes (2,8) into $y = ax(x-6)^2$ and attempts to find a	dM1	3.1a
	$y = \frac{1}{4}x(x-6)^2$ or $f(x) = \frac{1}{4}x(x-6)^2$ o.e	A1	2.1
		(3)	
(6 marks)			
Notes: Watch for answers written by the question. If they are beside the question and in the answer space, the one in the answer space takes precedence			

(a)

 B1: Deduces $2 < x < 6$ o.e. such as $x > 2, x < 6$ $x > 2$ and $x < 6$ $\{x : x > 2\} \cap \{x : x < 6\}$ $x \in (2, 6)$

 Condone attempts in which set notation is incorrectly attempted but correct values can be seen or implied E.g. $\{x > 2\} \cap \{x < 6\}$ $\{x > 2, x < 6\}$. Allow just the open interval $(2, 6)$

 Do not allow for incorrect inequalities such as e.g. $x > 2$ or $x < 6$, $\{x : x > 2\} \cup \{x : x < 6\}$ $x \in [2, 6]$

(b)

M1: Establishes a correct method by finding one of the (correct) inequalities

 States either $k > 8$ (condone $k \geq 8$) or $k < 0$ (condone $k \leq 0$)

 Condone for this mark $y \leftrightarrow k$ or $f(x) \leftrightarrow k$ and $8 < k < 0$

 A1: Fully correct solution in the form $\{k : k > 8\} \cup \{k : k < 0\}$ or $\{k | k > 8\} \cup \{k | k < 0\}$ either way around

 but condone $\{k < 0\} \cup \{k > 8\}$, $\{k : k < 0 \cup k > 8\}$, $\{k < 0 \cup k > 8\}$. It is not necessary to

 mention $\mathbb{R}$, e.g. $\{k : k \in \mathbb{R}, k > 8\} \cup \{k : k \in \mathbb{R}, k < 0\}$ Look for $\{ \}$ and $\cup$

 Do not allow solutions not in set notation such as $k < 0$ or $k > 8$.

(c)

 M1: Realises that the equation of C is of the form $y = ax(x-6)^2$. Condone with $a = 1$ for this mark.

 So award for sight of $ax(x-6)^2$ even with $a = 1$

 dM1: Substitutes (2,8) into the form $y = ax(x-6)^2$ and attempts to find the value for a .

 It is dependent upon having an equation, which the ($y = \dots$) may be implied, of the correct form.

 A1: Uses all of the information to form a correct **equation** for C $y = \frac{1}{4}x(x-6)^2$ o.e.

 ISW after a correct answer. Condone $f(x) = \frac{1}{4}x(x-6)^2$ but not $C = \frac{1}{4}x(x-6)^2$

Examples of alternative methods

.....
Alternative I part (c):

Using the form $y = ax^3 + bx^2 + cx$ and setting up then solving simultaneous equations.

There are various versions of this but can be marked similarly

M1: Realises that the equation of C is of the form $y = ax^3 + bx^2 + cx$ and forms two equations in a , b and c . Condone with $a = 1$ for this mark.

Note that the form $y = ax^3 + bx^2 + cx + d$ is M0 until d is set equal to 0.

There are four equations that could be formed, only two are necessary for this mark.

Condone slips

$$\text{Using } (6, 0) \quad \Rightarrow 216a + 36b + 6c = 0$$

$$\text{Using } (2, 8) \quad \Rightarrow 8a + 4b + 2c = 8$$

$$\text{Using } \frac{dy}{dx} = 0 \text{ at } x = 2 \quad \Rightarrow 12a + 4b + c = 0$$

$$\text{Using } \frac{dy}{dx} = 0 \text{ at } x = 6 \Rightarrow 108a + 12b + c = 0$$

dM1: Forms and solves three different equations, one of which must be using $(2, 8)$ to find values for a , b and c . A calculator can be used to solve the equations

A1: Uses all of the information to form a correct equation for C $y = \frac{1}{4}x^3 - 3x^2 + 9x$ o.e.

$$\text{ISW after a correct answer. Condone } f(x) = \frac{1}{4}x^3 - 3x^2 + 9x$$

.....
Alternative II part (c)

Using the gradient and integrating

M1: Realises that the gradient of C is zero at 2 and 6 so sets $f'(x) = k(x-2)(x-6)$ oe **and** attempts to integrate. Condone with $k = 1$

dM1: Substitutes $x = 2, y = 8$ into $f(x) = k(\dots x^3 + \dots x + \dots)$ and finds a value for k

A1: Uses all of the information to form a correct equation for C $y = \frac{3}{4}\left(\frac{1}{3}x^3 - 4x^2 + 12x\right)$ o.e.

$$\text{ISW after a correct answer. Condone } f(x) = \frac{3}{4}\left(\frac{1}{3}x^3 - 4x^2 + 12x\right)$$



Question	Scheme	Marks	AOs
2(a)	(i) $\frac{dy}{dx} = 2x - 2 - 12x^{-\frac{1}{2}}$	M1 A1	1.1b 1.1b
	(ii) $\frac{d^2y}{dx^2} = 2 + 6x^{-\frac{3}{2}}$	B1ft	1.1b
		(3)	
(b)	Substitutes $x = 4$ into their $\frac{dy}{dx} = 2 \times 4 - 2 - 12 \times 4^{-\frac{1}{2}} = \dots$	M1	1.1b
	Shows $\frac{dy}{dx} = 0$ and states "hence there is a stationary point" oe	A1	2.1
		(2)	
(c)	Substitutes $x = 4$ into their $\frac{d^2y}{dx^2} = 2 + 6 \times 4^{-\frac{3}{2}} = (2.75)$	M1	1.1b
	$\frac{d^2y}{dx^2} = 2.75 > 0$ and states "hence minimum"	A1ft	2.2a
		(2)	
(7 marks)			
(a)(i) M1: Differentiates to $\frac{dy}{dx} = Ax + B + Cx^{\frac{1}{2}}$ A1: $\frac{dy}{dx} = 2x - 2 - 12x^{-\frac{1}{2}}$ (Coefficients may be unsimplified) (a)(ii) B1ft: Achieves a correct $\frac{d^2y}{dx^2}$ for their $\frac{dy}{dx}$ (Their $\frac{dy}{dx}$ must have a negative or fractional index) (b) M1: Substitutes $x = 4$ into their $\frac{dy}{dx}$ and attempts to evaluate. There must be evidence $\frac{dy}{dx} \Big _{x=4} = \dots$ Alternatively substitutes $x = 4$ into an equation resulting from $\frac{dy}{dx} = 0$ Eg. $\frac{36}{x} = (x-1)^2$ and equates A1: There must be a reason and a minimal conclusion. Allow ✓, QED for a minimal conclusion Shows $\frac{dy}{dx} = 0$ and states "hence there is a stationary point" oe Alt Shows that $x = 4$ is a root of the resulting equation and states "hence there is a stationary point" All aspects of the proof must be correct including a conclusion (c) M1: Substitutes $x = 4$ into their $\frac{d^2y}{dx^2}$ and calculates its value, or implies its sign by a statement such as when $x = 4 \Rightarrow \frac{d^2y}{dx^2} > 0$. This must be seen in (c) and not labelled (b). Alternatively calculates the gradient of C either side of $x = 4$ or calculates the value of y either side of $x = 4$. A1ft: For a correct calculation, a valid reason and a correct conclusion. Ignore additional work where candidate finds $\frac{d^2y}{dx^2}$ left and right of $x = 4$. Follow through on an incorrect $\frac{d^2y}{dx^2}$ but it is dependent upon having a negative or fractional index. Ignore any references to the word convex. The nature of the turning point is "minimum". Using the gradient look for correct calculations, a valid reason.... goes from negative to positive, and a correct conclusion ...minimum.			

Question	Scheme	Marks	AOs
5(a)(i)	$\frac{dy}{dx} = 20x^3 - 72x^2 + 84x - 32$	M1 A1	1.1b 1.1b
(ii)	$\frac{d^2y}{dx^2} = 60x^2 - 144x + 84$	A1ft	1.1b
		(3)	
(b)(i)	$x = 1 \Rightarrow \frac{dy}{dx} = 20 - 72 + 84 - 32$	M1	1.1b
	$\frac{dy}{dx} = 0$ so there is a stationary point at $x = 1$	A1	2.1
	Alternative for (b)(i)		
	$20x^3 - 72x^2 + 84x - 32 = 4(x-1)^2(5x-8) = 0 \Rightarrow x = \dots$	M1	1.1b
	When $x = 1$, $\frac{dy}{dx} = 0$ so there is a stationary point	A1	2.1
(b)(ii)	Note that in (b)(ii) there are no marks for <u>just</u> evaluating $\left(\frac{d^2y}{dx^2}\right)_{x=1}$		
	E.g. $\left(\frac{d^2y}{dx^2}\right)_{x=0.8} = \dots \left(\frac{d^2y}{dx^2}\right)_{x=1.2} = \dots$	M1	2.1
	$\left(\frac{d^2y}{dx^2}\right)_{x=0.8} > 0, \left(\frac{d^2y}{dx^2}\right)_{x=1.2} < 0$ Hence point of inflection	A1	2.2a
		(4)	
	Alternative 1 for (b)(ii)		
	$\left(\frac{d^2y}{dx^2}\right)_{x=1} = 60x^2 - 144x + 84 = 0$ (is inconclusive) $\left(\frac{d^3y}{dx^3}\right) = 120x - 144 \Rightarrow \left(\frac{d^3y}{dx^3}\right)_{x=1} = \dots$	M1	2.1
	$\left(\frac{d^2y}{dx^2}\right)_{x=1} = 0$ and $\left(\frac{d^3y}{dx^3}\right)_{x=1} \neq 0$ Hence point of inflection	A1	2.2a
	Alternative 2 for (b)(ii)		
	E.g. $\left(\frac{dy}{dx}\right)_{x=0.8} = \dots \left(\frac{dy}{dx}\right)_{x=1.2} = \dots$	M1	2.1
	$\left(\frac{dy}{dx}\right)_{x=0.8} < 0, \left(\frac{dy}{dx}\right)_{x=1.2} < 0$ Hence point of inflection	A1	2.2a
(7 marks)			
Notes			
(a)(i) M1: $x^n \rightarrow x^{n-1}$ for at least one power of x A1: $\frac{dy}{dx} = 20x^3 - 72x^2 + 84x - 32$ (a)(ii)			

A1ft: Achieves a correct $\frac{d^2y}{dx^2}$ for their $\frac{dy}{dx} = 20x^3 - 72x^2 + 84x - 32$

(b)(i)

M1: Substitutes $x = 1$ into their $\frac{dy}{dx}$

A1: Obtains $\frac{dy}{dx} = 0$ following a correct derivative and makes a conclusion which can be minimal

e.g. tick, QED etc. which may be in a preamble e.g. stationary point when $\frac{dy}{dx} = 0$ and then shows $\frac{dy}{dx} = 0$

Alternative:

M1: Attempts to solve $\frac{dy}{dx} = 0$ by factorisation. This may be by using the factor of $(x - 1)$ or possibly using a calculator to find the roots and showing the factorisation. Note that they may divide by 4 before factorising which is acceptable. Need to either see either $4(x - 1)^2(5x - 8)$ or $(x - 1)^2(5x - 8)$ for the factorisation or $x = \frac{8}{5}$ and $x = 1$ seen as the roots.

A1: Obtains $x = 1$ and makes a conclusion as above

(b)(ii)

M1: Considers the value of the second derivative either side of $x = 1$. Do not be too concerned with the interval for the method mark.

(NB $\frac{d^2y}{dx^2} = (x - 1)(60x - 84)$ so may use this factorised form when considering $x < 1$, $x > 1$ for sign change of second derivative)

A1: Fully correct work including a correct $\frac{d^2y}{dx^2}$ with a reasoned conclusion indicating that the stationary point is a point of inflection. Sufficient reason is e.g. “sign change”/“ > 0 , < 0 ”. If values are given they should be correct (but be generous with accuracy) but also just allow “ > 0 ” and “ < 0 ” provided they are correctly paired. The interval must be where $x < 1.4$

Alternative 1 for (b)(ii)

M1: Shows that second derivative at $x = 1$ is zero and **then finds the third derivative at $x = 1$**

A1: Fully correct work including a correct $\frac{d^2y}{dx^2}$ with a reasoned conclusion indicating that stationary point is a point of inflection. Sufficient reason is “ $\neq 0$ ” but must follow a correct third derivative and a correct value if evaluated. For reference $\left(\frac{d^3y}{dx^3}\right)_{x=1} = -24$

Alternative 2 for (b)(ii)

M1: Considers the value of the first derivative either side of $x = 1$. Do not be too concerned with the interval for the method mark.

A1: Fully correct work with a reasoned conclusion indicating that stationary point is a point of inflection. Sufficient reason is e.g. “same sign”/“both negative”/“ < 0 , < 0 ”. If values are given they should be correct (but be generous with accuracy). The interval must be where $x < 1.4$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$f'(x)$	-32	-24.3	-17.92	-12.74	-8.64	-5.5	-3.2	-1.62	-0.64	-0.14	0
$f''(x)$	84	70.2	57.6	46.2	36	27	19.2	12.6	7.2	3	0

x	1.1	1.2	1.3	1.4	1.5	1.6	1.7
$f'(x)$	-0.1	-0.32	-0.54	-0.64	-0.5	0	0.98
$f''(x)$	-1.8	-2.4	-1.8	0	3	7.2	12.6

Question	Scheme	Marks	AOs
11	$\frac{1+11x-6x^2}{(x-3)(1-2x)} \equiv A + \frac{B}{(x-3)} + \frac{C}{(1-2x)}$		
(a) Way 1	$1+11x-6x^2 \equiv A(1-2x)(x-3) + B(1-2x) + C(x-3) \Rightarrow B=..., C=...$	M1	2.1
	$A=3$	B1	1.1b
	Uses substitution or compares terms to find either $B=...$ or $C=...$	M1	1.1b
	$B=4$ and $C=-2$ which have been found using a correct identity	A1	1.1b
		(4)	
(a) Way 2	{long division gives} $\frac{1+11x-6x^2}{(x-3)(1-2x)} \equiv 3 + \frac{-10x+10}{(x-3)(1-2x)}$		
	$-10x+10 \equiv B(1-2x) + C(x-3) \Rightarrow B=..., C=...$	M1	2.1
	$A=3$	B1	1.1b
	Uses substitution or compares terms to find either $B=...$ or $C=...$	M1	1.1b
	$B=4$ and $C=-2$ which have been found using $-10x+10 \equiv B(1-2x) + C(x-3)$	A1	1.1b
		(4)	
(b)	$f(x) = 3 + \frac{4}{(x-3)} - \frac{2}{(1-2x)}$ { $= 3 + 4(x-3)^{-1} - 2(1-2x)^{-1}$ } ; $x > 3$		
	$f'(x) = -4(x-3)^{-2} - 4(1-2x)^{-2} \left\{ = -\frac{4}{(x-3)^2} - \frac{4}{(1-2x)^2} \right\}$	M1	2.1
	Correct $f'(x)$ and as $(x-3)^2 > 0$ and $(1-2x)^2 > 0$, then $f'(x) = -(+ve) - (+ve) < 0$, so $f(x)$ is a decreasing function	A1ft	1.1b
		A1	2.4
		(3)	
(7 marks)			
Notes for Question 11			
(a)			
M1:	Way 1: Uses a correct identity $1+11x-6x^2 \equiv A(1-2x)(x-3) + B(1-2x) + C(x-3)$ in a complete method to find values for B and C . Note: Allow one slip in copying $1+11x-6x^2$ Way 2: Uses a correct identity $-10x+10 \equiv B(1-2x) + C(x-3)$ (which has been found from long division) in a complete method to find values for B and C		
B1:	$A=3$		
M1:	Attempts to find the value of either B or C from their identity This can be achieved by <i>either</i> substituting values into their identity <i>or</i> by comparing coefficients and solving the resulting equations simultaneously		
A1:	See scheme		
Note:	Way 1: Comparing terms: $x^2: -6 = -2A$; $x: 11 = 7A - 2B + C$; constant: $1 = -3A + B - 3C$ Way 1: Substituting: $x=3: -20 = -5B \Rightarrow B=4$; $x=\frac{1}{2}: 5 = -\frac{5}{2}C \Rightarrow C=-2$		
Note:	Way 2: Comparing terms: $x: -10 = -2B + C$; constant: $10 = B - 3C$ Way 2: Substituting: $x=3: -20 = -5B \Rightarrow B=4$; $x=\frac{1}{2}: 5 = -\frac{5}{2}C \Rightarrow C=-2$		



Note:	$A=3, B=4, C=-2$ from no working scores M1B1M1A1
Note:	The final A1 mark is effectively dependent upon both M marks

Notes for Question 11 Continued	
(a) ctd	
Note:	Writing $1+11x-6x^2 \equiv B(1-2x)+C(x-3) \Rightarrow B=4, C=-2$ will get 1 st M0, 2 nd M1, 1 st A0
Note:	Way 1: You can imply a correct identity $1+11x-6x^2 \equiv A(1-2x)(x-3)+B(1-2x)+C(x-3)$ from seeing $\frac{1+11x-6x^2}{(x-3)(1-2x)} \equiv \frac{A(1-2x)(x-3)+B(1-2x)+C(x-3)}{(x-3)(1-2x)}$
Note:	Way 2: You can imply a correct identity $-10x+10 \equiv B(1-2x)+C(x-3)$ from seeing $\frac{-10x+10}{(x-3)(1-2x)} \equiv \frac{B(1-2x)+C(x-3)}{(x-3)(1-2x)}$
(b)	
M1:	Differentiates to give $\{f'(x) = \} \pm \lambda(x-3)^{-2} \pm \mu(1-2x)^{-2}; \lambda, \mu \neq 0$
A1ft:	$f'(x) = -4(x-3)^{-2} - 4(1-2x)^{-2}$, which can be simplified or un-simplified
Note:	Allow A1ft for $f'(x) = -(their\ B)(x-3)^{-2} + (2)(their\ C)(1-2x)^{-2}; (their\ B), (their\ C) \neq 0$
A1:	$f'(x) = -4(x-3)^{-2} - 4(1-2x)^{-2}$ or $f'(x) = -\frac{4}{(x-3)^2} - \frac{4}{(1-2x)^2}$ and a correct explanation e.g. $f'(x) = -(+ve) - (+ve) < 0$, so $f(x)$ is a decreasing {function}
Note:	The final A mark can be scored in part (b) from an incorrect $A=...$ or from $A=0$ or no value of A found in part (a)

Notes for Question 11 Continued - Alternatives					
(a)					
Note:	Be aware of the following alternative solutions, by initially dividing by "$(x-3)$" or "$(1-2x)$" $\frac{1+11x-6x^2}{(x-3)(1-2x)} \equiv \frac{-6x-7}{(1-2x)} - \frac{20}{(x-3)(1-2x)} \equiv 3 - \frac{10}{(1-2x)} - \frac{20}{(x-3)(1-2x)}$ $\frac{20}{(x-3)(1-2x)} \equiv \frac{D}{(x-3)} + \frac{E}{(1-2x)} \Rightarrow 20 \equiv D(1-2x) + E(x-3) \Rightarrow D = -4, E = -8$ $\Rightarrow 3 - \frac{10}{(1-2x)} - \left(\frac{-4}{(x-3)} + \frac{-8}{(1-2x)} \right) \equiv 3 + \frac{4}{(x-3)} - \frac{2}{(1-2x)}; A=3, B=4, C=-2$ $\frac{1+11x-6x^2}{(x-3)(1-2x)} \equiv \frac{3x-4}{(x-3)} + \frac{5}{(x-3)(1-2x)} \equiv 3 + \frac{5}{(x-3)} + \frac{5}{(x-3)(1-2x)}$ $\frac{5}{(x-3)(1-2x)} \equiv \frac{D}{(x-3)} + \frac{E}{(1-2x)} \Rightarrow 5 \equiv D(1-2x) + E(x-3) \Rightarrow D = -1, E = -2$ $\Rightarrow 3 + \frac{5}{(x-3)} + \left(\frac{-1}{(x-3)} + \frac{-2}{(1-2x)} \right) \equiv 3 + \frac{4}{(x-3)} - \frac{2}{(1-2x)}; A=3, B=4, C=-2$				
(b)					
Alternative Method 1:					
$f(x) = \frac{1+11x-6x^2}{(x-3)(1-2x)}, x > 3 \Rightarrow f(x) = \frac{1+11x-6x^2}{-2x^2+7x-3}; \begin{cases} u=1+11x-6x^2 & v=-2x^2+7x-3 \\ u'=11-12x & v'=-4x+7 \end{cases}$					
$f'(x) = \frac{(-2x^2+7x-3)(11-12x) - (1+11x-6x^2)(-4x+7)}{(-2x^2+7x-3)^2}$	<table> <tr> <td>Uses quotient rule to find $f'(x)$</td><td>M1</td></tr> <tr> <td>Correct differentiation</td><td>A1</td></tr> </table>	Uses quotient rule to find $f'(x)$	M1	Correct differentiation	A1
Uses quotient rule to find $f'(x)$	M1				
Correct differentiation	A1				
$f'(x) = \frac{-20((x-1)^2+1)}{(-2x^2+7x-3)^2}$ and a correct explanation, e.g. $f'(x) = -\frac{(+ve)}{(+ve)} < 0$, so $f(x)$ is a decreasing {function}	A1				
Alternative Method 2:					
Allow M1A1A1 for the following solution:					
Given $f(x) = 3 + \frac{4}{(x-3)} - \frac{2}{(1-2x)} = 3 + \frac{4}{(x-3)} + \frac{2}{(2x-1)}$ as $\frac{4}{(x-3)}$ decreases when $x > 3$ and $\frac{2}{(2x-1)}$ decreases when $x > 3$ then $f(x)$ is a decreasing {function}					



Question	Scheme	Marks	AOs
10(a)	$8(4) - 4^{\frac{5}{2}} = 32 - 32 = 0$	B1	1.1b
		(1)	
(b)	$8 - \frac{5}{2}x^{\frac{3}{2}}$	B1	1.1b
	$x = 4 \Rightarrow \left\{ \frac{dy}{dx} \right\} 8 - \frac{5}{2} \times 8 = -12$ $\Rightarrow y\{-0\} = "-12"(x-4)$	M1	1.1b
	$12x + y = 48$ *	A1*	1.1b
		(3)	
(c)	Attempts to find one of the coordinates of the point of intersection $y = 8x, 12x + y = 48 \Rightarrow y = 19.2$ (or $x = 2.4$)	M1	1.1b
	Triangle area is $\frac{1}{2} \times 4 \times "19.2" \left(= 38.4 \text{ or } \frac{192}{5} \right)$ or $\int_0^{"2.4"} 8x \, dx + \int_{"2.4"}^4 "(48 - 12x)" \, dx$	dM1	3.1a
	$\int \left(8x - x^{\frac{5}{2}} \right) dx = 4x^2 - \frac{2}{7}x^{\frac{7}{2}}$	B1	1.1b
	$A = 38.4 - \left[4x^2 - \frac{2}{7}x^{\frac{7}{2}} \right]_0^4 = 38.4 - 64 + \frac{256}{7}$	ddM1	3.1a
	$= \frac{384}{35}$	A1	1.1b
		(5)	

(9 marks)

Notes

(a)

B1: Substitutes $x = 4$ into the equation of the curve and verifies that $y = 0$. Accept " $8(4) - 4^{\frac{5}{2}} = 0$ "

Alternatively, sets $8x - x^{\frac{5}{2}} = 0$ and solves with correct processing to achieve $x = 4$.

As a minimum accept e.g. $8x - x^{\frac{5}{2}} = 0 \Rightarrow x^{\frac{3}{2}} = 8 \Rightarrow \{x = \} 4$ which may follow factorisation.

(b)

B1: Correct differentiation. The $\frac{dy}{dx} =$ need not be present.

M1: Correct method for finding the equation of the tangent at $A(4, 0)$.

Requires substitution of $x = 4$ into their $\frac{dy}{dx}$ and an attempt at the equation of the line using this gradient. If using $y - y_1 = m(x - x_1)$ then condone the omission of the $- 0$.

If $y = mx + c$ is used they must proceed as far as $c = \dots$

Accept $\frac{dy}{dx} = -12$ or $m = -12$ without explicit substitution of $x = 4$ provided $8 - \frac{5}{2}x^{\frac{3}{2}}$ is seen.



A1*: Correct work leading to the given equation having scored B1M1.

Condone $y + 12x = 48$ and apply isw once seen.

Do not condone $12x + y - 48 = 0$ (unless a correct equation $= 48$ is seen).

(c) **Note:** Condone poor notation such as missing dx or spurious $\int$ symbols throughout.

M1: Attempts to find either the x or y coordinate of the intersection of line l_1 and line l_2

You may need to check the diagram or limits to their integrals.

dM1: Correct method for the area of the triangle. e.g. Triangle area is $\frac{1}{2} \times 4 \times "19.2" \left(= 38.4 \text{ or } \frac{192}{5} \right)$

If integration is attempted then condone slips in their rearrangement of $12x + y = 48$ to $y = 48 - 12x$ and note that their integrals do need not to be evaluated, so for example

$$\text{look for } \int_0^{2.4} 8x \, dx + \int_{2.4}^4 "(48 - 12x)" \, dx \quad \left\{ = \frac{576}{25} + \frac{384}{25} = 23.04 + 15.36 \right\}$$

B1: Correct integration of **curve** ignoring limits, i.e. $4x^2 - \frac{2}{7}x^{\frac{7}{2}}$ but condone e.g. $\frac{8x^{1+1}}{2} - \frac{x^{\frac{5}{2}+1}}{\frac{7}{2}}$

ddM1: Fully correct strategy including substitution which would lead to an **exact** area.

Does not need to reach a value. Dependent on both previous M marks.

Implied by $38.4 - \frac{192}{7}$ or a correct final answer $\frac{384}{35}$

Note that the decimal approximation that might be seen is 10.97142857 and implies ddM0 unless there is evidence of substitution (which need not be evaluated).

A1: Correct exact value. Either $\frac{384}{35}$ or $10\frac{34}{35}$

Alternative using lines – curve:

M1: Attempts to find either the x or y coordinate of the intersection of line l_1 and line l_2

You may need to check the diagram or limits to their integrals.

dM1: Correct method for at least one part (0 to "2.4" or "2.4" to 4) of the area of R including limits.

Condone slips in their rearrangement of $12x + y = 48$ to $y = 48 - 12x$ and note that their integrals do need not to be evaluated, so for example

$$\text{look for } \int_0^{2.4} 8x - \left(8x - x^{\frac{5}{2}} \right) dx \text{ or } \int_{2.4}^4 "(48 - 12x) - \left(8x - x^{\frac{5}{2}} \right) dx \text{ (or a sum of both)}$$

B1: Correct integration of **both** regions ignoring limits. May be completed as a sum or separately.

Condone e.g. $\frac{x^{\frac{5}{2}+1}}{\frac{7}{2}}$ in place of $\frac{2}{7}x^{\frac{7}{2}}$ Note that each integral may have been simplified.

$$\int_{\dots}^{\dots} x^{\frac{5}{2}} dx \text{ and } \{+\} \int_{\dots}^{\dots} 48 - 20x + x^{\frac{5}{2}} dx \rightarrow \left[\frac{2}{7}x^{\frac{7}{2}} \right]_{\dots}^{\dots} \text{ and } \{+\} \left[48x - 10x^2 + \frac{2}{7}x^{\frac{7}{2}} \right]_{\dots}^{\dots}$$

ddM1: Fully correct strategy including substitution which would lead to an **exact** area.

Does not need to reach a value. Dependent on both previous M marks.

This approach requires:

- substitution of 0, their 2.4 and 4 in the correct places

- the $\frac{2}{7}(2.4)^{\frac{7}{2}} - \frac{2}{7}(2.4)^{\frac{7}{2}}$ to be cancelled (may be implied by a correct final answer $\frac{384}{35}$)

Note that the decimal approximation that might be seen is 10.97142857 and implies ddM0

unless there is evidence that the $\frac{2}{7}(2.4)^{\frac{7}{2}} - \frac{2}{7}(2.4)^{\frac{7}{2}}$ has been cancelled e.g. ~~6.118...~~ - ~~6.118...~~



Question	Scheme	Marks	AOs
7(a)	$y = x^3 - 10x^2 + 27x - 23 \Rightarrow \frac{dy}{dx} = 3x^2 - 20x + 27$	B1	1.1b
	$\left(\frac{dy}{dx}\right)_{x=5} = 3 \times 5^2 - 20 \times 5 + 27 (= 2)$	M1	1.1b
	$y + 13 = 2(x - 5)$	M1	2.1
	$y = 2x - 23$	A1	1.1b
		(4)	
(b)	Both C and l pass through $(0, -23)$ and so C meets l again on the y -axis	B1	2.2a
		(1)	
(c)	$\pm \int (x^3 - 10x^2 + 27x - 23 - (2x - 23)) dx$ $= \pm \left(\frac{x^4}{4} - \frac{10}{3}x^3 + \frac{25}{2}x^2 \right)$	M1 A1ft	1.1b 1.1b
	$\left[\frac{x^4}{4} - \frac{10}{3}x^3 + \frac{25}{2}x^2 \right]_0^5$ $= \left(\frac{625}{4} - \frac{1250}{3} + \frac{625}{2} \right) (-0)$	dM1	2.1
	$= \frac{625}{12}$	A1	1.1b
		(4)	
(c) Alternative 1:			
	$\pm \int (x^3 - 10x^2 + 27x - 23) dx$ $= \pm \left(\frac{x^4}{4} - \frac{10}{3}x^3 + \frac{27}{2}x^2 - 23x \right)$	M1 A1	1.1b 1.1b
	$\left[\frac{x^4}{4} - \frac{10}{3}x^3 + \frac{27}{2}x^2 - 23x \right]_0^5 + \frac{1}{2} \times 5(23 + 13)$ $= -\frac{455}{12} + 90$	dM1	2.1
	$= \frac{625}{12}$	A1	1.1b
(c) Alternative 2:			
	$\int (x^3 - 10x^2 + 27x) dx = \left(\frac{x^4}{4} - \frac{10}{3}x^3 + \frac{27}{2}x^2 \right)$	M1 A1	1.1b 1.1b
	$\left[\frac{x^4}{4} - \frac{10}{3}x^3 + \frac{27}{2}x^2 \right]_0^5 - \frac{1}{2} \times 5 \times 10$	dM1	2.1
	$= \frac{625}{12}$	A1	1.1b
(9 marks)			

Notes

(a)

B1: Correct derivative

M1: Substitutes $x = 5$ into their derivative. This may be implied by their value for $\frac{dy}{dx}$ M1: Fully correct straight line method using $(5, -13)$ and their $\frac{dy}{dx}$ at $x = 5$

A1: cao. Must see the full equation in the required form.

(b)

B1: Makes a suitable deduction.

Alternative via equating l and C and factorising e.g.

$$x^3 - 10x^2 + 27x - 23 = 2x - 23$$

$$x^3 - 10x^2 + 25x = 0$$

$$x(x^2 - 10x + 25) = 0 \Rightarrow x = 0$$

So they meet on the y -axis

(c)

M1: For an attempt to integrate $x^n \rightarrow x^{n+1}$ for $\pm "C - l"$

A1ft: Correct integration in any form which may be simplified or unsimplified. (follow through their equation from (a))

If they attempt as 2 separate integrals e.g. $\int (x^3 - 10x^2 + 27x - 23) dx - \int (2x - 23) dx$ then

award this mark for the correct integration of the curve as in the alternative.

If they combine the curve with the line first then the subsequent integration must be correct or a correct ft for their line and allow for $\pm "C - l"$

dM1: Fully correct strategy for the area. Award for use of 5 as the limit and condone the omission of the -0 . **Depends on the first method mark.**

A1: Correct exact value

Alternative 1:M1: For an attempt to integrate $x^n \rightarrow x^{n+1}$ for $\pm C$ A1: Correct integration for $\pm C$

dM1: Fully correct strategy for the area e.g. correctly attempts the area of the trapezium and subtracts the area enclosed between the curve and the x -axis. Need to see the use of 5 as the limit condoning the omission of the -0 **and** a correct attempt at the trapezium **and** the subtraction.

May see the trapezium area attempted as $\int (2x - 23) dx$ in which case the integration and

use of the limits needs to be correct or correct follow through for their straight line equation.

Depends on the first method mark.

A1: Correct exact value

Note if they do $l - C$ rather than $C - l$ and the working is otherwise correct allow full marks if their final answer is given as a positive value. E.g. correct work with $l - C$ leading to $-\frac{625}{12}$ and

then e.g. hence area is $\frac{625}{12}$ is acceptable for full marks.

If the answer is left as $-\frac{625}{12}$ then score A0

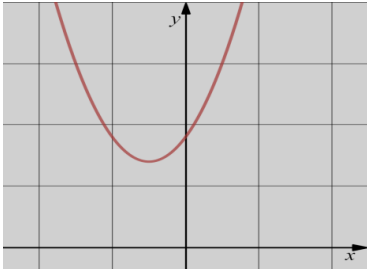
Alternative 2:M1: For an attempt to integrate $x^n \rightarrow x^{n+1}$ for $(C + 23)$ A1: Correct integration for $(C + 23)$ 

subtracts from the area under the curve

Need to see the use of 5 as the limit condoning the omission of the “– 0” **and** a correct attempt at the triangle **and** the subtraction.

Depends on the first method mark.



Question	Scheme	Marks	AOs
5 (a)	$2x^2 + 4x + 9 = 2(x \pm k)^2 \pm \dots$ $a = 2$	B1	1.1b
	Full method $2x^2 + 4x + 9 = 2(x+1)^2 \pm \dots$ $a = 2$ & $b = 1$	M1	1.1b
	$2x^2 + 4x + 9 = 2(x+1)^2 + 7$	A1	1.1b
		(3)	
(b)	 U shaped curve any position but not through (0,0) y - intercept at (0,9) Minimum at (-1,7)	B1	1.2
		B1	1.1b
		B1ft	2.2a
		(3)	
(c)	(i) Deduces translation with one correct aspect.	M1	3.1a
	Translate $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$	A1	2.2a
	(ii) $h(x) = \frac{21}{2(x+1)^2 + 7} \Rightarrow$ (maximum) value $\frac{21}{7} (= 3)$	M1	3.1a
	$0 < h(x) \leq 3$	A1ft	1.1b
		(4)	
(10 marks)			

(a)

B1: Achieves $2x^2 + 4x + 9 = 2(x \pm k)^2 \pm \dots$ or states that $a = 2$
M1: Deals correctly with first two terms of $2x^2 + 4x + 9$.

 Scored for $2x^2 + 4x + 9 = 2(x+1)^2 \pm \dots$ or stating that $a = 2$ and $b = 1$
A1: $2x^2 + 4x + 9 = 2(x+1)^2 + 7$

 Note that this may be done in a variety of ways including equating $2x^2 + 4x + 9$ with the expanded form of $a(x+b)^2 + c \equiv ax^2 + 2abx + ab^2 + c$


(b)

B1: For a U-shaped curve in any position not passing through $(0,0)$. Be tolerant of slips of the pen but do not allow if the curve bends back on itself

B1: A curve with a y -intercept on the +ve y axis of 9. The curve cannot just stop at $(0,9)$

Allow the intercept to be marked 9, $(0,9)$ but not $(9,0)$

B1ft: For a minimum at $(-1,7)$ in quadrant 2. This may be implied by -1 and 7 marked on the axes in the correct places. The curve must be a U shape and not a cubic say.

Follow through on a minimum at $(-b,c)$, marked in the correct quadrant, for their $a(x+b)^2 + c$

(c)(i)

M1: Deduces translation with one correct aspect or states $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ with no reference to 'translate'.

Allow instead of the word translate, shift or move. $g(x) = f(x-2) - 4$ can score M1

For example, possible methods of arriving at this deduction are:

- $f(x) \rightarrow g(x)$ is $2x^2 + 4x + 9 \rightarrow 2(x-2)^2 + 4(x-2) + 5$ So $g(x) = f(x-2) - 4$
- $g(x) = 2(x-1)^2 + 3$ New curve has its minimum at $(1,3)$ so $(-1,7) \rightarrow (1,3)$
- Using a graphical calculator to sketch $y=g(x)$ and compares to the sketch of $y=f(x)$

In almost all cases you will not allow if the candidate gives two **different types of** transformations.

Eg, stretch and

A1: Requires both 'translate' and $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$, Allow 'shift' or move' instead of translate.

So condone " Move shift 2 (units) to the right and move 4 (units) down

However, for M1 A1, it is possible to reflect in $x=0$ and translate $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$, so please consider all responses.

SC: If the candidate writes translate $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ or " move 2 (units) to the left and 4 (units) up" score M1 A0

(c)(ii)

M1: Correct attempt at finding the maximum value (although it may not be stated as a maximum)

- Uses part (a) to write $h(x) = \frac{21}{2(x+1)^2 + 7}$ and attempts to find $\frac{21}{\text{their "7"}}$
- Attempts to differentiate, sets $4x+4=0 \rightarrow x=-1$ and substitutes into $h(x) = \frac{21}{2x^2 + 4x + 9}$
- Uses a graphical calculator to sketch $y=h(x)$ and establishes the 'maximum' value $(...,3)$

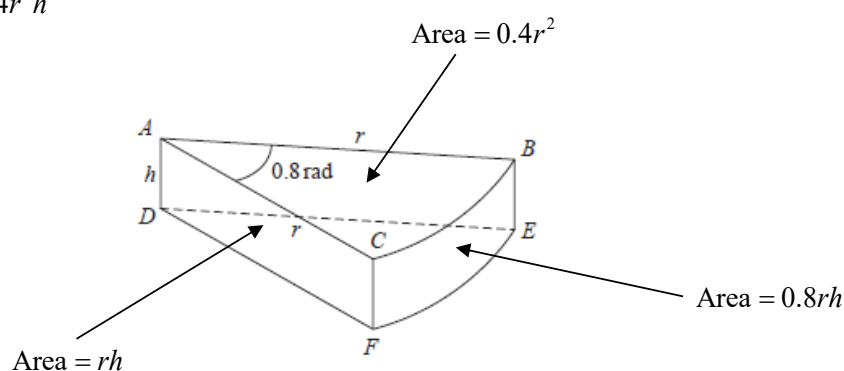
A1ft: $0 < h(x) \leq 3$ Allow for $0 < h \leq 3$ $[0,3]$ and $0 < y \leq 3$ but not $0 < x \leq 3$

Follow through on their $a(x+b)^2 + c$ so award for $0 < h(x) \leq \frac{21}{c}$



Question	Scheme	Marks	AOs
15 (a)	Sets up an allowable equation using volume = 240 E.g. $\frac{1}{2}r^2 \times 0.8h = 240 \Rightarrow h = \frac{600}{r^2}$ o.e.	M1 A1	3.4 1.1b
	Attempts to substitute their $h = \frac{600}{r^2}$ into $(S =) \frac{1}{2}r^2 \times 0.8 + \frac{1}{2}r^2 \times 0.8 + 2rh + 0.8rh$	dM1	3.4
	$S = 0.8r^2 + 2.8rh = 0.8r^2 + 2.8 \times \frac{600}{r} = 0.8r^2 + \frac{1680}{r}$ *	A1*	2.1
		(4)	
(b)	$\left(\frac{dS}{dr}\right) = 1.6r - \frac{1680}{r^2}$	M1 A1	3.1a 1.1b
	Sets $\frac{dS}{dr} = 0 \Rightarrow r^3 = 1050$ $r = \text{awrt } 10.2$	dM1 A1	2.1 1.1b
		(4)	
(c)	Attempts to substitute their positive r into $\left(\frac{d^2S}{dr^2}\right) = 1.6 + \frac{3360}{r^3}$ and considers its value or sign	M1	1.1b
	E.g. Correct $\frac{d^2S}{dr^2} = 1.6 + \frac{3360}{r^3}$ with $\frac{d^2S}{dr^2}_{r=10.2} = 5 > 0$ proving a minimum value of S	A1	1.1b
		(2)	
(10 marks)			
Notes:			

$$\text{Volume} = 0.4r^2h$$



$$\text{Total surface area} = 2rh + 0.8r^2 + 0.8rh$$

(a)

M1: Attempts to use the fact that the volume of the toy is 240 cm^3

Sight of $\frac{1}{2}r^2 \times 0.8 \times h = 240$ leading to $h = \dots$ or $rh = \dots$ scores this mark

But condone an equation of the correct form so allow for $kr^2h = 240 \Rightarrow h = \dots$ or $rh = \dots$

A1: A correct expression for $h = \frac{600}{r^2}$ or $rh = \frac{600}{r}$ which may be left unsimplified.

This may be implied when you see an expression for S or part of S E.g. $2rh = 2r \times \frac{600}{r^2}$

dM1: Attempts to substitute their $h = \frac{a}{r^2}$ o.e. such as $hr = \frac{a}{r}$ into a **correct** expression for S

Sight of $\frac{1}{2}r^2 \times 0.8 + \frac{1}{2}r^2 \times 0.8 + rh + rh + 0.8rh$ with an appropriate substitution

Simplified versions such as $0.8r^2 + 2rh + 0.8rh$ used with an appropriate substitution is fine.

A1*: Correct work leading to the given result.

$S =$, $SA =$ or surface area = must be seen at least once in the correct place

The method must be made clear so expect to see evidence. For example

$S = 0.8r^2 + 2rh + 0.8rh \Rightarrow S = 0.8r^2 + 2r \times \frac{600}{r^2} + 0.8r \times \frac{600}{r^2} \Rightarrow S = 0.8r^2 + \frac{1680}{r}$ would be fine.

(b) There is no requirement to see $\frac{dS}{dr}$ in part (b). It may even be called $\frac{dy}{dx}$.

M1: Achieves a derivative of the form $pr \pm \frac{q}{r^2}$ where p and q are non-zero constants

A1: Achieves $\left(\frac{dS}{dr}\right) = 1.6r - \frac{1680}{r^2}$

dM1: Sets or implies that their $\frac{dS}{dr} = 0$ and proceeds to $mr^3 = n$, $m \times n > 0$. It is dependent upon a

correct attempt at differentiation. This mark may be implied by a correct answer to their $pr - \frac{q}{r^2} = 0$

A1: $r = \text{awrt } 10.2$ or $\sqrt[3]{1050}$

(c)

M1: Attempts to substitute their positive r (found in (b)) into $\left(\frac{d^2S}{dr^2}\right) = e \pm \frac{f}{r^3}$ where e and f are non zero

and finds its value or sign.

Alternatively considers the sign of $\left(\frac{d^2S}{dr^2}\right) = e \pm \frac{f}{r^3}$ (at their positive r found in (b))

Condone the $\frac{d^2S}{dr^2}$ to be $\frac{d^2y}{dx^2}$ or being absent, but only for this mark.

A1: States that $\frac{d^2S}{dr^2}$ or $S'' = 1.6 + \frac{3360}{r^3} = \text{awrt } 5 > 0$ proving a minimum value of S

This is dependent upon having achieved $r = \text{awrt } 10$ and a correct $\frac{d^2S}{dr^2} = 1.6 + \frac{3360}{r^3}$

It can be argued without finding the value of $\frac{d^2S}{dr^2}$. E.g. $\frac{d^2S}{dr^2} = 1.6 + \frac{3360}{r^3} > 0$ as $r > 0$, so

minimum value of S. For consistency it is also dependent upon having achieved $r = \text{awrt } 10$



$$\text{or } {}^7C_4 a^3 2^4, \frac{7!}{4!3!} a^3 2^4, \binom{7}{4} a^3 2^4, 35a^3 2^4, 560a^3 \text{ etc.}$$

$$\text{or } {}^7C_3 a^3 (2x)^4, \frac{7!}{4!3!} a^3 (2x)^4, \binom{7}{3} a^3 (2x)^4, 35a^3 (2x)^4, 560a^3 x^4, \binom{7}{3} a^3 16x^4 \text{ etc.}$$

$$\text{or } {}^7C_3 a^3 2^4, \frac{7!}{4!3!} a^3 2^4, \binom{7}{3} a^3 2^4, 35a^3 2^4, 560a^3$$

You can condone missing brackets around the “2x” so allow e.g. $\frac{7!}{4!3!} a^3 2x^4$

An alternative is to attempt to expand $a^7 \left(1 + \frac{2x}{a}\right)^7$ to give $a^7 \left(\dots \frac{7 \times 6 \times 5 \times 4}{4!} \left(\frac{2x}{a}\right)^4 \dots \right)$

Allow M1 for e.g. $a^7 \left(\dots \frac{7 \times 6 \times 5 \times 4}{4!} \left(\frac{2x}{a}\right)^4 \dots \right), a^7 \left(\dots \binom{7}{4} \left(\frac{2x}{a}\right)^4 \dots \right), a^7 \left(\dots 35 \left(\frac{2x}{a}\right)^4 \dots \right)$ etc.

but condone missing brackets around the $\frac{2x}{a}$

Note that ${}^7C_3, \binom{7}{3}$ etc. are equivalent to ${}^7C_4, \binom{7}{4}$ etc. and are equally acceptable.

If the candidate attempts $(a + 2x)(a + 2x)(a + 2x) \dots$ etc. then it must be a complete method to reach the required term. Send to review if necessary.

dM1: For “560” $a^3 = 15120 \Rightarrow a = \dots$ Condone slips on copying the 15120 but their “560” must be an attempt at

${}^7C_4 \times 2$ or ${}^7C_4 \times 2^4$ and must be attempting the cube root of $\frac{15120}{560}$. **Depends on the first mark.**

A1: $a = 3$ and no other values i.e. ± 3 scores A0

Note that this is fairly common:

$${}^7C_4 a^3 2x^4 = 70a^3 x^4 \Rightarrow 70a^3 = 15120 \Rightarrow a^3 = 216 \Rightarrow a = 6$$

and scores M1 dM1 A0

7(a)	$\ln x \rightarrow \frac{1}{x}$	B1	1.1a
	Method to differentiate $\frac{4x^2 + x}{2\sqrt{x}}$ – see notes	M1	1.1b
	E.g. $2 \times \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2} \times \frac{1}{2}x^{-\frac{1}{2}}$	A1	1.1b
	$\frac{dy}{dx} = 3\sqrt{x} + \frac{1}{4\sqrt{x}} - \frac{4}{x} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$ *	A1*	2.1
		(4)	
(b)	$12x^2 + x - 16\sqrt{x} = 0 \Rightarrow 12x^{\frac{3}{2}} + x^{\frac{1}{2}} - 16 = 0$	M1	1.1b
	E.g. $12x^{\frac{3}{2}} = 16 - \sqrt{x}$	dM1	1.1b
	$x^{\frac{3}{2}} = \frac{4}{3} - \frac{\sqrt{x}}{12} \Rightarrow x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{\frac{2}{3}}$ *	A1*	2.1
		(3)	
(c)	$x_2 = \sqrt[3]{\left(\frac{4}{3} - \frac{\sqrt{2}}{12}\right)^2}$	M1	1.1b
	$x_2 = \text{awrt } 1.13894$	A1	1.1b
	$x = 1.15650$	A1	2.2a
		(3)	
(10 marks)			

Notes:

(a)

B1: Differentiates $\ln x \rightarrow \frac{1}{x}$ seen or implied

M1: Correct method to differentiate $\frac{4x^2 + x}{2\sqrt{x}}$:

Look for $\frac{4x^2 + x}{2\sqrt{x}} \rightarrow \dots x^{\frac{3}{2}} + \dots x^{\frac{1}{2}}$ being then differentiated to $Px^{\frac{1}{2}} + \dots$ **or** $\dots + Qx^{-\frac{1}{2}}$

Alternatively uses the quotient rule on $\frac{4x^2 + x}{2\sqrt{x}}$.

Condone slips but if rule is not quoted expect $\left(\frac{dy}{dx}\right) = \frac{2\sqrt{x}(Ax + B) - (4x^2 + x)Cx^{-\frac{1}{2}}}{(2\sqrt{x})^2} (A, B, C > 0)$

But a correct rule may be implied by their u, v, u', v' followed by applying $\frac{vu' - uv'}{v^2}$ etc.

Alternatively uses the product rule on $(4x^2 + x)(2\sqrt{x})^{-1}$

Condone slips but expect $\left(\frac{dy}{dx}\right) = Ax^{-\frac{1}{2}}(Bx + C) + D(4x^2 + x)x^{-\frac{3}{2}} (A, B, C > 0)$

In general condone missing brackets for the M mark. If they quote $u = 4x^2 + x$ and $v = 2\sqrt{x}$ and don't make the differentiation easier, they can be awarded this mark for applying the correct rule. Also allow this mark if they quote the correct quotient rule but only have v rather than v^2 in the denominator.

A1: Correct differentiation of $\frac{4x^2 + x}{2\sqrt{x}}$ although may not be simplified.



Question	Scheme	Marks	AOs
7 (a)	Either $x \leq -1$ or $2 \leq x \leq 5$	M1	2.2a
	Both $\{x : x \in \mathbb{R}, x \leq -1\} \cup \{x : x \in \mathbb{R}, 2 \leq x \leq 5\}$ o.e.	A1	2.5
		(2)	
(b)	States $(y =) \alpha(x+1)^2(x-5)^2$ or $(f(x) =) \alpha(x+1)^2(x-5)^2$	M1	1.1b
	Substitutes $(0, -75)$ into $y = \alpha(x+1)^2(x-5)^2$ and attempts to find the value for α	dM1	3.1a
	$y = -3(x+1)^2(x-5)^2$ o.e.	A1	2.1
		(3)	
(c)	Substitutes $x = 2$ into their $y = -3(x+1)^2(x-5)^2 \Rightarrow y = (-243)$	M1	2.1
	$0 < k < 243$	A1ft	1.1b
		(2)	

(7 marks)

Notes:**(a)**

M1: Either

- $x \leq -1$ o.e. e.g. $-1 \geq x$
- $2 \leq x \leq 5$ o.e.

but condone use of strict inequalities anywhere for this mark.

e.g. $2 < x < 5$ or $2 < x \leq 5$ or $2 \leq x < 5$ May also write e.g. $x < 5$ and $x > 2$ which scores M1 but not " $x < 5$ or $x > 2$ "Allow interval notation such as e.g. $[2, 5]$ or $(-\infty, -1]$ or condone e.g. $(2, 5)$

Ignore incorrect inequality statements not related to the one which is valid.

e.g. " $2 \leq x < 5$ and $x > -1$ " which scores M1 for the first inequality.A1: Requires $\{ \}$ and $\cup$ $\{x : x \leq -1\} \cup \{x : 2 \leq x \leq 5\}$ or $\{x | x \leq -1\} \cup \{x | 2 \leq x \leq 5\}$ either way roundbut condone $\{x \leq -1\} \cup \{2 \leq x \leq 5\}$, $\{x \leq -1 \cup 2 \leq x \leq 5\}$.Allow e.g. $\{x : x \leq -1\} \cup \{x : 2 \leq x \cap x \leq 5\}$ Use of $\cap$ to join the two separate regions is A0It is acceptable (but not required) to mention $\mathbb{R}$ e.g. $\{x : x \in \mathbb{R}, x \leq -1\} \cup \{x : x \in \mathbb{R}, 2 \leq x \leq 5\}$ Condone use of a lower limit written as e.g. $\{x : -\infty \leq x \leq -1\} \cup \{x : 2 \leq x \leq 5\}$ 

(b) **Note a correct equation written down scores all 3 marks.**

A correct expression but missing e.g. $y = \dots$ or $f(x) = \dots$ scores M1dM1A0

M1: Forms the equation of the form $(y =) \alpha(x+1)^2(x-5)^2$. Condone $\alpha = 1$

Award for sight of $\alpha(x+1)^2(x-5)^2$ even with $\alpha = 1$ i.e. $(x+1)^2(x-5)^2$

dM1: Substitutes $(0, -75)$ into the form $y = \alpha(x+1)^2(x-5)^2$ and attempts to find the value for α . It is dependent on the previous method mark.

A1: $y = -3(x+1)^2(x-5)^2$ o.e. (e.g. $y = -3x^4 + 24x^3 - 18x^2 - 120x - 75$)

isw after a correct answer. Condone $f(x) = -3(x+1)^2(x-5)^2$ but not

$C = -3(x+1)^2(x-5)^2$

A correct equation scores all 3 marks. Allow if seen in (c)

isw if they attempt to multiply out.

Alternative I part (b):

Using the form $y = ax^4 + bx^3 + cx^2 + dx + e$, then setting up and solving simultaneous equations.

There are various versions of this but can be marked similarly.

M1: Sets e equal to -75 (may just be seen in their equation) and forms **three correct** different equations in a, b, c and d which may be unsimplified.

Note that the form $y = ax^4 + bx^3 + cx^2 + dx + e$ is M0 until e is set equal to -75

There are 5 equations that can be formed, only 3 are necessary for this mark.

Do not condone slips.

Using $(-1, 0) \Rightarrow 0 = a - b + c - d - 75$ o.e.

Using $(5, 0) \Rightarrow 0 = 625a + 125b + 25c + 5d - 75$ o.e.

Using $\frac{dy}{dx} = 0$ at $x = 2 \Rightarrow 0 = 32a + 12b + 4c + d$ o.e.

Using $\frac{dy}{dx} = 0$ at $x = -1 \Rightarrow 0 = -4a + 3b - 2c + d$ o.e.

Using $\frac{dy}{dx} = 0$ at $x = 5 \Rightarrow 0 = 500a + 75b + 10c + d$ o.e.

dM1: Forms **four correct** different equations and solves to find values for a, b, c and d . You do not need to be concerned by the process of solving. A calculator can be used to solve the equations.

A1: $y = -3x^4 + 24x^3 - 18x^2 - 120x - 75$ o.e. isw if they attempt to factorise but withhold this mark if they e.g. divide all terms by 3.

Condone $f(x) = \dots$ but not $C = \dots$

A correct equation scores all 3 marks. Allow if seen in (c)



Alternative II part (b): Uses the form $y = (x+1)(x-5)(ax^2 + bx + c)$

M1: Substitutes $x = 0, y = -75$ $-75 = -5c \Rightarrow c = 15$, multiplies out, differentiates

$$\Rightarrow \frac{dy}{dx} = (2x-4)(ax^2 + bx + 15) + (x^2 - 4x - 5)(2ax + b)$$

and forms a **correct equation** in a and b which may be unsimplified.

$$\text{Using } \frac{dy}{dx} = 0 \text{ at } x = 2 \quad \Rightarrow 0 = 4a + b \quad \text{o.e.}$$

$$\text{Using } \frac{dy}{dx} = 0 \text{ at } x = -1 \quad \Rightarrow 0 = a - b + 15 \quad \text{o.e.}$$

$$\text{Using } \frac{dy}{dx} = 0 \text{ at } x = 5 \quad \Rightarrow 0 = 5a + b + 3 = 0 \quad \text{o.e.}$$

dm1: Forms **two correct** different equations and solves to find values for a and b . You do not need to be concerned by the process of solving. A calculator can be used to solve the equations.

A1: $y = (x+1)(x-5)(-3x^2 + 12x + 15)$ o.e. isw if they attempt to multiply out or factorise
Condone $f(x) = \dots$ but not $C = \dots$ but withhold this mark if they e.g. divide all terms by 3. A correct equation scores all 3 marks. Allow if seen in (c)

Alternative III part (b): Uses $\frac{dy}{dx} = \beta(x+1)(x-2)(x-5)$ (β may be 1) and integrates.

M1: Integrates $\left(\frac{dy}{dx} = \beta(x+1)(x-2)(x-5)\right)$ to $(y =) \beta\left(\frac{1}{4}x^4 - 2x^3 + \frac{3}{2}x^2 + 10x + k\right)$ and forms

one correct equation using either $(0, -75): -75 = \beta k$ (allow $-75 = k$)

$$(-1, 0): 0 = \beta\left(\frac{1}{4} + 2 + \frac{3}{2} - 10 + k\right) \quad (5, 0): 0 = \beta\left(\frac{625}{4} - 250 + \frac{75}{2} + 50 + k\right)$$

dm1: Forms a different equation using one of $(0, -75), (-1, 0), (5, 0)$ and solves to find values for β and k . You do not need to be concerned by the process of solving. A calculator can be used to solve the equations.

A1: $y = -12\left(\frac{1}{4}x^4 - 2x^3 + \frac{3}{2}x^2 + 10x + \frac{25}{4}\right)$ o.e. isw if they attempt to multiply out or factorise

Condone $f(x) = \dots$ but not $C = \dots$ but withhold this mark if they e.g. divide all terms by 3. A correct equation scores all 3 marks. Allow if seen in (c)

(c)

M1: Substitutes $x = 2$ into their $y = -3(x+1)^2(x-5)^2$ (must be a quartic in any form) and proceeds to find a value for y . Sight of their $\pm y$ (or ± 243) scores M1.
You may need to check this on your calculator if only a value is seen.

A1ft: $0 < k < 243$ o.e. ft on their negative y value at $x = 2$.

Allow use of set notation, interval notation and allow e.g. $k < 243, k > 0$ but do not allow OR or $\cup$. Do not accept $0 \leq k \leq 243$ o.e.

If there are multiple attempts at describing the region, mark what appears to be their final answer.

This mark can only be scored if they have a negative quartic graph function

i.e. $\alpha < 0$ for their $y = \alpha(x+1)^2(x-5)^2$ or $a < 0$ for their $y = ax^4 + bx^3 + cx^2 + dx + e$

Question	Scheme	Marks	AOs
9 (a)	$0 = 15T - T e^{0.2T} \Rightarrow e^{0.2T} = 15$	M1	3.4
	$(T =)$ awrt 13.5	A1	1.1b
		(2)	
(b)	Attempts to differentiate using the product rule $\left(\frac{d}{dt} (t e^{0.2t}) = \right) 0.2t e^{0.2t} + e^{0.2t}$	M1	1.1b
	$v = 15t - t e^{0.2t} \Rightarrow \left(\frac{dv}{dt} = \right) 15 - (0.2t e^{0.2t} + e^{0.2t})$	A1	1.1b
	Sets their $\frac{dv}{dt} = 0 \Rightarrow e^{0.2t} (0.2t + 1) = 15 \Rightarrow e^{0.2t} = \frac{15}{0.2t + 1}$	dM1	3.1b
	$\Rightarrow t = 5 \ln \left(\frac{75}{t + 5} \right) *$	A1*	2.1
		(4)	
(c)	(i) Attempts $t_2 = 5 \ln \left(\frac{75}{8 + 5} \right)$	M1	1.1b
	awrt 8.478	A1	1.1b
	(ii) awrt 8.55 seconds (including units)	A1	3.2a
		(3)	
(9 marks)			
Notes:			
(a) May use t or another variable which is acceptable. M1: Uses the model with $v = 0$ and proceeds to $e^{0.2T} = 15$ Do not be concerned by the use of an inequality sign instead of an equals. May be implied by awrt 13.5 A1: $(T =)$ awrt 13.5 units are not required but if given they must be seconds (or e.g. secs or s)			

(b) If no attempt is seen for (b) then allow differentiation seen in (a) to score in (b)

M1: Attempts to use the product rule to differentiate $t e^{0.2t}$ achieving the form $Ae^{0.2t} + Bte^{0.2t}$ (A and B both non zero but may be 1) which may be unsimplified. It is likely to be part of an expression.

A1: $\left(\frac{dv}{dt}\right) = 15 - (0.2t e^{0.2t} + e^{0.2t})$ o.e. which may be unsimplified. (Condone a missing trailing bracket)

Do not allow recovery of signs to score this mark if they initially write

e.g. $15 - 0.2t e^{0.2t} + e^{0.2t} = 0$ and on a later line correct this. e.g. $15 - 0.2t e^{0.2t} - e^{0.2t} = 0$

dM1: Sets $15 \pm Ae^{0.2t} \pm Bte^{0.2t} = 0$ (the $=0$ may be implied), attempts to make $e^{\pm 0.2t}$ (or $Ce^{\pm 0.2t}$) the subject **and** proceeds to the form

$$Ce^{0.2t} = \frac{D}{E + Ft} \text{ or } Ce^{-0.2t} = \frac{E + Ft}{D} \text{ (where } C \text{ can be 1 and } A, B, C, D, E, F \neq 0 \text{)}$$

It is dependent on the previous method mark.

May see $15 - "0.2t e^{0.2t} - e^{0.2t}" = 0 \Rightarrow e^{0.2t} = \frac{15}{0.2t + 1}$ which scores dM1

They must take out a factor of $e^{\pm 0.2t}$ (or $Ce^{\pm 0.2t}$) and divide by their bracket. Condone sign slips in their rearrangement, however, if they take logs of both sides first, the rearrangement must be correct (with no sign slips).

Allow invisible brackets to be implied by further work which is not the given answer.

A1*: Achieves the given answer with **no errors seen including use of invisible brackets** (but condone a missing trailing bracket) All previous marks in (b) must have been scored. The $= 0$ must have been seen somewhere in their solution. Do not allow this mark to be scored for proceeding directly from

$$e^{0.2t} (0.2t + 1) = 15 \Rightarrow t = 5 \ln \left(\frac{75}{t + 5} \right) \text{ which is A0*}$$

We must see either $e^{0.2t} = \frac{15}{0.2t + 1}$ o.e. or an unsimplified expression for t

e.g. $t = 5 \ln \left(\frac{15}{0.2t + 1} \right)$ before achieving the given answer. Condone $t = 5 \ln \frac{75}{t + 5}$

(c) (i) Check by the question. If there is a contradiction between answers, the answer in the main body of the script takes precedence.

M1: Attempts to use the iteration formula at least once. $t_2 = 5 \ln \left(\frac{75}{8 + 5} \right)$. May be implied by awrt 8.76 or awrt 8.48 or awrt 8.58. It is not implied by awrt 8.55

A1: awrt 8.478 (on its own can score M1A1)

(c)(ii) This mark can only be scored provided in (c)(i) M1 has been scored so M0A0A1 is not a possible mark profile.

A1: awrt 8.55 **seconds** (e.g. s or secs) **Requires units.**

If the candidate lists their iterations but does not select an answer then take the final value, which still requires units to be stated (which in many cases is likely to be omitted)

Note that awrt 8.55 does not imply M1.



Question	Scheme	Marks	AOs
15 (a)	$\frac{1}{2}r^2\theta + \frac{1}{10}r^2 = 240 \Rightarrow r\theta = \frac{240 - \frac{1}{10}r^2}{\frac{1}{2}r}$ or $\theta = \frac{240 - \frac{1}{10}r^2}{\frac{1}{2}r^2}$	M1 A1	3.4 1.1b
	Substitutes into the expression for P $r\theta = \frac{240 - \frac{1}{10}r^2}{\frac{1}{2}r}$ into $(P =) r\theta + 2r + \frac{1}{5}r$	dM1	3.4
	$P = \frac{240 - \frac{1}{10}r^2}{\frac{1}{2}r} + 2r + \frac{1}{5}r = \frac{480}{r} - \frac{1}{5}r + 2r + \frac{1}{5}r = 2r + \frac{480}{r}$ *	A1*	2.1
		(4)	
(b)	$\left(\frac{dP}{dr}\right) = 2 - \frac{480}{r^2}$	M1	1.1b
	Sets $\frac{dP}{dr} = 0 \Rightarrow r^2 = 240$ $r = \text{awrt } 15.5$	dM1 A1	2.1 1.1b
		(3)	
(c)	$\left(\frac{d^2P}{dr^2}\right) = \frac{960}{r^3}$	M1	1.1b
	$\left(\frac{d^2P}{dr^2}\right) = \text{awrt } 0.26 > 0$ proving a minimum value of P	A1	1.1b
		(2)	
(9 marks)			
Notes:			
(a)	Note that just finding a correct equation for the area and/or a correct equation for the perimeter (before any substitution) is insufficient to score any marks. M1: Uses area formulae to form an equation of the form $\alpha r^2\theta + \beta r^2 = 240$ o.e. ($\alpha, \beta \neq 0$) and rearranges to make $r\theta$, θ or $r\theta + \frac{1}{5}r$ the subject. Look for: $r\theta = \frac{M \pm Nr^2}{r} \left(= \frac{M}{r} \pm Nr \right) \text{ o.e. or } \theta = \frac{M \pm Nr^2}{r^2} \left(= \frac{M}{r^2} \pm N \right) \text{ o.e. where } M, N \neq 0$ or $r\theta + \frac{1}{5}r = \frac{L}{r}$ $L \neq 0$ o.e. May work in degrees. A1: A correct rearrangement for θ or $r\theta$ or $r\theta + \frac{1}{5}r$ which may be unsimplified (may be in degrees) $r\theta = \frac{240 - \frac{1}{10}r^2}{\frac{1}{2}r} \text{ o.e. e.g. } r\theta = \frac{2400 - r^2}{5r} \text{ or } r\theta = \frac{480 - 0.2r^2}{r}$ or $r\theta + \frac{1}{5}r = \frac{480}{r}$ o.e.		



$$\text{or } \theta = \frac{240 - \frac{1}{10}r^2}{\frac{1}{2}r^2} \text{ o.e. e.g. } \theta = \frac{2400 - r^2}{5r^2} \text{ or } \theta = \frac{480}{r^2} - \frac{1}{5} \text{ or } \theta = 480r^{-2} - 0.2$$

dM1: Substitutes their $r\theta = \frac{M \pm Nr^2}{r}$ o.e. or $\theta = \frac{M \pm Nr^2}{r^2}$ o.e. or $r\theta + \frac{1}{5}r = \frac{L}{r}$ into an

expression of the form $(P =) r\theta + Qr$, $Q \neq 0$ (typically $P = r\theta + \frac{11}{5}r$) which may be unsimplified or in degrees. It is dependent on the previous method mark. It is acceptable for their valid expression for θ , $r\theta$ or $r\theta + \frac{1}{5}r$ to be substituted into the perimeter expression directly (without first seeing them in the perimeter expression).

A1*: $P = 2r + \frac{480}{r}$ following a correct method (condone slips to be recovered) and all previous marks scored. Condone invisible brackets to be recovered.
 $P =$, Perimeter = must be seen at least once in their solution in the correct place.

(b) Mark (b) and (c) together. There is no requirement to see the notation $\frac{dP}{dr}$ in part

(b). It may even be called $\frac{dy}{dx}$. Allow use of e.g. P' or e.g. y'

M1: $\left(\frac{dP}{dr} =\right) p \pm \frac{q}{r^2}$ where p and q are non-zero constants

dM1: Sets or implies that their $\frac{dP}{dr} = 0$ and proceeds to $mr^{\pm 2} = n$, $m \times n > 0$. It is dependent on the previous method mark. Do not be concerned by the mechanics of the rearrangement. This mark may be implied by a correct answer to their $p - \frac{q}{r^2} = 0$. You may need to check this on your calculator.

A1: $r = \text{awrt } 15.5$ or $\sqrt{240}$ ($= 4\sqrt{15}$) Do not accept $\pm$ (ignore any units if given)

(c) Condone other letters used instead of P and r for $\frac{d^2P}{dr^2}$ e.g. $\frac{d^2y}{dx^2}$ for M1 only.

Just using $\frac{dP}{dr}$ and considering a sign change is M0A0

M1: Differentiates and finds $\left(\frac{d^2P}{dr^2} =\right) \pm \frac{f}{r^3}$ (do not be concerned about the sign)

A1: **Note if they score A0 in (b) then this mark cannot be scored.**
 Requires

- a correct a correct expression for $\frac{d^2P}{dr^2}$
- a correct value for $\left(\frac{d^2P}{dr^2} =\right) \frac{960}{r^3} = \text{awrt } 0.26$ using awrt 15.5 (but allow 0.23(43..) if using 16)
- a correct comparison with 0 **and** a conclusion e.g. minimum

The expression for the second derivative does not need to be labelled but if it is then it must be $\frac{d^2P}{dr^2}$ o.e. or accept e.g. P'' BUT $\frac{d^2y}{dx^2}$ used in their conclusion is A0



Question	Scheme	Marks	AOs
8(a)	$x^4 \rightarrow \dots x^3$ or $\dots x^3 \rightarrow \dots x^2$ or $\dots x^2 \rightarrow \dots x$ or $ax \rightarrow a$	M1	1.1b
	$(f'(x) =) 4x^3 + x^2 - 16x + a$	A1	1.1b
	$\left(f'\left(-\frac{1}{4}\right) =\right) 4\left(-\frac{1}{4}\right)^3 + \left(\pm\frac{1}{4}\right)^2 - 16\left(-\frac{1}{4}\right) + a = 0$ $\Rightarrow a = \dots$	dM1	3.1a
	$a = -4^*$	A1*	2.1
		(4)	
(b)	$\frac{4735}{768}$	B1	1.1b
		(1)	

Notes

(a) The first two marks are the same for all approaches.

M1: Reduces the power by one for any term in $f(x)$. Look for $x^n \rightarrow x^{n-1}$ and allow for $ax \rightarrow a$ including $ax^1 \rightarrow ax^0$ or $-4x \rightarrow -4$

A1: Correct differentiation $(f'(x) =) 4x^3 + x^2 - 16x + a$ or $4x^3 + x^2 - 16x - 4$

Ignore the absence of the LHS. May be in terms of a or have -4 substituted at this point as above.

Main Scheme:

dM1: Substitutes $x = -\frac{1}{4}$ into their $f'(x)$, sets $= 0$ ($= 0$ may be implied) and solves for a .

Condone $4 + a = 0$ as evidence of substitution for the dM1 only.

This mark is not available if their $f'(x)$ does not have a term with a in it using this approach.

A1*: cso Requires:

- a correct derivative
- correct substitution of $x = -\frac{1}{4}$ into $f'(x)$ and set $= 0$ (now the $= 0$ must be seen somewhere for this mark). Each term does not need to be evaluated.
- achieves $a = -4$

For the A1*, evidence of correct substitution might be $-\frac{1}{16} + \frac{1}{16} + 4 + a = 0$ but not $4 + a = 0$

Condone invisible brackets around $-\frac{1}{4}$ provided this is recovered before the given answer e.g.

$$-\frac{1}{4}^2 \text{ recovered to } +\frac{1}{16}$$

Alt 1: Verification

dM1: Substitutes $x = -\frac{1}{4}$ and $a = -4$ into their $f'(x)$, and finds a value.

The $a = -4$ may already have been substituted into $f(x)$ as above.

A1*: cso Requires:

- a correct derivative



- correct substitution of $x = -\frac{1}{4}$ into $f'(x)$ and use of $a = -4$ (at some stage) to achieve $f'\left(-\frac{1}{4}\right) = \dots = 0$. The substitution must be embedded in $f'(x)$ **or** evaluated in an intermediate step e.g. $-\frac{1}{16} + \frac{1}{16} + 4 - 4$ before the $= 0$
- a minimal conclusion e.g. “hence proven” or “ $\therefore a = -4$ ” or e.g. a tick. but **not** e.g. “max = $-\frac{1}{4}$ ” or just “ $0 = 0$ ”

Alt 2: Algebraic division

dM1: Attempts to divide their $f'(x)$ by $(4x+1)$ $\left(\text{or } \left(x + \frac{1}{4}\right)\right)$, achieves a linear remainder in a only, sets $= 0$ and achieves a value for a . Condone slips in their calculations. As a minimum, expect to see $x^2 + \lambda x + \mu$ (or $4x^2 + \lambda x + \mu$) as their quotient (with $\mu \neq 0$) leading to a linear remainder in a only set $= 0$ leading to a value for a .

A1*: cso Requires:

- a correct derivative
- correct quotient $x^2 - 4$ (or $(4x^2 - 16)$) and correct remainder $a + 4$ set $= 0$
- achieves $a = -4$

(b)

B1: $\frac{4735}{768}$ cao Allow exact equivalents e.g. $6\frac{127}{768}$ or the recurring decimal $6.16536458\dot{3}$
There is no need to see a calculation. Decimal approximations e.g. $6.1653645833\dots$ score B0.



Question	Scheme	Marks	AOs
8(c)	$(f'(x) =) 4x^3 + x^2 - 16x - 4 = (4x+1)(x^2 + \dots)$	M1	3.1a
	$= (4x+1)(x^2 - 4)$ or e.g. $\frac{4x^3 + x^2 - 16x - 4}{4x+1} = x^2 - 4$	A1	1.1b
	$f(-2) = \dots(-5)$ $\Rightarrow -5 < k < \frac{4735}{768}$	M1	2.1
	$\left\{ k \in \mathbb{R} : -5 < k < \frac{4735}{768} \right\}$	A1ft	2.5
		(4)	

(9 marks)

Notes

(c) **Note: Candidates who do not score the first M mark may score maximum M0A0M1A1ft via the special case at the end of the mark scheme for part (c).**

M1: For the key step in using the factor theorem to take $(4x+1)$ or $\left(x + \frac{1}{4}\right)$ out as a factor of their

$f'(x)$, **not** $f(x)$. Look for $(4x+1)(x^2 + \lambda x - 4)$ or $(4x+1)(x^2 \pm \mu)$ or $\left(x + \frac{1}{4}\right)(4x^2 \pm \mu)$

or $\left(x + \frac{1}{4}\right)(4x^2 \pm \lambda x - 16)$ (where the λ or μ is a constant) either by inspection or division.

If they attempt division in (a) then they must use their quotient in (c) to score the marks.

They may “spot” that one of $f'(\pm 2) = 0$ and factor out $(x \pm 2)(4x^2 + \lambda x \pm 2)$ without evidence that $f'(\pm 2) = 0$

There must be some factorisation or algebraic division present – not just roots stated from a calculator.

A1: Correct factorisation of the cubic to a linear and quadratic product, so

$(4x+1)(x^2 - 4)$ or $\left(x + \frac{1}{4}\right)(4x^2 - 16)$ or $(x-2)(4x^2 + 9x + 2)$ or $(x+2)(4x^2 - 7x - 2)$

or for the correct quadratic seen e.g. $x^2 - 4$ or $4x^2 - 16$ (which may be the quotient in their algebraic division and may come from their work in (a) provided it is used in (c)).

There is no need to complete the long division if the appropriate quotient is found.

Note that proceeding directly to $(4x+1)(x-2)(x+2)$ does not score either mark without sight of the e.g. $(4x+1)(x^2 - 4)$ but they can score the final 2 marks via the special case.

Similarly, proceeding directly to $\left(x + \frac{1}{4}\right)(2x-4)(2x+4)$ or $4\left(x + \frac{1}{4}\right)(x-2)(x+2)$ does not

score either mark without sight of $\left(x + \frac{1}{4}\right)(4x^2 - 16)$ but they can score marks via the SC.

M1: Solves their quadratic factor = 0, substitutes one of these solutions (not $x = -\frac{1}{4}$) into $f(x)$

to find a value **and** selects the inside region between this and their answer to (b).

If their quadratic is of the form $Ax^2 - B$ then they can write down their solution for x but if it is a 3TQ then they must find their value of x using the usual non-calculator rules.

Condone “incorrect” factorisation following the correct quadratic leading to $x = \pm 2$



e.g. $\left(x + \frac{1}{4}\right)(4x^2 - 16) \rightarrow \left(x + \frac{1}{4}\right)(x - 2)(x + 2) = 0 \rightarrow x = \pm 2$

y (or k) $= -5$ (or $-\frac{47}{3}$) with nothing else can imply substitution of $x = -2$ (or 2)

Note that if they “spot” that $f'(\pm 2) = 0$ earlier then they can go straight to substitution.

Condone the use of y but not x in their region. Condone e.g. $k > -5$, $k < \frac{4735}{768}$ for this mark.

Condone the use of $\leq$ in place of $<$ for this mark. Note that $\left(2, -\frac{47}{3}\right)$ is the other minimum.

A1ft: $\left\{k : -5 < k < \frac{4735}{768}\right\}$ Requires correct use of set notation but condone absence of $\in \mathbb{R}$

Follow through on their upper limit from (b) which may be a decimal but must be positive. Must be strict inequalities i.e. not include the end points and must now be in terms of k .

Other acceptable notation includes: $k \in \left(-5, \frac{4735}{768}\right)$, $\{k, k > -5\} \cap \left\{k, k < \frac{4735}{768}\right\}$

Condone $\left\{k : k > -5 \cap k < \frac{4735}{768}\right\}$ or $\left\{-5 < k < \frac{4735}{768}\right\}$

Do not allow e.g. $\{k : k > -5\} \cup \left\{k : k < \frac{4735}{768}\right\}$ or just $-5 < k < \frac{4735}{768}$

SC: Candidates that have not scored the first M1 may go on to score the final two marks by finding an acceptable range between -5 and $\frac{4735}{768}$.

Use of their “ -5 ” or “ $-\frac{47}{3}$ ”, coming from $x = \pm 2$, (or -5 or $-\frac{47}{3}$ directly from a calculator)

with $\frac{4735}{768}$ will score M1 if the inside region is selected. A correct $\left\{k : -5 < k < \frac{4735}{768}\right\}$ will score A1ft

There is no need to see any working to achieve the -5 (or $-\frac{47}{3}$).

All the relevant guidance regarding the region in the M1 and A1ft notes above continues to apply. e.g. for the M1 we will condone the use of y but not x in their region while for the A1ft we require use of k .

For example:

- $\left\{k : -5 < k < \frac{4735}{768}\right\}$ without scoring the first M scores M0A0M1A1ft
- “ -5 ” $< k < \frac{4735}{768}$ without scoring the first M scores M0A0M1A0ft
- $\left\{k : -\frac{47}{3} < k < \frac{4735}{768}\right\}$ without scoring the first M scores M0A0M1A0ft

Question	Scheme	Marks	AOs
3 (a)	Correct method used in attempting to differentiate $y = \frac{5x^2 + 10x}{(x+1)^2}$	M1	3.1a
	$\frac{dy}{dx} = \frac{(x+1)^2 \times (10x+10) - (5x^2 + 10x) \times 2(x+1)}{(x+1)^4}$ oe	A1	1.1b
	Factorises/Cancel term in $(x+1)$ and attempts to simplify $\frac{dy}{dx} = \frac{(x+1) \times (10x+10) - (5x^2 + 10x) \times 2}{(x+1)^3} = \frac{A}{(x+1)^3}$	M1	2.1
	$\frac{dy}{dx} = \frac{10}{(x+1)^3}$	A1	1.1b
		(4)	
(b)	For $x < -1$ Follow through on their $\frac{dy}{dx} = \frac{A}{(x+1)^n}, n=1,3$	B1ft	2.2a
		(1)	
(5 marks)			

(a)

M1: Attempts to use a correct rule to differentiate Eg: Use of quotient (& chain) rules on $y = \frac{5x^2 + 10x}{(x+1)^2}$

Alternatively uses the product (and chain) rules on $y = (5x^2 + 10x)(x+1)^{-2}$

Condone slips but expect $\left(\frac{dy}{dx}\right) = \frac{(x+1)^2 \times (Ax+B) - (5x^2 + 10x) \times (Cx+D)}{(x+1)^4}$ ($A, B, C, D > 0$) or

$\left(\frac{dy}{dx}\right) = \frac{(x+1)^2 \times (Ax+B) - (5x^2 + 10x) \times (Cx+D)}{((x+1)^2)^2}$ ($A, B, C, D > 0$) using the quotient rule

or $\left(\frac{dy}{dx}\right) = (x+1)^{-2} \times (Ax+B) + (5x^2 + 10x) \times C(x+1)^{-3}$ ($A, B, C \neq 0$) using the product rule.

Condone missing brackets and slips for the M mark. For instance if they quote $u = 5x^2 + 10$, $v = (x+1)^2$ and don't make the differentiation easier, they can be awarded this mark for applying the correct rule. Also allow where they quote the correct formula, give values of u and v , but only have v rather than v^2 the denominator.

A1: A correct (unsimplified) answer

Question	Scheme	Marks	AOs
Alt via division	Writes $y = \frac{5x^2 + 10x}{(x+1)^2}$ in form $y = A \pm \frac{B}{(x+1)^2}$ $A, B \neq 0$	M1	3.1a
	Writes $y = \frac{5x^2 + 10x}{(x+1)^2}$ in the form $y = 5 - \frac{5}{(x+1)^2}$	A1	1.1b
	Uses the chain rule $\Rightarrow \frac{dy}{dx} = \frac{C}{(x+1)^3}$ (May be scored from $A = 0$)	M1	2.1
	$\frac{dy}{dx} = \frac{10}{(x+1)^3}$ which cannot be awarded from incorrect value of A	A1	1.1b
		(4)	
(b)	For $x < -1$ or correct follow through	B1ft	2.2a
		(1)	
(5 marks)			
Question	Scheme	Marks	AOs

Question	Scheme	Marks	AOs
12(a)	$f(x) = \frac{e^{3x}}{4x^2 + k} \Rightarrow f'(x) = \frac{(4x^2 + k)3e^{3x} - 8xe^{3x}}{(4x^2 + k)^2}$	M1	1.1b
	or	A1	1.1b
	$f(x) = e^{3x}(4x^2 + k)^{-1} \Rightarrow f'(x) = 3e^{3x}(4x^2 + k)^{-1} - 8xe^{3x}(4x^2 + k)^{-2}$		
	$f'(x) = \frac{(12x^2 - 8x + 3k)e^{3x}}{(4x^2 + k)^2}$	A1	2.1
		(3)	
(b)	If $y = f(x)$ has at least one stationary point then		
	$12x^2 - 8x + 3k = 0$ has at least one root	B1	2.2a
	Applies $b^2 - 4ac (\geq) 0$ with $a = 12, b = -8, c = 3k$	M1	2.1
	$0 < k \leq \frac{4}{9}$	A1	1.1b
		(3)	
(6 marks)			
Notes:			

(a)

M1: Attempts the quotient rule to obtain an expression of the form $\frac{\alpha(4x^2 + k)e^{3x} - \beta xe^{3x}}{(4x^2 + k)^2}$, ~~$\alpha \neq 0$~~

condoning bracketing errors/omissions as long as the intention is clear.

If the quotient rule formula is quoted it must be correct.

Condone e.g. $f'(x) = \frac{(4x^2 + k)3e^{3x} - 8xe^{3x}}{(4x^2 + k)}$ provided an incorrect formula is not quoted.

May also see product rule applied to $e^{3x}(4x^2 + k)^{-1}$ to obtain an expression of the form

$\alpha e^{3x}(4x^2 + k)^{-1} + \beta xe^{3x}(4x^2 + k)^{-2}$ ~~$\alpha, \beta \neq 0$~~ condoning bracketing errors/omissions as

long as the intention is clear. If the product rule formula is quoted it must be correct.

A1: Correct differentiation in any form with correct bracketing which may be implied by subsequent work.

A1: Obtains $f'(x) = (12x^2 - 8x + 3k)g(x)$ where $g(x) = \frac{e^{3x}}{(4x^2 + k)^2}$ **or equivalent**

e.g. $g(x) = e^{3x}(4x^2 + k)^{-2}$

Allow recovery from “invisible” brackets earlier and apply isw here once a correct answer is seen.

Note that the complete form of the answer is not given so allow candidates to go from e.g.

$\frac{(4x^2 + k)3e^{3x} - 8xe^{3x}}{(4x^2 + k)^2}$ or $3e^{3x}(4x^2 + k)^{-1} - 8xe^{3x}(4x^2 + k)^{-2}$ to $\frac{(12x^2 - 8x + 3k)e^{3x}}{(4x^2 + k)^2}$ for the final mark.



The " $f'(x) =$ " must appear at some point but allow e.g. " $\frac{dy}{dx} =$ "

(b) Note that B0M1A1 is not possible in (b)

B1: Deduces that if $y = f(x)$ has at least one stationary point then $12x^2 - 8x + 3k = 0$ has at least one

root. There is no requirement to formally state $\frac{e^{3x}}{(4x^2 + k)^2} > 0$

This may be implied by an attempt at $b^2 - 4ac \geq 0$ or $b^2 - 4ac > 0$ condoning slips.

M1: Attempts $b^2 - 4ac \dots 0$ with $a = 12$, $b = -8$, $c = 3k$ where ... is e.g. "=", "<", ">", etc.

Alternatively attempts to complete the square and sets rhs ...0

E.g. $12x^2 - 8x + 3k = 0 \Rightarrow x^2 - \frac{2}{3}x + \frac{1}{4}k = 0 \Rightarrow \left(x - \frac{1}{3}\right)^2 = \frac{1}{9} - \frac{1}{4}k$ leading to $\frac{1}{9} - \frac{1}{4}k \geq 0$

A1: $0 < k \leq \frac{4}{9}$ but condone $k \leq \frac{4}{9}$ and condone $0 \leq k \leq \frac{4}{9}$

Must be in terms of k not x so do not allow e.g. $0 < x \leq \frac{4}{9}$ but condone $\left(0, \frac{4}{9}\right]$ or $\left[0, \frac{4}{9}\right]$



Question	Scheme	Marks	AOs
4(a)	$f'(x) = 2x + \frac{4x-4}{2x^2-4x+5}$	M1 A1	1.1b 1.1b
	$2x + \frac{4x-4}{2x^2-4x+5} = 0 \Rightarrow 2x(2x^2-4x+5) + 4x-4 = 0$	dM1	1.1b
	$2x^3 - 4x^2 + 7x - 2 = 0^*$	A1*	2.1
		(4)	
(b)	(i) $x_2 = \frac{1}{7}(2 + 4(0.3)^2 - 2(0.3)^3)$	M1	1.1b
	$x_2 = 0.3294$	A1	1.1b
	(ii) $x_4 = 0.3398$	A1	1.1b
		(3)	
(c)	$h(x) = 2x^3 - 4x^2 + 7x - 2$ $h(0.3415) = 0.00366... \quad h(0.3405) = -0.00130...$	M1	3.1a
	States: - there is a change of sign $f'(x)$ is continuous $\alpha = 0.341$ to 3dp	A1	2.4
		(2)	
(9 marks)			
Notes			

(a)

M1: Differentiates $\ln(2x^2 - 4x + 5)$ to obtain $\frac{g(x)}{2x^2 - 4x + 5}$ where $g(x)$ could be 1

A1: For $f'(x) = 2x + \frac{4x - 4}{2x^2 - 4x + 5}$

dM1: Sets their $f'(x) = ax + \frac{g(x)}{2x^2 - 4x + 5} = 0$ and uses "correct" algebra, condoning slips, to obtain a

cubic equation. E.g Look for $ax(2x^2 - 4x + 5) \pm g(x) = 0$ o.e. , condoning slips, followed by some attempt to simplify

A1*: Achieves $2x^3 - 4x^2 + 7x - 2 = 0$ with no errors. (The dM1 mark must have been awarded)

(b)(i)

M1: Attempts to use the iterative formula with $x_1 = 0.3$. If no method is shown award for $x_2 = \text{awrt } 0.33$

A1: $x_2 = \text{awrt } 0.3294$ Note that $\frac{1153}{3500}$ is correct

Condone an incorrect suffix if it is clear that a correct value has been found

(b)(ii)

A1: $x_4 = \text{awrt } 0.3398$ Condone an incorrect suffix if it is clear that a correct value has been found

(c)

M1: Attempts to substitute $x = 0.3415$ and $x = 0.3405$ into a suitable function and gets one value correct (rounded or truncated to 1 sf). It is allowable to use a tighter interval that contains the root 0.340762654

Examples of suitable functions are $2x^3 - 4x^2 + 7x - 2$, $x - \frac{1}{7}(4x^2 - 2x^3 + 2)$ and $f'(x)$ as this has been

found in part (a) with $f'(0.3405) = -0.00067\dots$, $f'(0.3415) = (+) 0.0018$

There must be sufficient evidence for the function, which would be for example, a statement such as $h(x) = 2x^3 - 4x^2 + 7x - 2$ or sight of embedded values that imply the function, not just a value or values

even if both are correct. Condone $h(x)$ being mislabelled as f

$h(0.3415) = 2 \times 0.3415^3 - 4 \times 0.3415^2 + 7 \times 0.3415 - 2$

A1: Requires

- both calculations correct (rounded or truncated to 1sf)
- a statement that there is a change in sign and that the function is continuous
- a minimal conclusion e.g. ✓, proven, $\alpha = 0.341$, root

Question	Scheme	Marks	AOs
9(a)	$f(x) = 4(x^2 - 2)e^{-2x}$		
	Differentiates to $e^{-2x} \times 8x + 4(x^2 - 2) \times -2e^{-2x}$	M1 A1	1.1b 1.1b
	$f'(x) = 8e^{-2x} \{x - (x^2 - 2)\} = 8(2 + x - x^2)e^{-2x}$ *	A1*	2.1
		(3)	
(b)	States roots of $f'(x) = 0$ $x = -1, 2$	B1	1.1b
	Substitutes one x value to find a y value	M1	1.1b
	Stationary points are $(-1, -4e^2)$ and $(2, 8e^{-4})$	A1	1.1b
		(3)	
(c)	(i) Range $[-8e^2, \infty)$ o.e. such as $g(x) \geq -8e^2$	B1ft	2.5
	(ii) For - Either attempting to find $2f(0) - 3 = 2 \times -8 - 3 = (-19)$ and identifying this as the lower bound - Or attempting to find $2 \times 8e^{-4} - 3$ and identifying this as the upper bound	M1	3.1a
	Range $[-19, 16e^{-4} - 3]$	A1	1.1b
		(3)	
(9 marks)			
Notes:			

(a)

M1: Attempts the product rule and uses $e^{-2x} \rightarrow ke^{-2x}$, $k \neq 0$

If candidate states $u = 4(x^2 - 2)$, $v = e^{-2x}$ with $u' = \dots$, $v' = \dots e^{-2x}$ it can be implied by their $vu' + uv'$

If they just write down an answer without working award for $f'(x) = pxe^{-2x} \pm q(x^2 - 2)e^{-2x}$

They may multiply out first $f(x) = 4x^2e^{-2x} - 8e^{-2x}$. Apply in the same way condoning slips

Alternatively attempts the quotient rule on $f(x) = \frac{u}{v} = \frac{4(x^2 - 2)}{e^{2x}}$ with $v' = ke^{2x}$ and $f'(x) = \frac{vu' - uv'}{v^2}$

A1: A correct $f'(x)$ which may be unsimplified.

Via the quotient rule you can award for $f'(x) = \frac{8xe^{2x} - 8(x^2 - 2)e^{2x}}{e^{4x}}$ o.e.

A1*: Proceeds correctly to given answer showing all necessary steps.

The $f'(x)$ or $\frac{dy}{dx}$ must be present at some point in the solution

This is a "show that" question and there must not be any errors. All bracketing must be correct.

Allow a candidate to move from the **simplified** unfactorised answer of $f'(x) = 8xe^{-2x} - 8(x^2 - 2)e^{-2x}$

to the given answer in one step.

Do not allow it from an **unsimplified** $f'(x) = 4 \times 2xe^{-2x} + 4(x^2 - 2) \times -2e^{-2x}$

Allow the expression / bracketed expression to be written in a different order.

So, for example, $8(x - x^2 + 2)e^{-2x}$ is OK

(b)

B1: States or implies $x = -1, 2$ (as the roots of $f'(x) = 0$)

M1: Substitutes one x value of their solution to $f'(x) = 0$ in $f(x)$ to find a y value.

Allow decimals here (3sf). FYI, to 3 sf, $-4e^2 = -29.6$ and $8e^{-4} = 0.147$

Some candidates just write down the x coordinates but then go on in part (c) to find the ranges using the y coordinates. Allow this mark to be scored from work in part (c)

A1: Obtains $(-1, -4e^2)$ and $(2, 8e^{-4})$ as the stationary points. This must be scored in (b). Remember to isw

after a correct answer. Allow these to be written separately. E.g. $x = -1, y = -4e^2$

Extra solutions, e.g. from $x = 0$ will be penalised on this mark.

(c)(i)

B1ft: For a correct range written using correct notation.

Follow through on $2 \times$ their minimum "y" value from part (b), providing it is negative.

Condone a decimal answer if this is consistent with their answer in (b) to 3sf or better.

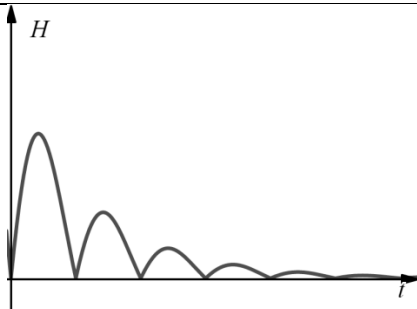
Examples of correct responses are $[-8e^2, \infty)$, $y \geq -8e^2$, $y \geq -8e^2$, $\{q \in \mathbb{R}, q \geq -8e^2\}$

(c)(ii)

M1: See main scheme. Follow through on $2 \times$ their " $8e^{-4}$ " $- 3$ for the upper bound.

A1: Range $[-19, 16e^{-4} - 3]$ o.e. such as $-19 \leq y \leq 16e^{-4} - 3$ but must be exact



Question	Scheme	Marks	AOs
12 (a)	$f(x) = 10e^{-0.25x} \sin x$		
	$\Rightarrow f'(x) = -2.5e^{-0.25x} \sin x + 10e^{-0.25x} \cos x$ oe	M1 A1	1.1b 1.1b
	$f'(x) = 0 \Rightarrow -2.5e^{-0.25x} \sin x + 10e^{-0.25x} \cos x = 0$	M1	2.1
	$\frac{\sin x}{\cos x} = \frac{10}{2.5} \Rightarrow \tan x = 4^*$	A1*	1.1b
		(4)	
(b)	 "Correct" shape for 2 loops Fully correct with decreasing heights	M1 A1	1.1b 1.1b
		(2)	
(c)	Solves $\tan x = 4$ and substitutes answer into $H(t)$	M1	3.1a
	$H(4.47) = 10e^{-0.25 \times 4.47} \sin 4.47 $	M1	1.1b
	awrt 3.18 (metres)	A1	3.2a
		(3)	
(d)	The times between each bounce should not stay the same when the heights of each bounce is getting smaller	B1	3.5b
		(1)	
(10 marks)			

(a)

M1: For attempting to differentiate using the product rule condoning slips, for example the power of e .

 So for example score expressions of the form $\pm \dots e^{-0.25x} \sin x \pm \dots e^{-0.25x} \cos x$ M1

 Sight of $vdu - u dv$ however is M0

A1: $f'(x) = -2.5e^{-0.25x} \sin x + 10e^{-0.25x} \cos x$ which may be unsimplified

M1: For clear reasoning in setting their $f'(x) = 0$, factorising/ cancelling out the $e^{-0.25x}$ term leading to a trigonometric equation in only $\sin x$ and $\cos x$

 Do not allow candidates to substitute $x = \arctan 4$ into $f'(x)$ to score this mark.

A1*: Shows the steps $\frac{\sin x}{\cos x} = \frac{10}{2.5}$ or equivalent leading to $\Rightarrow \tan x = 4^*$. $\frac{\sin x}{\cos x}$ must be seen.

(b)

M1: Draws at least two "loops". The height of the second loop should be lower than the first loop.

Condone the sight of rounding where there should be cusps

A1: At least 4 loops with decreasing heights and no rounding at the cusps.

The intention should be that the graph should 'sit' on the x-axis but be tolerant.

It is possible to overwrite Figure 3, but all loops must be clearly seen.

(c)

M1: Understands that to solve the problem they are required to substitute an answer to $\tan t = 4$ into $H(t)$ This can be awarded for an attempt to substitute $t = \text{awrt } 1.33$ or $t = \text{awrt } 4.47$ into $H(t)$ $H(t) = 6.96$ implies the use of $t = 1.33$ Condone for this mark only, an attempt to substitute $t = \text{awrt } 76^\circ$ or $\text{awrt } 256^\circ$ into $H(t)$ **M1:** Substitutes $t = \text{awrt } 4.47$ into $H(t) = 10e^{-0.25t} \sin t$. Implied by awrt 3.2**A1:** Awrt 3.18 metres. Condone the lack of units. If two values are given the correct one must be seen to have been chosen

It is possible for candidates to sketch this on their graphical calculators and gain this answer. If there is no incorrect working seen and 3.18 is given, then award 111 for such an attempt.

(d)

B1: Makes reference to the fact that the time between each bounce should not stay the same when the heights of each bounce is getting smaller.

Look for "time (or gap) between the bounces will change"

‘bounces would not be equal times apart’

‘bounces would become more frequent’

But do not accept ‘the times between each bounce would be longer or slower’

Do not accept explanations such as there are other factors that would affect this such as "wind resistance", friction etc

Question	Scheme	Marks	AOs	
11 (a) Way 1	$\{y = x^x \Rightarrow\} \ln y = x \ln x$	B1	1.1a	
	$\frac{1}{y} \frac{dy}{dx} = 1 + \ln x$	M1	1.1b	
		A1	2.1	
	$\left\{ \frac{dy}{dx} = 0 \Rightarrow \right\} \frac{x}{x} + \ln x = 0 \text{ or } 1 + \ln x = 0 \Rightarrow \ln x = k \Rightarrow x = \dots$	M1	1.1b	
	$x = e^{-1} \text{ or awrt } 0.368$	A1	1.1b	
	Note: $k \neq 0$	(5)		
(a) Way 2	$\{y = x^x \Rightarrow\} y = e^{x \ln x}$	B1	1.1a	
	$\frac{dy}{dx} = \left(\frac{x}{x} + \ln x \right) e^{x \ln x}$	M1	1.1b	
		A1	2.1	
	$\left\{ \frac{dy}{dx} = 0 \Rightarrow \right\} \frac{x}{x} + \ln x = 0 \text{ or } 1 + \ln x = 0 \Rightarrow \ln x = k \Rightarrow x = \dots$	M1	1.1b	
	$x = e^{-1} \text{ or awrt } 0.368$	A1	1.1b	
	Note: $k \neq 0$	(5)		
(b) Way 1	Attempts both $1.5^{1.5} = 1.8\dots$ and $1.6^{1.6} = 2.1\dots$ and at least one result is correct to awrt 1 dp	M1	1.1b	
	$1.8\dots < 2$ and $2.1\dots > 2$ and as C is continuous then $1.5 < \alpha < 1.6$	A1	2.1	
		(2)		
(c)	Attempts $x_{n+1} = 2x_n^{1-x_n}$ at least once with $x_1 = 1.5$	M1	1.1b	
	Can be implied by $2(1.5)^{1-1.5}$ or awrt 1.63			
	$\{x_4 = 1.67313\dots \Rightarrow\} x_4 = 1.673 \text{ (3 dp) cao}$	A1	1.1b	
		(2)		
(d)	Give 1 st B1 for any of - oscillates - periodic - non-convergent - divergent - fluctuates - goes up and down 1, 2, 1, 2, 1, 2 - alternates (condone)	Give B1 B1 for any of - periodic {sequence} with period 2 - oscillates between 1 and 2	B1	2.5
		Condone B1 B1 for any of - fluctuates between 1 and 2 - keep getting 1, 2 - alternates between 1 and 2 - goes up and down between 1 and 2 1, 2, 1, 2, 1, 2, ...	B1	2.5
			(2)	
(11 marks)				
Note	A common solution			
	A maximum of 3 marks (i.e. B1 1 st M1 and 2 nd M1) can be given for the solution			
	$\log y = x \log x \Rightarrow \frac{1}{y} \frac{dy}{dx} = 1 + \log x$			
	$\left\{ \frac{dy}{dx} = 0 \Rightarrow \right\} 1 + \log x = 0 \Rightarrow x = 10^{-1}$			
	1st B1 for $\log y = x \log x$ 1st M1 for $\log y \rightarrow \lambda \frac{1}{y} \frac{dy}{dx}; \lambda \neq 0 \text{ or } x \log x \rightarrow 1 + \log x \text{ or } \frac{x}{x} + \log x$ 2nd M1 can be given for $1 + \log x = 0 \Rightarrow \log x = k \Rightarrow x = \dots; k \neq 0$			

Question	Scheme	Marks	AOs
11 (b) Way 2	For $x^x - 2$, attempts both $1.5^{1.5} - 2 = -0.16...$ and $1.6^{1.6} - 2 = 0.12...$ and at least one result is correct to awrt 1 dp	M1	1.1b
	$-0.16... < 0$ and $0.12... > 0$ and as C is continuous then $1.5 < \alpha < 1.6$	A1	2.1
		(2)	
11 (b) Way 3	For $\ln y = x \ln x$, attempts both $1.5 \ln 1.5 = 0.608...$ and $1.6 \ln 1.6 = 0.752...$ and at least one result is correct to awrt 1 dp	M1	1.1b
	$0.608... < 0.69...$ and $0.752... > 0.69...$ and as C is continuous then $1.5 < \alpha < 1.6$	A1	2.1
		(2)	
11 (b) Way 4	For $\log y = x \log x$, attempts both $1.5 \log 1.5 = 0.264...$ and $1.6 \log 1.6 = 0.326...$ and at least one result is correct to awrt 2 dp	M1	1.1b
	$0.264... < 0.301...$ and $0.326... > 0.301...$ and as C is continuous then $1.5 < \alpha < 1.6$	A1	2.1
		(2)	
Notes for Question 11			
(a)	Way 1		
B1:	$\ln y = x \ln x$. Condone $\log_x y = x \log_x x$ or $\log_x y = x$		
M1:	For either $\ln y \rightarrow \frac{1}{y} \frac{dy}{dx}$ or $x \ln x \rightarrow 1 + \ln x$ or $\frac{x}{x} + \ln x$		
A1:	Correct differentiated equation. i.e. $\frac{1}{y} \frac{dy}{dx} = 1 + \ln x$ or $\frac{1}{y} \frac{dy}{dx} = \frac{x}{x} + \ln x$ or $\frac{dy}{dx} = y(1 + \ln x)$ or $\frac{dy}{dx} = x^x(1 + \ln x)$		
M1:	Sets $1 + \ln x = 0$ and rearranges to make $\ln x = k \Rightarrow x = ...$; k is a constant and $k \neq 0$		
A1:	$x = e^{-1}$ or awrt 0.368 only (with no other solutions for x)		
Note:	Give no marks for no working leading to 0.368		
Note:	Give M0 A0 M0 A0 for $\ln y = x \ln x \rightarrow x = 0.368$ with no intermediate working		
(a)	Way 2		
B1:	$y = e^{x \ln x}$		
M1:	For either $y = e^{x \ln x} \Rightarrow \frac{dy}{dx} = f(\ln x) e^{x \ln x}$ or $x \ln x \rightarrow 1 + \ln x$ or $\frac{x}{x} + \ln x$		
A1:	Correct differentiated equation. i.e. $\frac{dy}{dx} = \left(\frac{x}{x} + \ln x \right) e^{x \ln x}$ or $\frac{dy}{dx} = (1 + \ln x) e^{x \ln x}$ or $\frac{dy}{dx} = x^x(1 + \ln x)$		
M1:	Sets $1 + \ln x = 0$ and rearranges to make $\ln x = k \Rightarrow x = ...$; k is a constant and $k \neq 0$		
A1:	$x = e^{-1}$ or awrt 0.368 only (with no other solutions for x)		
Note:	Give B1 M1 A0 M1 A1 for the following solution: $\{y = x^x \Rightarrow \ln y = x \ln x \Rightarrow \frac{dy}{dx} = 1 + \ln x \Rightarrow 1 + \ln x = 0 \Rightarrow x = e^{-1} \text{ or awrt 0.368}$		

Notes for Question 11 Continued	
(b)	Way 1
M1:	Attempts both $1.5^{1.5} = 1.8...$ and $1.6^{1.6} = 2.1...$ and at least one result is correct to awrt 1 dp
A1:	Both $1.5^{1.5} = \text{awrt } 1.8...$ and $1.6^{1.6} = \text{awrt } 2.1...$, reason (e.g. $1.8... < 2$ and $2.1... > 2$ or states C cuts through $y = 2$), C continuous and conclusion
(b)	Way 2
M1:	Attempts both $1.5^{1.5} - 2 = -0.16...$ and $1.6^{1.6} - 2 = 0.12...$ and at least one result is correct to awrt 1 dp
A1:	Both $1.5^{1.5} - 2 = -0.16...$ and $1.6^{1.6} - 2 = 0.12...$ correct to awrt 1 dp, reason (e.g. $-0.16... < 0$ and $0.12... > 0$, sign change or states C cuts through $y = 0$), C continuous and conclusion
(b)	Way 3
M1:	Attempts both $1.5 \ln 1.5 = 0.608...$ and $1.6 \ln 1.6 = 0.752...$ and at least one result is correct to awrt 1 dp
A1:	Both $1.5 \ln 1.5 = 0.608...$ and $1.6 \ln 1.6 = 0.752...$ correct to awrt 1 dp, reason (e.g. $0.608... < 0.69...$ and $0.752... > 0.69...$ or states they are either side of $\ln 2$), C continuous and conclusion.
(b)	Way 4
M1:	Attempts both $1.5 \log 1.5 = 0.264...$ and $1.6 \log 1.6 = 0.326...$ and at least one result is correct to awrt 2 dp
A1:	Both $1.5 \log 1.5 = 0.264...$ and $1.6 \log 1.6 = 0.326...$ correct to awrt 2 dp, reason (e.g. $0.264... < 0.301...$ and $0.326... > 0.301...$ or states they are either side of $\log 2$), C continuous and conclusion.
(c)	
M1:	An attempt to use the given or their formula once. Can be implied by $2(1.5)^{1-1.5}$ or awrt 1.63
A1:	States $x_4 = 1.673$ cao (to 3 dp)
Note:	Give M1 A1 for stating $x_4 = 1.673$
Note:	M1 can be implied by stating their final answer $x_4 = \text{awrt } 1.673$
Note:	$x_2 = 1.63299...$, $x_3 = 1.46626...$, $x_4 = 1.67313...$
(d)	
B1:	see scheme
B1:	see scheme
Note:	Only marks of B1B0 or B1B1 are possible in (d)
Note:	Give B0 B0 for “Converges in a cob-web pattern” or “Converges up and down to α ”

Question	Scheme	Marks	AOs
15(a)	$\dots xe^x + \dots e^x$	M1	1.1b
	$k(xe^x + e^x)$	A1	1.1b
	$\frac{d}{dx}(\sqrt{e^{3x}-2}) = \frac{1}{2} \times 3e^{3x}(e^{3x}-2)^{-\frac{1}{2}}$	B1	1.1b
	$(f'(x) =) \frac{(e^{3x}-2)^{\frac{1}{2}}(7xe^x + 7e^x) - \frac{3}{2}e^{3x}(e^{3x}-2)^{-\frac{1}{2}} \times 7xe^x}{e^{3x}-2}$	dM1	2.1
	$f'(x) = \frac{7e^x(e^{3x}(2-x) - 4x - 4)}{2(e^{3x}-2)^{\frac{3}{2}}}$	A1	1.1b
		(5)	
(b)	$e^{3x}(2-x) - 4x - 4 = 0 \Rightarrow x(\dots e^{3x} \pm \dots) = \dots e^{3x} \pm \dots$	M1	1.1b
	$\Rightarrow x = \frac{2e^{3x}-4}{e^{3x}+4} *$	A1*	2.1
		(2)	
(c)	Draws a vertical line $x=1$ up to the curve then across to the line $y=x$ then up to the curve finishing at the root (need to see a minimum of 2 vertical and horizontal lines tending to the root)	B1	2.1
		(1)	
(d)(i)	$x_2 = \frac{2e^3-4}{e^3+4} = 1.5017756\dots$	M1	1.1b
	$x_2 = \text{awrt } 1.502$	A1	1.1b
(ii)	$\beta = 1.968$	dB1	2.2b
		(3)	
(e)	$h(x) = \frac{2e^{3x}-4}{e^{3x}+4} - x$ $h(0.4315) = -0.000297\dots \quad h(0.4325) = 0.000947\dots$	M1	3.1a
	Both calculations correct and e.g. states: - There is a change of sign - e.g $f'(x)$ is continuous $\alpha = 0.432$ (to 3dp)	A1cao	2.4
		(2)	
(13 marks)			
Notes			
(a)			
M1:	Attempts the product rule on xe^x (or may be $7xe^x$) achieving an expression of the form $\dots xe^x \pm \dots e^x$. If it is clear that the quotient rule has been applied instead which may be quoted then M0.		
A1:	$k(xe^x + e^x)$ (e.g. $7(xe^x + e^x)$) or equivalent which may be unsimplified (may be implied by further work)		
B1:	$\left(\frac{d}{dx}(\sqrt{e^{3x}-2})\right) = \frac{1}{2} \times 3e^{3x}(e^{3x}-2)^{-\frac{1}{2}}$ (simplified or unsimplified)		

dM1: Attempts to use the quotient rule. It is dependent on the previous method mark.

Score for achieving an expression of the form

$$(f'(x) =) \frac{(e^{3x} - 2)^{\frac{1}{2}} ("7"xe^x + "7"e^x) - \frac{3}{2}e^{3x}(e^{3x} - 2)^{-\frac{1}{2}} \times "7"xe^x}{e^{3x} - 2} \text{ or equivalent (do not be}$$

concerned by the constants for their “7” or their “ $\frac{3}{2}$ ” which may be both 1)

If it is clear that the quotient rule has been applied the wrong way round then score M0.

Alternatively, applies the product rule. Score for achieving an expression of the form

$$(f'(x) =) (e^{3x} - 2)^{\frac{1}{2}} ("7"xe^x + "7"e^x) - \frac{3}{2}e^{3x}(e^{3x} - 2)^{\frac{3}{2}} \times "7"xe^x \text{ or equivalent (do not be}$$

concerned by the constants for their “7” or their “ $\frac{3}{2}$ ” which may be both 1)

Do not condone invisible brackets.

A1: $(f'(x) =) \frac{7e^x(e^{3x}(2-x) - 4x - 4)}{2(e^{3x} - 2)^{\frac{3}{2}}}$ following a fully correct differentiated expression.

You may need to check to see if (a) is continued after other parts for evidence of this.

Condone the lack of $f'(x) =$ on the left hand side or allow the use of $\frac{dy}{dx}$ or y' instead.

Alternative (a) attempt using the triple product rule

$$\begin{aligned} \text{e.g. } \frac{d}{dx} \left(7xe^x(e^{3x} - 2)^{-\frac{1}{2}} \right) &= 7e^x(e^{3x} - 2)^{-\frac{1}{2}} + 7xe^x(e^{3x} - 2)^{-\frac{1}{2}} + 7xe^x \times \left(-\frac{1}{2} \right) \times 3e^{3x}(e^{3x} - 2)^{-\frac{3}{2}} \\ &\Rightarrow \frac{(7e^x + 7xe^x)(e^{3x} - 2) + 7xe^x \times \left(-\frac{1}{2} \right) \times 3e^{3x}}{(e^{3x} - 2)^{\frac{3}{2}}} = \frac{7e^x \left(e^{3x} - 2 + xe^{3x} - 2x - \frac{3}{2}xe^{3x} \right)}{(e^{3x} - 2)^{\frac{3}{2}}} \Rightarrow \frac{7e^x(e^{3x}(2-x) - 4x - 4)}{2(e^{3x} - 2)^{\frac{3}{2}}} \end{aligned}$$

M1: Attempts the product rule on $xe^x \rightarrow \dots xe^x \pm \dots e^x$ which may be seen within the expression $\dots e^x(e^{3x} - 2)^{-\frac{1}{2}} \pm \dots xe^x(e^{3x} - 2)^{-\frac{1}{2}} + \dots$ simplified or unsimplified.

A1: $k(xe^x + e^x)$ which may be seen within the expression $k \left(e^x(e^{3x} - 2)^{-\frac{1}{2}} + xe^x(e^{3x} - 2)^{-\frac{1}{2}} \right) + \dots$ simplified or unsimplified.

B1: $\left(-\frac{1}{2} \right) \times 3e^{3x}(e^{3x} - 2)^{-\frac{3}{2}}$ which may be seen within the expression $\dots + k \left(xe^x \times \left(-\frac{1}{2} \right) \times 3e^{3x}(e^{3x} - 2)^{-\frac{3}{2}} \right)$ simplified or unsimplified.

dM1: A complete method using all three products (which may appear all on one line). Do not condone invisible brackets.

A1: As above in main scheme notes.

(b) Note that if they do not have values $A = -4$, $B = -4$ in (a) (which may be seen later) then maximum score is M1A0*

M1: Sets their $e^{3x}(2-x) - 4x - 4$ equal to zero, collects terms in x on one side of the equation and non x terms on the other and attempts to factorise the side with x as a common factor. Condone sign slips only for this mark. Allow A and B to be used instead of “-4” and “-4”

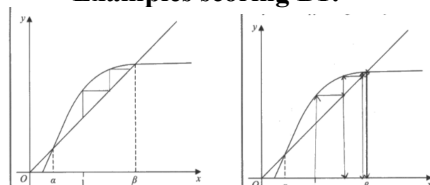


A1*: Achieves the given answer with no errors including invisible brackets. If they do not reach the printed answer then it is A0. If they subsequently write $x_{n+1} = \frac{2e^{3x_n} - 4}{e^{3x_n} + 4}$ then isw

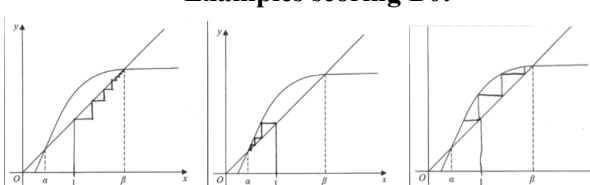
(c)

B1: Starting at $x_1 = 1$ look for at least 2 sets of vertical and horizontal lines drawn (may be dashes) tending to β . Condone a lack of arrows on the lines but the sequence of lines should finish at the point of intersection where the root is. Condone the initial vertical line not starting from the x -axis. Mark the intention to draw horizontal and vertical lines. If they have any lines to the left of $x = 1$ this is B0. If they use both diagrams and do not indicate which one they want marking, then the “copy of Diagram 1” should be marked.

Examples scoring B1:



Examples scoring B0:



(d)(i)

M1: Substitutes 1 into the iterative formula. The values embedded in the formula is sufficient for this mark. May be implied by awrt 1.50

A1: awrt 1.502 isw

(d)(ii)

dB1: 1.968 cao (which can only be scored if M1 is scored in (d)(i))

SC: If (d)(i) is rounded to 1.50 then allow 1.97 in (d)(ii) to score M1A0dB1 for (d)

(e)

M1: Attempts to substitute $x = 0.4315$ and 0.4325 into a suitable function and gets one value correct (rounded or truncated to 1sf). It is allowable to use a tighter interval that contains the root $0.4317388728...$

If no function is stated then may be implied by their answers to e.g. $f'(0.4315)$, $f'(0.4325)$

You will need to check their calculation is correct.

Other possible functions include:

- $h(x) = x - \frac{2e^{3x} - 4}{e^{3x} + 4}$ (other way round to MS) $h(0.4315) = 0.0002974...$, $h(0.4325) = -0.0009479...$

- their $f'(x) = \pm \left(\frac{7e^x (e^{3x} (2-x) - 4x - 4)}{2(e^{3x} - 2)^{\frac{3}{2}}} \right)$

(If correct A and B then $f'(0.4315) = \mp 0.005789...$, $f'(0.4325) = \pm 0.01831...$)

- their $g(x) = \pm (e^{3x} (2-x) - 4x - 4)$

(If correct A and B then $g(0.4315) = \mp 0.002275...$, $g(0.4325) = \pm 0.007261...$)

A1: Requires

- Both calculations correct (rounded or truncated to 1sf)
- A statement that there is a change in sign and that their **function** is continuous (must refer to the function used for the substitution (which is not $f(x)$))
Accept equivalent statements for $f'(0.4315) < 0$, $f'(0.4325) > 0$ e.g.
 $f'(0.4315) \times f'(0.4325) < 0$, “one negative one positive”. A minimum is “change of sign and continuous” but do **not** allow this mark if the comment about continuity is clearly incorrect e.g. “because x is continuous” or “because the interval is continuous”
- A minimal conclusion e.g. “hence $\alpha = 0.432$ ”, “so rounds to 0.432”. Do not allow “hence root”

Question	Scheme	Marks	AOs
15(a)	$(f'(x) =) \frac{\lambda x(1+x^2)^2 - \mu x(1-x^2)(1+x^2)}{(1+x^2)^4}$	M1	1.1b
	$= \frac{-2x(1+x^2)^2 - 4x(1-x^2)(1+x^2)}{(1+x^2)^4}$	A1	1.1b
	e.g. $\frac{-2x(1+x^2) - 4x(1-x^2)}{(1+x^2)^3} = 0$ $\Rightarrow (1+x^2) + 2(1-x^2) = 0$ $\Rightarrow 3 - x^2 = 0$ or e.g. $= \frac{-2x(1+x^2)[(1+x^2) + 2(1-x^2)]}{(1+x^2)^4}$ $= \frac{-2x(1+x^2)(3-x^2)}{(1+x^2)^4}$	dM1	3.1a
	e.g. $3 - x^2 = 0$ or $2x(1+x^2)(x^2 - 3) = 0$ $\Rightarrow x = \dots \Rightarrow y = \dots$	ddM1	2.1
	$\left(-\sqrt{3}, -\frac{1}{8}\right)$	A1	2.3
		(5)	
Notes			
(a) There may be other valid ways to differentiate or to solve the resulting equation. M1: Attempts the quotient rule to achieve $\frac{\pm \lambda x(1+x^2)^2 \pm \mu x(1-x^2)(1+x^2)}{(1+x^2)^4}$ but condone $\frac{\pm \lambda x(1+x^2)^2 \pm \mu x(1-x^2)(1+x^2)}{(1+x^2)^2}$ provided an incorrect quotient rule is not seen. Alternatively, attempts the product rule on $f(x) = (1-x^2)(1+x^2)^{-2}$ to achieve $\pm \lambda x(1+x^2)^{-2} \pm \mu x(1-x^2)(1+x^2)^{-3}$ There may be attempts at splitting the numerator to $f(x) = (1+x^2)^{-2} - x^2(1+x^2)^{-2}$ which should differentiate to $\pm \lambda x(1+x^2)^{-3} \pm \mu x(1+x^2)^{-2} \pm \gamma x^3(1+x^2)^{-3}$ In all cases, do not penalise signs of λ, μ and/or γ unless an incorrect quotient rule or an incorrect product rule is stated. Any occurrences of $(1+x^2)^2$ may be replaced with $1+2x^2+x^4$ There is no need for a LHS e.g. $f'(x) =$ to be present so ignore an incorrect LHS. Invisible brackets may be recovered/implied by later work. A1: Correct differentiation. May be unsimplified. Ignore absence of (or incorrect) LHS. The correct derivative using the quotient rule is $= \frac{-2x(1+x^2)^2 - 4x(1-x^2)(1+x^2)}{(1+x^2)^4}$ o.e. using the product rule is $-2x(1+x^2)^{-2} - 4x(1-x^2)(1+x^2)^{-3}$ o.e. and splitting the numerator is $-4x(1+x^2)^{-3} - 2x(1+x^2)^{-2} + 4x^3(1+x^2)^{-3}$ o.e.			

You may need to check carefully for equivalent derivatives and ISW after a correct expression is seen. Note that some candidates may set $= 0$ at the start and may omit the denominator as a result – send to review if you are unsure in such cases.

dM1: Attempts to reduce the expression to a suitable form so that the roots can be found, either by cancelling a factor of $(1+x^2)$ and simplifying the other brackets

$$\text{e.g. } \frac{-2x(1+x^2)-4x(1-x^2)}{(1+x^2)^3} \rightarrow \text{e.g. } \frac{-6x+2x^3}{(1+x^2)^3} \text{ or } \frac{2x(x^2-3)}{(1+x^2)^3} \text{ or } 2x(x^2-3) (=0)$$

or by attempting to take x , $(1+x^2)$ or $\pm 2x(1+x^2)$ out as a factor and simplify the other brackets. If they only take out a factor of x then they must have $x(\pm Ax^4 \pm Bx^2 \pm C)$

$$\text{e.g. } \frac{-2x(1+x^2)[(1+x^2)+2(1-x^2)]}{(1+x^2)^4} \rightarrow \text{e.g. } \frac{-2x(1+x^2)(3-x^2)}{(1+x^2)^4} \text{ or } \frac{x(2x^4-4x^2-6)}{(1+x^2)^4}$$

Depends on the first M mark.

Note: they might attempt to expand the brackets first, or set $= 0$ and multiply through by $(1+x^2)^k$ and/or divide through by x or $2x$ at any stage.

Do not be concerned if any factors of $(1+x^2)$, x or 2 go “missing” during their work.

The $= 0$ does not explicitly need to be seen.

$$\text{Look for } \pm \frac{2x(1+x^2)(Ax^2+B)}{(1+x^2)^k} \text{ or } \pm \frac{(1+x^2)(Ax^3+Bx)}{(1+x^2)^k} \text{ or } \pm \frac{x(Ax^4+Bx^2+C)}{(1+x^2)^k} \text{ where}$$

- k may be 0, 1, 2, 3 or 4
- A and B (and C if present) are non-zero
- any of $2x$ or $(1+x^2)$ in the numerator and/or $(1+x^2)^k$ in the denominator may be absent

ddM1: Attempts to solve their numerator set $= 0$ using a valid non-calculator method **and** uses a solution for x to find a corresponding value for y .

The substitution may be implied by their value of y but, if not, the substitution must be seen.

- For either Ax^2+B or Ax^3+Bx , we require $A \neq B$ and $A \times B < 0$ From either expression

they can write down their $x = \pm \sqrt{\frac{-B}{A}}$ without working.

- For Ax^4+Bx^2+C they must show a valid non-calculator method for solving the quartic, treating it as a quadratic in x^2 and using the usual rules for solving a quadratic algebraically, e.g. $a = x^2 \rightarrow 2a^2 - 4a - 6 (=0) \rightarrow (a+1)(2a-6) (=0) \rightarrow x = \pm\sqrt{3}$. They must reach a value for x (and not just x^2) as well as finding a value for y .

Dependent on both previous method marks. They must use a value for x that is not -1 , 0 or 1 .

A1: Deduces the correct **exact** coordinates for $P \left(-\sqrt{3}, -\frac{1}{8} \right)$ o.e. $(-\sqrt{3}, -0.125)$

Requires all the previous marks to have been scored.

Condone $x = -\sqrt{3}$, $y = -0.125$ or e.g. $-\sqrt{3}$, $-\frac{1}{8}$

If there is more than one pair of coordinates given, then the correct coordinates must be clearly selected or any others clearly rejected.

Ignore any mistakes that occur if they multiply out the denominator $(1+x^2)^4$ after differentiation.



Question	Scheme	Marks	AOs
15(b)	$x = -1 \Rightarrow \alpha = -\frac{\pi}{4}$ and $x = 1 \Rightarrow \beta = \frac{\pi}{4}$	B1	2.2a
	$\frac{dx}{d\theta} = \sec^2 \theta$	B1	1.1b
	$\int \frac{1-x^2}{(1+x^2)^2} dx = \int \frac{1-\tan^2 \theta}{(1+\tan^2 \theta)^2} " \sec^2 \theta " d\theta$ $\text{or} = \int \frac{1-\tan^2 \theta}{\sec^4 \theta} " \sec^2 \theta " d\theta$	M1	1.1b
	$= \int (1 - \tan^2 \theta) \cos^2 \theta d\theta$ $= \int \left(1 - \frac{\sin^2 \theta}{\cos^2 \theta} \right) \cos^2 \theta d\theta$ $= \int (\cos^2 \theta - \sin^2 \theta) d\theta$ $= \int \frac{2 - 1 - \tan^2 \theta}{\sec^2 \theta} d\theta$ $= \int \left(\frac{2}{\sec^2 \theta} - \frac{1 + \tan^2 \theta}{\sec^2 \theta} \right) d\theta$ $= \int (2 \cos^2 \theta - 1) d\theta$	dM1	3.1a
	$= \int \cos 2\theta d\theta *$	A1*	2.1
		(5)	

Notes

(b) Mark parts (b) and (c) together.

B1: Deduces the correct limits for the integral in θ which may be seen separately as side working or within their integral work. This mark cannot be scored working in degrees.

Allow $\frac{3\pi}{4}$ instead of $-\frac{\pi}{4}$ but not decimal approximations.

B1: $\frac{dx}{d\theta} = \sec^2 \theta$ or equivalent e.g. $\frac{dx}{d\theta} = 1 + x^2$ or $\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta}$ (coming from $x = \frac{\sin \theta}{\cos \theta}$) but must be correct, so not e.g. $\frac{dx}{d\theta} = \sec^2 x$ unless recovered.

M1: Makes a complete attempt at using the substitution $x = \tan \theta$

Requires:

- An attempt at using $\frac{dx}{d\theta}$ to replace dx with $d\theta$ either way round, so if $\frac{dx}{d\theta} = g(\theta)$ allow either $dx = g(\theta)\{d\theta\}$ **or** $dx = \frac{1}{g(\theta)}\{d\theta\}$. $\frac{dx}{d\theta}$ must be a function of θ and not a constant.
 - All terms in x replaced with $\tan \theta$ (or $1 + x^2$ with $\sec^2 \theta$)
- Condone the absence of $d\theta$ but dx must no longer be present.
 Condone if they fail to square the denominator or e.g. a slip in missing a θ
 Use of notation such as $d(\tan \theta)$ is correct but does not score the M1 until replaced with $\sec^2 \theta \{d\theta\}$ or their derivative of $\tan \theta$ (which must not be a constant multiple of $\tan \theta$).

dM1: Uses trigonometric identities e.g. $\pm 1 \pm \tan^2 \theta = \pm \sec^2 \theta$, $\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\sec \theta = \frac{1}{\cos \theta}$,

$\pm \cos^2 \theta \pm \sin^2 \theta = \pm 1$ to simplify the integral to an expression of the form
 $\pm a \sin^2 \theta \pm b \cos^2 \theta \pm c$ where one of a , b or c may be 0.

The algebra should essentially be correct but condone e.g. sign slips or errors collecting terms.



Alternatively, e.g. $\int \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} d\theta = \int \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta} d\theta = \int \frac{\cos^2 \theta - \sin^2 \theta}{1} d\theta$

They must have replaced dx with $d\theta$ the correct way round for this mark, so if $\frac{dx}{d\theta} = g(\theta)$

then they must have $dx = g(\theta)\{d\theta\}$ and **not** $dx = \frac{1}{g(\theta)}\{d\theta\}$

It is acceptable to replace e.g. $\tan^2 \theta \cos^2 \theta$ with $\sin^2 \theta$

Dependent on the previous method mark.

A1*: cso Arrives at $\int \cos 2\theta d\theta$ with sufficient working shown and no incorrect work ignoring limits. The final line must be fully correct, including the integral sign and $d\theta$ (ignoring limits). Note that this may be seen in part (c) and may score the mark.

All trigonometric identities must be fully correct.

Condone one or two slips in a missing θ but not frequent omissions.

Condone missing integral signs in their intermediate work, but it must be present on the final line.

dx must be replaced with $d\theta$ at some stage before the final line but does not need to be present in every intermediate line of working.

Condone notational errors throughout e.g. $\sin \theta^2$ provided they are recovered.

This mark is independent from any work to do with limits, i.e., B0B1M1dM1A1* is possible.

Question	Scheme	Marks	AOs
15(c)	$\int \cos 2\theta \, d\theta = \frac{1}{2} \sin 2\theta$	M1	1.1b
	$\left[\frac{1}{2} \sin 2\theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{2} \sin \left(2 \times \frac{\pi}{4} \right) - \frac{1}{2} \sin \left(2 \times -\frac{\pi}{4} \right)$	dM1	1.1b
	$= 1$	A1	2.1
		(3)	
(13 marks)			
Notes			
(c)	Mark parts (b) and (c) together.		
M1:	$\int \cos 2\theta \, d\theta \rightarrow \pm \frac{1}{2} \sin 2\theta \{+c\}$		
dM1:	Substitutes their changed limits (not 1 and -1) into $\pm \frac{1}{2} \sin 2\theta$ and subtracts either way round. May be implied e.g. $\frac{1}{2} + \frac{1}{2}$ provided they have the correct limits. Dependent on the previous method mark.		
A1:	Area of 1 found following correct work. This mark requires clear substitution of the correct limits into $\frac{1}{2} \sin 2\theta$ the correct way round. If they have the limits the wrong way round and achieve an answer of -1 they cannot just make the answer positive for this mark. They may use limits from 0 to $\frac{\pi}{4}$ and multiply their result by 2. $\frac{3\pi}{4}$ may be used in place of $-\frac{\pi}{4}$ as the lower limit which is acceptable. Condone any spurious integral symbol accompanying $\frac{1}{2} \sin 2\theta$ The substitution may be implied by e.g. $\frac{1}{2} - -\frac{1}{2}$ or $\frac{1}{2} \sin \left(\frac{\pi}{2} \right) - \frac{1}{2} \sin \left(-\frac{\pi}{2} \right)$ but must follow sight of $\frac{1}{2} \sin 2\theta$ Condone use of limits -45 and 45 (degrees) the correct way round for full marks in part (c). Note that use of a calculator on the original integral will give the correct answer. An answer of 1 scores no marks without evidence of scoring both method marks above including the substitution.		

Question	Scheme	Marks	AOs
10(a)	$x = 4, y = 2 \Rightarrow t = -1$	B1	2.2a
	$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -3t^2 \times \frac{1}{2(t+3)}$	M1	1.1b
	$\frac{dy}{dx} = -3(-1)^2 \times \frac{1}{2(-1+3)} = -\frac{3}{4}$	M1	1.1b
	$\Rightarrow y - 2 = -\frac{3}{4}(x - 4)$ or $\Rightarrow y = -\frac{3}{4}x + c \rightarrow 2 = -\frac{3}{4} \times 4 + c \Rightarrow c = \dots$	ddM1	2.1
	$y - 2 = -\frac{3}{4}(x - 4) \Rightarrow 4y - 8 = -3x + 12$ or $c = 5 \Rightarrow y = -\frac{3}{4}x + 5$ $\Rightarrow 3x + 4y = 20^*$	A1*	1.1b
		(5)	
(b)	Maximum height is 9m	B1	3.4
		(1)	
(6 marks)			
Notes			
(a) If parametric differentiation is not used in part (a) (e.g. uses Cartesian form) then only the B mark is available but see alternative below.			
B1: Uses the given Cartesian coordinates to deduce the correct value for t . If more than one value for t e.g. $t = -5$ is given and $t = -1$ is not “selected” score B0 but if just $t = -1$ is used subsequently allow recovery and score B1			
M1: Attempts to use $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ or equivalent with their differentiated equations. There must be an attempt to differentiate both parameters, however poor, and divide or multiply correctly so using $\frac{dy}{dx} = \frac{y}{x}$ scores M0. Both parameters must be “changed”. Condone confusion with the variables e.g. referring to $\frac{dy}{dt}$ as $\frac{dy}{dx}$ if the intention is clear. This may be implied by e.g. $\frac{dy}{dt} = -3t^2$, $\frac{dx}{dt} = 2(t+3)$, $t = -1$, $\frac{dy}{dt} = -3$, $\frac{dx}{dt} = 4 \Rightarrow \frac{dy}{dx} = -\frac{3}{4}$			
M1: Uses their numerical value of t (not 4) in their $\frac{dy}{dx}$ to obtain a value. Condone attempts with different values of t e.g. $t = -1$ and $t = -5$			
ddM1: Applies a correct straight line method with their value of $\frac{dy}{dx}$ which has come from an attempt to use parametric differentiation with their value of t (not 4) and with $x = 4$ and $y = 2$ correctly placed. An attempt at the equation of the normal is M0. If using $y = mx + c$ they must reach as far as $c = \dots$ Depends on both previous M marks.			
A1*: Correct equation as printed with no errors but condone $4y + 3x = 20^*$ Allow equivalents e.g. $20 = 4y + 3x^*$ or $3x + 4y = 20^*$ This is a printed answer so there must be at least one intermediate step as shown in the main scheme.			

Alternative for (a) using parametric differentiation but avoids the need for a value for t :

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} = -3t^2 \times \frac{1}{2(t+3)} \\ -3t^2 \times \frac{1}{2(t+3)} &= -3(1-y)^{\frac{2}{3}} \times \frac{1}{2\sqrt{x}} = -3(1-2)^{\frac{2}{3}} \times \frac{1}{2\sqrt{4}} = -\frac{3}{4} \\ \text{or} \\ -3t^2 \times \frac{1}{2(t+3)} &= -3(\sqrt{x}-3)^2 \times \frac{1}{2\sqrt{x}} = -3(2-3)^2 \times \frac{1}{2\sqrt{4}} = -\frac{3}{4} \\ \text{or} \\ -3t^2 \times \frac{1}{2(t+3)} &= -3(1-y)^{\frac{2}{3}} \times \frac{1}{2\left((1-y)^{\frac{1}{3}}+3\right)} = -3(1-2)^2 \times \frac{1}{2 \times 2} = -\frac{3}{4} \\ \Rightarrow y-2 &= -\frac{3}{4}(x-4) \Rightarrow 3x+4y=20^*\end{aligned}$$

B1: **Either** a correct expression for $\frac{dy}{dx}$ in terms of x and/or y following a correct $\frac{dy}{dx}$ in terms of t **or** for $t = -1$ seen anywhere.

M1: Attempts to use $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ with their differentiated equations.

There must be an attempt to differentiate both parameters, however poor, and divide or multiply correctly so using $\frac{dy}{dx} = \frac{y}{x}$ scores M0

Condone confusion with the variables e.g. referring to $\frac{dy}{dt}$ as $\frac{dy}{dx}$ if the intention is clear.

This may be implied by e.g. $\frac{dy}{dt} = -3t^2$, $\frac{dx}{dt} = 2(t+3)$, $t = -1$, $\frac{dy}{dt} = -3$, $\frac{dx}{dt} = 4 \Rightarrow \frac{dy}{dx} = -\frac{3}{4}$

M1: Attempts to express their $\frac{dy}{dx}$ which is in terms of t , in terms of x and/or y **and** uses $x = 4$ and $y = 2$ correctly placed in an attempt to find the gradient of the tangent.

ddM1: Applies a correct straight line method with their value of $\frac{dy}{dx}$ which has come from an attempt to use parametric differentiation with their gradient and with $x = 4$ and $y = 2$ correctly placed.

If using $y = mx + c$ they must reach as far as $c = \dots$

Depends on both previous M marks.

A1*: Correct equation as printed with no errors.

This is a printed answer so there must be at least one intermediate step as shown in the main scheme.

(b)

B1: 9m or equivalent **including correct units**. Accept e.g. 9 metres, 900cm etc.



Question	Scheme	Marks	AOs
13(a)	$y = \operatorname{cosec}^3 \theta \Rightarrow \frac{dy}{d\theta} = -3 \operatorname{cosec}^2 \theta \operatorname{cosec} \theta \cot \theta$	B1	1.1b
	$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$	M1	1.1b
	$\frac{dy}{dx} = \frac{-3 \operatorname{cosec}^3 \theta \cot \theta}{2 \cos 2\theta}$	A1	1.1b
		(3)	
(b)	$y = 8 \Rightarrow \operatorname{cosec}^3 \theta = 8 \Rightarrow \sin^3 \theta = \frac{1}{8} \Rightarrow \sin \theta = \frac{1}{2}$	M1	3.1a
	$\theta = \frac{\pi}{6} \Rightarrow \frac{dy}{dx} = \frac{-3 \operatorname{cosec}^3 \left(\frac{\pi}{6} \right) \cot \left(\frac{\pi}{6} \right)}{2 \cos \left(\frac{2\pi}{6} \right)} = \dots$ or $\sin \theta = \frac{1}{2} \Rightarrow \frac{dy}{dx} = \frac{\frac{-3}{\sin^3 \theta} \times \frac{\cos \theta}{\sin \theta}}{2(1 - 2 \sin^2 \theta)} = \frac{-3 \times 8 \times \frac{\sqrt{3}/2}{1/2}}{2 \left(1 - 2 \times \frac{1}{4} \right)}$	M1	2.1
	$= -24\sqrt{3}$	A1	2.2a
		(3)	
(6 marks)			
Notes			
(a) B1: Correct expression for $\frac{dy}{d\theta}$ seen or implied in any form e.g. $\frac{-3 \cos \theta}{\sin^4 \theta}$ M1: Obtains $\frac{dx}{d\theta} = k \cos 2\theta$ or $\alpha \cos^2 \theta + \beta \sin^2 \theta$ (from product rule on $\sin \theta \cos \theta$) and attempts $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$ A1: Correct expression in any form. May see e.g. $\frac{-3 \cos \theta}{2 \sin^4 \theta \cos 2\theta}, -\frac{3}{4 \sin^4 \theta \cos \theta - 2 \sin^3 \theta \tan \theta}$ (b) M1: Recognises the need to find the value of $\sin \theta$ or θ when $y = 8$ and uses the y parameter to establish its value. This should be correct work leading to $\sin \theta = \frac{1}{2}$ or e.g. $\theta = \frac{\pi}{6}$ or 30°. M1: Uses their value of $\sin \theta$ or θ in their $\frac{dy}{dx}$ from part (a) (working in exact form) in an attempt to obtain an exact value for $\frac{dy}{dx}$. May be implied by a correct exact answer. If no working is shown but an exact answer is given you may need to check that this follows their $\frac{dy}{dx}$. A1: Deduces the correct gradient			

Question	Scheme	Marks	AOs
15 (a)	$x^2 \tan y = 9 \Rightarrow 2x \tan y + x^2 \sec^2 y \frac{dy}{dx} = 0$	M1 A1	3.1a 1.1b
	Full method to get $\frac{dy}{dx}$ in terms of x using $\sec^2 y = 1 + \tan^2 y = 1 + f(x)$	M1	1.1b
	$\frac{dy}{dx} = \frac{-2x \times \frac{9}{x^2}}{x^2 \left(1 + \frac{81}{x^4}\right)} = \frac{-18x}{x^4 + 81} \quad *$	A1*	2.1
		(4)	
(b)	$\frac{dy}{dx} = \frac{-18x}{x^4 + 81}$ $\Rightarrow \frac{d^2y}{dx^2} = \frac{-18 \times (x^4 + 81) - (-18x)(4x^3)}{(x^4 + 81)^2} = \frac{54(x^4 - 27)}{(x^4 + 81)^2} \text{ o.e.}$	M1 A1	1.1b 1.1b
	States that when $x < \sqrt[4]{27} \Rightarrow \frac{d^2y}{dx^2} < 0$ when $x = \sqrt[4]{27} \Rightarrow \frac{d^2y}{dx^2} = 0$ AND when $x > \sqrt[4]{27} \Rightarrow \frac{d^2y}{dx^2} > 0$ giving a point of inflection when $x = \sqrt[4]{27}$	A1	2.4
		(3)	
(7 marks)			
Notes:			

(a)

M1: Attempts to differentiate $\tan y$ implicitly. Eg. $\tan y \rightarrow \sec^2 y \frac{dy}{dx}$ or $\cot y \rightarrow -\operatorname{cosec}^2 y \frac{dy}{dx}$

You may well see an attempt $\tan y = \frac{9}{x^2} \Rightarrow \sec^2 y \frac{dy}{dx} = \dots$

When a candidate writes $x^2 \tan y = 9 \Rightarrow x = 3 \tan^{\frac{1}{2}} y$ the mark is scored for $\tan^{\frac{1}{2}} y \rightarrow \dots \tan^{\frac{3}{2}} y \sec^2 y$

A1: Correct differentiation $2x \tan y + x^2 \sec^2 y \frac{dy}{dx} = 0$

Allow also $\sec^2 y \frac{dy}{dx} = -\frac{18}{x^3}$ or $2x = -9 \operatorname{cosec}^2 y \frac{dy}{dx}$ amongst others

M1: Full method to get $\frac{dy}{dx}$ in terms of x using $\sec^2 y = 1 + \tan^2 y = 1 + f(x)$

A1*: Proceeds correctly to the given answer of $\frac{dy}{dx} = \frac{-18x}{x^4 + 81}$



(b)

M1: Attempts to differentiate the given expression using the product or quotient rule.

For example look for a correct attempt at $\frac{vu' - uv'}{v^2}$ with $u = -18x, v = x^4 + 81, u' = \pm 18, v' = \dots x^3$

If no method is seen or implied award for $\frac{\pm 18 \times (x^4 + 81) \pm 18x(ax^3)}{(x^4 + 81)^2}$

Using the product rule award for $\pm 18(x^4 + 81)^{-1} \pm 18x(x^4 + 81)^{-2} \times cx^3$

A1: Correct **simplified** $\frac{d^2y}{dx^2} = \frac{54(x^4 - 27)}{(x^4 + 81)^2}$ o.e. such as $\frac{d^2y}{dx^2} = \frac{54x^4 - 1458}{(x^4 + 81)^2}$

Alternatively score for showing that when a correct (unsimplified) $\frac{d^2y}{dx^2} = 0 \Rightarrow x^4 = 27 \Rightarrow x = \sqrt[4]{27}$

Or for substituting $x = \sqrt[4]{27}$ into an unsimplified but correct $\frac{d^2y}{dx^2}$ and showing that it is 0

A1: Correct explanation with a minimal conclusion and correct second derivative.

See scheme.

It can be also be argued from $x^4 < 27$, $x^4 = 27$ and $x^4 > 27$ provided the conclusion states that the point of inflection is at $x = \sqrt[4]{27}$

Alternatively substitutes values of x either side of $\sqrt[4]{27}$ and at $\sqrt[4]{27}$, into $\frac{d^2y}{dx^2}$, finds all three values and makes a minimal conclusion.

A different method involves finding $\frac{d^3y}{dx^3}$ and showing that $\frac{d^3y}{dx^3} \neq 0$ and $\frac{d^2y}{dx^2} = 0$ when $x = \sqrt[4]{27}$

$$\text{FYI } \frac{d^3y}{dx^3} = \frac{23328x^3}{(x^4 + 81)^3} = 0.219 \text{ when } x = \sqrt[4]{27}$$

Alternative part (a) using arctan

M1: Sets $y = \arctan \frac{9}{x^2} \rightarrow \frac{dy}{dx} = \frac{1}{1 + \left(\frac{9}{x^2}\right)^2} \times \dots$ where $\dots$ could be 1

A2: $y = \arctan \frac{9}{x^2} \rightarrow \frac{dy}{dx} = \frac{1}{1 + \left(\frac{9}{x^2}\right)^2} \times -\frac{18}{x^3}$

A1*: $\frac{dy}{dx} = \frac{1}{1 + \frac{81}{x^4}} \times -\frac{18}{x^3} = \frac{-18x}{x^4 + 1}$ showing correct intermediate step and no errors.

Question	Scheme	Marks	AOs
7(a)	$x^3 \rightarrow \dots x^2$ and $3y^2 \rightarrow \dots y \frac{dy}{dx}$	M1	1.1b
	$2xy \rightarrow 2y + 2x \frac{dy}{dx}$	B1	1.1b
	$3x^2 + 2x \frac{dy}{dx} + 2y + 6y \frac{dy}{dx} = \dots \Rightarrow \frac{dy}{dx} = \dots$	M1	2.1
	$\frac{dy}{dx} = -\frac{2y + 3x^2}{2x + 6y}$	A1	1.1b
		(4)	
(b)	$\frac{dy}{dx} = -\frac{2(5) + 3(-2)^2}{2(-2) + 6(5)}$ or e.g. $3(-2)^2 + 2(-2) \frac{dy}{dx} + 2 \times 5 + 6 \times 5 \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \dots \left(-\frac{11}{13} \right)$	M1	1.1b
	$y - 5 = \frac{13}{11}(x + 2)$	dM1	1.1b
	$13x - 11y + 81 = 0$	A1	2.2a
		(3)	

(7 marks)**Notes**

(a) Allow equivalent notation for the $\frac{dy}{dx}$ e.g. y'

M1: Attempts to differentiate $x^3 \rightarrow \dots x^2$ **and** $3y^2 \rightarrow \dots y \frac{dy}{dx}$ where ... are constants

B1: Correct application of the product rule on $2xy$: $2xy \rightarrow 2x \frac{dy}{dx} + 2y$

Note that some candidates have a spurious $\frac{dy}{dx} = \dots$ at the start (as their intention to differentiate) and this can be ignored for the first 2 marks

M1: For a valid attempt to make $\frac{dy}{dx}$ the subject, with exactly 2 different terms in $\frac{dy}{dx}$ coming from $3y^2$ and

$2xy$. Look for $(\dots \pm \dots) \frac{dy}{dx} = \dots \Rightarrow \frac{dy}{dx} = \dots$ which may be implied by their working.

Condone slips provided the intention is clear.

For those candidates who had a spurious $\frac{dy}{dx} = \dots$ at the start, they may incorporate this in their

rearrangement in which case they will have 3 terms in $\frac{dy}{dx}$ and so score M0.

If they ignore it, then this mark is available for the condition as described above.

A1: $\frac{dy}{dx} = -\frac{2y + 3x^2}{2x + 6y}$ oe e.g. $\frac{dy}{dx} = \frac{-2y - 3x^2}{2x + 6y}, \frac{2y + 3x^2}{-2x - 6y}$ Isw once a correct expression is seen.

Note that it is sometimes unclear if the minus sign(s) is/are correctly placed and you may have to use your judgement. Evidence may be available in part (b) to help you decide if they have the correct expression.



(b)

M1: Substitutes $x = -2$ and $y = 5$ into $\frac{dy}{dx} = -\frac{2y+3x^2}{2x+6y}$

They must have x 's and y 's in their $\frac{dy}{dx}$ but condone slips in substitution provided the intention is clear.

As a minimum look for at least one x and at least one y substituted correctly.

Note that this mark may be implied by their value for $\frac{dy}{dx}$ and may be implied if, for example, they find

the negative reciprocal or the reciprocal of $-\frac{2y+3x^2}{2x+6y}$ and then substitute $x = -2$ and $y = 5$

Alternatively, substitutes $x = -2$ and $y = 5$ into their attempt to differentiate and then rearranges to find a value or numerical expression for $\frac{dy}{dx}$

dM1: Attempts to find the equation of the normal using their gradient of the tangent and $x = -2$ and $y = 5$ correctly placed. Score for an expression of the form $(y-5) = \frac{13}{11}(x+2)$ or if they use $y = mx + c$

they must proceed as far as $c = \dots$. Must be using the **negative reciprocal** of the tangent gradient.

Note that $y-5 = \frac{2x+6y}{2y+3x^2}(x+2)$ is not a correct method unless the gradient is evaluated first *before* expanding.

A1: $13x - 11y + 81 = 0$ or any integer multiple of this equation including the " $= 0$ ", not just a, b, c given. e.g., $26x - 22y + 162 = 0$ is likely if they don't cancel down their gradient.



Question	Scheme	Marks	AOs
9(a)	Either $3y^2 \rightarrow Ay \frac{dy}{dx}$ or $2xy \rightarrow 2x \frac{dy}{dx} + 2y$	M1	2.1
	$2x - 2x \frac{dy}{dx} - 2y + 6y \frac{dy}{dx} = 0$	A1	1.1b
	$(6y - 2x) \frac{dy}{dx} = 2y - 2x$	M1	2.1
	$\frac{dy}{dx} = \frac{2y - 2x}{6y - 2x} = \frac{y - x}{3y - x} *$	A1*	1.1b
		(4)	
(b)	$\left(\text{At } P \text{ and } Q \frac{dy}{dx} \rightarrow \infty \Rightarrow \right)$ Deduces that $3y - x = 0$	M1	2.2a
	Solves $y = \frac{1}{3}x$ and $x^2 - 2xy + 3y^2 = 50$ simultaneously	M1	3.1a
	$\Rightarrow x = (\pm)5\sqrt{3}$ OR $\Rightarrow y = (\pm)\frac{5}{3}\sqrt{3}$	A1	1.1b
	Using $y = \frac{1}{3}x \Rightarrow x = ..$ AND $y = ..$	dM1	1.1b
	$P = \left(-5\sqrt{3}, -\frac{5}{3}\sqrt{3} \right)$	A1	2.2a
		(5)	
(c)	Explains that you need to solve $y = x$ and $x^2 - 2xy + 3y^2 = 50$ simultaneously and choose the positive solution	B1ft	2.4
		(1)	

(10 marks)**Notes:****(a)**

M1: For selecting the appropriate method of differentiating either $3y^2 \rightarrow Ay \frac{dy}{dx}$ or $2xy \rightarrow 2x \frac{dy}{dx} + 2y$

It may be quite difficult awarding it for the product rule but condone $-2xy \rightarrow -2x \frac{dy}{dx} + 2y$ unless you see evidence that they have used the incorrect law $vu' - uv'$

A1: Fully correct derivative $2x - 2x \frac{dy}{dx} - 2y + 6y \frac{dy}{dx} = 0$

Allow attempts where candidates write $2xdx - 2xdy - 2ydx + 6ydy = 0$

but watch for students who write $\frac{dy}{dx} = 2x - 2x \frac{dy}{dx} - 2y + 6y \frac{dy}{dx}$ This, on its own, is A0 unless you are

convinced that this is just their notation. Eg $\frac{dy}{dx} = 2x - 2x \frac{dy}{dx} - 2y + 6y \frac{dy}{dx} = 0$



M1: For a valid attempt at making $\frac{dy}{dx}$ the subject, with two terms in $\frac{dy}{dx}$ coming from $3y^2$ and $2xy$

Look for $(\dots \pm \dots) \frac{dy}{dx} = \dots$. It is implied by $\frac{dy}{dx} = \frac{2y-2x}{6y-2x}$

This cannot be scored from attempts such as $\frac{dy}{dx} = 2x - 2x \frac{dy}{dx} - 2y + 6y$ which only has one correct term.

A1*: $\frac{dy}{dx} = \frac{y-x}{3y-x}$ with no errors or omissions.

The previous line $\frac{dy}{dx} = \frac{2y-2x}{6y-2x}$ or equivalent must be seen.

(b)

M1: Deduces that $3y - x = 0$ oe

M1: Attempts to find either the x or y coordinates of P and Q by solving their $y = \frac{1}{3}x$ with

$x^2 - 2xy + 3y^2 = 50$ simultaneously. Allow for finding a quadratic equation in x or y and solving to find at least one value for x or y .

This may be awarded when candidates make the numerator = 0 ie using $y = x$

A1: $\Rightarrow x = (\pm)5\sqrt{3}$ OR $\Rightarrow y = (\pm)\frac{5}{3}\sqrt{3}$

dM1: Dependent upon the previous M, it is for finding the y coordinate from their x (or vice versa)

This may also be scored following the numerator being set to 0 ie using $y = x$

A1: Deduces that $P = \left(-5\sqrt{3}, -\frac{5}{3}\sqrt{3}\right)$ OE. Allow to be $x = \dots$ $y = \dots$

(c)

B1ft: Explains that this is where $\frac{dy}{dx} = 0$ and so you need to solve $y = x$ and $x^2 - 2xy + 3y^2 = 50$

simultaneously and choose the positive solution (or larger solution).

Allow a follow through for candidates who mix up parts (b) and (c)

Alternatively candidates could complete the square $(x-y)^2 + 2y^2 = 50$ and state that y would reach a maximum value when $x = y$ and choose the positive solution from $2y^2 = 50$

Question	Scheme	Marks	AOs
15(a)	$3(x+y)^2 \left(1 + \frac{dy}{dx}\right) = 6x - 3 \frac{dy}{dx}$	M1 A1 A1	3.1a 1.1b 1.1b
	$(3(x+y)^2 + 3) \frac{dy}{dx} = 6x - 3(x+y)^2 \Rightarrow \frac{dy}{dx} = \dots$	M1	2.1
	$\frac{dy}{dx} = \frac{6x - 3(x+y)^2}{3(x+y)^2 + 3} \left(\text{oe e.g. } \frac{2x - (x+y)^2}{(x+y)^2 + 1} \right)$	A1	1.1b
		(5)	
	Alternative – expands $(x+y)^3$ before differentiating $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$		
	$\Rightarrow 3x^2 + 3x^2 \frac{dy}{dx} + 6xy + 6xy \frac{dy}{dx} + 3y^2 + 3y^2 \frac{dy}{dx} = 6x - 3 \frac{dy}{dx}$	M1 A1 A1	3.1a 1.1b 1.1b
	$(3x^2 + 6xy + 3y^2 + 3) \frac{dy}{dx} = 6x - 3x^2 - 6xy - 3y^2 \Rightarrow \frac{dy}{dx} = \dots$	M1	2.1
	$(3x^2 + 6xy + 3y^2 + 3) \frac{dy}{dx} = 6x - 3x^2 - 6xy - 3y^2$ $\Rightarrow \frac{dy}{dx} = \frac{6x - 3x^2 - 6xy - 3y^2}{3x^2 + 6xy + 3y^2 + 3} \left(\text{oe e.g. } \frac{2x - x^2 - 2xy - y^2}{x^2 + 2xy + y^2 + 1} \right)$	A1	1.1b

(a) Notes

- (a) Some candidates have a spurious " $\frac{dy}{dx} =$ " appearing as their intention to differentiate e.g.

$$\left(\frac{dy}{dx} = \right) 3(x+y)^2 \left(1 + \frac{dy}{dx} \right) = 6x - 3 \frac{dy}{dx}$$

This can be condoned for the first 3 marks in both versions.

Allow equivalent notation for the $\frac{dy}{dx}$ e.g. y'

M1: Award this mark for one of:

- $(x+y)^3 \rightarrow k(x+y)^2 \left(\lambda + \frac{dy}{dx} \right)$ where λ is 1, x or 0 but condone missing brackets e.g.

$$3(x+y)^2 1 + \frac{dy}{dx}$$

- $3x^2 - 3y - 2 \rightarrow \dots x + \dots \frac{dy}{dx}$ but condone $3x^2 - 3y - 2 \rightarrow \dots x + \dots \frac{dy}{dx} - 2$

A1: **Either** $3(x+y)^2 \left(1 + \frac{dy}{dx} \right)$ **or** $6x - 3 \frac{dy}{dx}$ **oe**

May be implied if e.g. they collect terms to one side initially.

Do not condone missing brackets unless they are implied by subsequent work.

A1: $3(x+y)^2 \left(1 + \frac{dy}{dx} \right)$ **and** $6x - 3 \frac{dy}{dx}$ (seen separately or equated)

If they collect terms to one side initially then the signs must be correct.

M1: A valid attempt to make $\frac{dy}{dx}$ the subject with exactly 2 **different** terms in $\frac{dy}{dx}$, one coming from the differentiation of $(x+y)^3$ and the other coming from the differentiation of “ $-3y$ ”

Note that here, 2 **different** terms means terms such as $3\frac{dy}{dx}$ and $3(x+y)^2\frac{dy}{dx}$ and not

e.g. $3\frac{dy}{dx}$ and $-8\frac{dy}{dx}$

Look for $(\dots \pm \dots)\frac{dy}{dx} = \dots \Rightarrow \frac{dy}{dx} = \dots$ which may be implied by their working.

Condone slips provided the intention is clear.

For those candidates who had a spurious $\frac{dy}{dx} = \dots$ at the start, they may incorporate this in

their rearrangement in which case they will have 3 terms in $\frac{dy}{dx}$ and so score M0.

If they ignore it, then this mark is available for the condition as described above.

Note that from $3(x+y)^2\left(1+\frac{dy}{dx}\right) = 6x - 3\frac{dy}{dx}$, candidates may expand the brackets before

rearranging, in which case they would need 4 **different** $\frac{dy}{dx}$ terms coming from the appropriate places.

Note that the different $\frac{dy}{dx}$ terms do not have to be correct as long as the above conditions are satisfied.

A1: Fully correct expression for $\frac{dy}{dx}$. Allow any equivalent correct forms.

Apply isw as soon as a correct expression is seen.

(a) alternative by expanding:

M1: Award this mark for one of:

- $3x^2 - 3y - 2 \rightarrow \dots x + \dots \frac{dy}{dx}$ but condone $3x^2 - 3y - 2 \rightarrow \dots x + \dots \frac{dy}{dx} - 2$
- Expanding $(x+y)^3$ to obtain either an x^2y term or an xy^2 term and then uses the product rule to obtain $\dots x^2y \rightarrow \dots x^2 \frac{dy}{dx} + \dots xy$ or $\dots xy^2 \rightarrow \dots xy \frac{dy}{dx} + \dots y^2$

A1: **Either** $3x^2 + 3x^2 \frac{dy}{dx} + 6xy + 6xy \frac{dy}{dx} + 3y^2 + 3y^2 \frac{dy}{dx}$ **or** $6x - 3\frac{dy}{dx}$.

May be implied if e.g. they collect terms to one side initially.

A1: $3x^2 + 3x^2 \frac{dy}{dx} + 6xy + 6xy \frac{dy}{dx} + 3y^2 + 3y^2 \frac{dy}{dx}$ **and** $6x - 3\frac{dy}{dx}$ oe. (seen separately or equated) If they collect terms to one side initially then the signs must be correct.

M1: A valid attempt to make $\frac{dy}{dx}$ the subject with exactly 4 **different** terms in $\frac{dy}{dx}$, 3 coming from the differentiation of $(x+y)^3$ and the other coming from the differentiation of “ $-3y$ ”

Note that here, 4 **different** terms means terms such as $x^2 \frac{dy}{dx}$ and $6xy \frac{dy}{dx}$ and not e.g.

$$3 \frac{dy}{dx} \text{ and } -8 \frac{dy}{dx}$$

Look for $(\dots \pm \dots \pm \dots \pm \dots) \frac{dy}{dx} = \dots \Rightarrow \frac{dy}{dx} = \dots$ which may be implied by their working.

Condone slips provided the intention is clear.

For those candidates who had a spurious $\frac{dy}{dx} = \dots$ at the start, they may incorporate this in

their rearrangement in which case they will have 5 terms in $\frac{dy}{dx}$ and so score M0.

If they ignore it, then this mark is available for the condition as described above.

Note that the different $\frac{dy}{dx}$ terms do not have to be correct as long as the above conditions

are satisfied. E.g. if they have an incorrect term such as $6x \frac{dy}{dx}$, this mark is still available.

A1: Fully correct expression for $\frac{dy}{dx}$. Allow any equivalent correct forms.

Condone e.g. $3x2y$ for $6xy$.

Apply isw as soon as a correct expression is seen.

Alternative making y the subject in (a):

$$(x + y)^3 = 3x^2 - 3y - 2$$

$$x + y = (3x^2 - 3y - 2)^{\frac{1}{3}} \Rightarrow y = (3x^2 - 3y - 2)^{\frac{1}{3}} - x$$

$$\frac{dy}{dx} = \frac{1}{3}(3x^2 - 3y - 2)^{-\frac{2}{3}} \left(6x - 3 \frac{dy}{dx} \right) - 1$$

$$\frac{dy}{dx} \left(1 + (3x^2 - 3y - 2)^{-\frac{2}{3}} \right) = 2x(3x^2 - 3y - 2)^{-\frac{2}{3}} - 1$$

$$\frac{dy}{dx} = \frac{2x(3x^2 - 3y - 2)^{-\frac{2}{3}} - 1}{1 + (3x^2 - 3y - 2)^{-\frac{2}{3}}}$$

Score as follows:

M1: Cube roots both sides and makes $x + y$ or y the subject then award for

- $(3x^2 - 3y - 2)^{\frac{1}{3}} \rightarrow \dots (3x^2 - 3y - 2)^{-\frac{2}{3}}$ or
- $3x^2 - 3y - 2 \rightarrow \dots x + \dots \frac{dy}{dx}$ but condone $3x^2 - 3y - 2 \rightarrow \dots x + \dots \frac{dy}{dx} - 2$

A1: For the $\frac{1}{3}(3x^2 - 3y - 2)^{-\frac{2}{3}}$ **or** $6x - 3 \frac{dy}{dx}$

A1: Fully correct

M1: A valid attempt to make $\frac{dy}{dx}$ the subject with exactly 2 **different** terms in $\frac{dy}{dx}$

A1: Correct expression

Using partial derivatives in (a):

$$(x+y)^3 = 3x^2 - 3y - 2 \rightarrow f(x, y) = (x+y)^3 - 3x^2 + 3y + 2$$

$$\frac{\partial f}{\partial x} = 3(x+y)^2 - 6x \quad \frac{\partial f}{\partial y} = 3(x+y)^2 + 3$$

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \div \frac{\partial f}{\partial y} = \frac{6x - 3(x+y)^2}{3(x+y)^2 + 3}$$

or

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = 3x^2 - 3y - 2$$

$$f(x, y) = x^3 + 3x^2y + 3xy^2 + y^3 - 3x^2 + 3y + 2$$

$$\frac{\partial f}{\partial x} = 3x^2 + 6xy + 3y^2 - 6x \quad \frac{\partial f}{\partial y} = 3x^2 + 6xy + 3y^2 + 3$$

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \div \frac{\partial f}{\partial y} = \frac{-3x^2 - 6xy - 3y^2 + 6x}{3x^2 + 6xy + 3y^2 + 3}$$

Score as follows:

M1: Correct structure for either partial derivative:

$$\text{Doesn't expand: } \frac{\partial f}{\partial x} = \dots(x+y)^2 + \dots x \quad \text{or} \quad \frac{\partial f}{\partial y} = \dots(x+y)^2 + \dots$$

or

$$\text{Expands: } \frac{\partial f}{\partial x} = \dots x^2 + \dots xy + \dots y^2 + \dots x \quad \text{or} \quad \frac{\partial f}{\partial y} = \dots x^2 + \dots xy + \dots y^2 + \dots$$

Where “...” are non-zero constants

A1: Correct $\frac{\partial f}{\partial x}$ **or** correct $\frac{\partial f}{\partial y}$

A1: Correct $\frac{\partial f}{\partial x}$ **and** correct $\frac{\partial f}{\partial y}$

M1: Attempts $\frac{dy}{dx} = -\frac{\partial f}{\partial x} \div \frac{\partial f}{\partial y}$

A1: Correct expression



(b)	$\frac{dy}{dx} = \frac{6(1) - 3(0+1)^2}{3(0+1)^2 + 3} = \frac{1}{2}$ or e.g. $\frac{dy}{dx} = \frac{6(1) - 3(1)^2 - 6(1)(0) - 3(0)^2}{3(1)^2 + 6(1)(0) + 3(0)^2 + 3} = \frac{1}{2}$ $\Rightarrow y - 0 = -2(x - 1)$ or $\Rightarrow y = -2x + c \Rightarrow 0 = -2 + c \Rightarrow c = \dots$	M1	2.1
	$y = -2x + 2^*$		
		A1*	1.1b
(2)			
(b) Notes			
(b) Note that the gradient of $\frac{1}{2}$ could have been deduced from the given equation so you will need to check their solution carefully. M1: Substitutes $x = 1$ and $y = 0$ into their $\frac{dy}{dx}$ to obtain the tangent gradient and then uses the negative reciprocal and $x = 1$ and $y = 0$ in a correct straight line method to obtain the normal equation with $x = 1$ and $y = 0$ correctly placed. Note that when finding the normal gradient, they may find the negative reciprocal of their expression from part (a) and then substitute $x = 1$ and $y = 0$ which is fine. If using $y = mx + c$ they must proceed as far as finding a value for c. If no substitution of $x = 1$ and $y = 0$ into their $\frac{dy}{dx}$ is seen you will need to check their value. If they just state a value for $\frac{dy}{dx}$ then it must follow their $\frac{dy}{dx}$ with $x = 1$ and $y = 0$ A1*: Correct equation with no errors following a correct $\frac{dy}{dx}$ from part (a) (unless they start again which is unlikely) Be aware that some incorrect expressions for $\frac{dy}{dx}$ from part (a) may fortuitously give $\frac{dy}{dx} = \frac{1}{2}$ and would generally score A0 In general A1* must follow the final A1 in (a) or correct differentiation in (a)			

(c)	$y = -2x + 2 \Rightarrow (x - 2x + 2)^3 = 3x^2 - 3(-2x + 2) - 2$ or $x = \frac{2-y}{2} \Rightarrow \left(\frac{2-y}{2} + y\right)^3 = 3\left(\frac{2-y}{2}\right)^2 - 3y - 2$	M1	1.1b
	$x^3 - 3x^2 + 18x - 16 = 0$ or $y^3 + 60y = 0$	A1	1.1b
	$\Rightarrow (x-1)(x^2 - 2x + 16) = 0$ (x = 1 is known) or $\Rightarrow y(y^2 + 60) = 0$ (y = 0 is known)	dM1	2.1
	For $x^2 - 2x + 16 = 0$, $b^2 - 4ac = 4 - 4 \times 1 \times 16$ or For $y^2 + 60 = 0$, $y^2 \neq -60$	ddM1	2.1
	As $b^2 - 4ac < 0$ or as $y^2 \neq -60$ there are no other real roots and so the normal does not meet C again.	A1	2.4
		(5)	
(c) Notes			
(c) M1: Uses the equation from part (a) and substitutes $y = \pm 2x \pm 2$ or $x = \frac{\pm 2 \pm y}{2}$ to obtain an equation in one variable (usually x) (not necessarily a cubic equation). Allow slips in rearranging to obtain x in terms of y (or y in terms of x) as long as the intention is clear. A1: Correct cubic equation with terms collected and “= 0” seen or implied. Note that both $-x^3 + 3x^2 - 18x + 16 = 0$ and $-y^3 - 60y = 0$ are correct equations. To access any of the following marks, candidates must attempt to use either the factor of (x – 1) with their cubic in x or the factor of y in their cubic in y to obtain a quadratic expression in x or y. Attempts that just use a calculator to solve the cubic equation score no more marks in this part. dM1: Uses the fact that (x – 1) or y is a factor in an attempt to establish the quadratic factor. For the cubic in x, it must be of the form $ax^3 + bx^2 + cx + d = 0$ $a, b, c, d \neq 0$ For the cubic in y, it must be of the form $ay^3 + by = 0$ $a, b \neq 0$ For the cubic in x, the attempt at the quadratic factor using (x – 1) may be via inspection or e.g. long division to obtain a 3 term quadratic expression. There may or may not be a remainder but they must obtain 3 terms. For the cubic in y, they would need to take out a factor of y (or divide through by y) to obtain a factor of the form $k(y^2 + \alpha)$			

ddM1: This mark requires:

- a correct cubic equation in x or y
- the correct quadratic factor or a multiple of it e.g. $k(x^2 - 2x + 16)$ or $k(y^2 + 60)$
- an attempt to show that the quadratic factor has no real roots

For the quadratic in x this could be:

Attempts discriminant: e.g. $b^2 - 4ac = 4 - 4 \times 1 \times 16$ (may be embedded in the quadratic formula)

Attempts to complete the square: e.g. $x^2 - 2x + 16 = (x-1)^2 - 1 + 16$

Uses calculus to find the turning point: e.g. $\frac{d(x^2 - 2x + 16)}{dx} = 2x - 2 = 0 \Rightarrow x = 1 \Rightarrow y = \dots$

Attempts to solve: e.g. $x^2 - 2x + 16 = 0 \Rightarrow x = \frac{2 \pm \sqrt{4 - 4 \times 16}}{2}$ or from a calculator $x = 1 \pm \sqrt{15}i$

For the quadratic in y this is likely to be:

Attempts to solve: e.g. $y^2 + 60 = 0 \Rightarrow y^2 = -60 \Rightarrow \dots$

A1: Fully correct argument that requires:

- fully correct work
- a justification depending on their strategy
- a conclusion depending on their strategy

Via discriminant: $4 - 4 \times 1 \times 16 < 0$ so no real roots so they do not meet again

Via completing the square: $\rightarrow (x-1)^2 + 15$ which has a minimum value of 15 so no real roots so they do not meet again.

Via calculus: $x = 1 \Rightarrow y = 15$ is the minimum so no real roots so they do not meet again.

Via solving: $x = 1 \pm \sqrt{15}i$ or e.g. $x = 1 \pm 3.87i$ or e.g. $x = 1 \pm \sqrt{-15}$ so math error so they do not meet again.

For y it is likely to be more straightforward e.g. $y^2 \neq -60$ which cannot be solved so they do not meet again.

Allow equivalent statements for “they do not meet again” e.g. so they only meet once.

(But do **not** condone incorrect statements such as “therefore P does not meet C again”)

A minimum justification could be:

$$x^2 - 2x + 16 = 0 \rightarrow b^2 - 4ac = (-2)^2 - 4 \times 1 \times 16 \quad \text{ddM1}$$

$$4 - 4 \times 1 \times 16 < 0 \quad \text{so no more roots so no more intersections} \quad \text{A1}$$

Do not allow e.g.

“ $x^2 - 2x + 16 = 0$ gives a math error so they do not meet again”

as there has been no attempt to show why the “math error” occurs – this scores M0A0



Alternative to (c) by showing the cubic is strictly increasing (or decreasing):

M1A1: As in the main scheme then

$$f(x) = x^3 - 3x^2 + 18x - 16 \Rightarrow f'(x) = 3x^2 - 6x + 18$$

$$3x^2 - 6x + 18 = 3(x^2 - 2x + 6) = 3(x-1)^2 + 15$$

$$3(x-1)^2 + 15 > 0 \text{ so } f(x) \text{ is an increasing function}$$

Hence there can only be one intersection (at $x = 1$) so the normal and curve do not intersect again.

dM1: Differentiates their cubic of the form $ax^3 + bx^2 + cx + d = 0$ $a, b, c, d \neq 0$ to obtain a 3 term quadratic expression with only coefficient errors on the non-constant terms.

ddM1: This mark requires:

- a correct cubic equation in x
- the correct derivative or a multiple of it
- an attempt to show that the quadratic expression is always positive (or negative)

A1: Fully correct concluding argument e.g. that as the derivative is always positive (or always negative) the function is strictly increasing (or decreasing) and therefore there can only be one intersection (at $x = 1$) so the normal and curve do not meet again.

(12 marks)



Question	Scheme	Marks	AOs
16(a)	Attempts $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\sec^2 t \tan t}{2\sec^2 t} (= 2 \tan t)$	M1 A1	1.1b 1.1b
	At $t = \frac{\pi}{4}$, $\frac{dy}{dx} = 2, x = 3, y = 7$	M1	2.1
	Attempts equation of normal $y - 7 = -\frac{1}{2}(x - 3)$	M1	1.1b
	$y = -\frac{1}{2}x + \frac{17}{2}$ *	A1*	2.1
		(5)	
(b)	Attempts to use $\sec^2 t = 1 + \tan^2 t \Rightarrow \frac{y-3}{2} = 1 + \left(\frac{x-1}{2}\right)^2$	M1	3.1a
	$\Rightarrow y - 3 = 2 + \frac{(x-1)^2}{2} \Rightarrow y = \frac{1}{2}(x-1)^2 + 5$ *	A1*	2.1
		(2)	
(b) Alternative 1:			
	$y = \frac{1}{2}(x-1)^2 + 5 = \frac{1}{2}(2 \tan t + 1 - 1)^2 + 5$ $= \frac{1}{2}4 \tan^2 t + 5 = 2(\sec^2 t - 1) + 5$	M1	3.1a
	$= 2\sec^2 t + 3 = y$ *	A1	2.1
(b) Alternative 2:			
	$x = 2 \tan t + 1 \Rightarrow t = \tan^{-1}\left(\frac{x-1}{2}\right) \Rightarrow y = 2\sec^2\left(\tan^{-1}\left(\frac{x-1}{2}\right)\right) + 3$ $\Rightarrow y = 2\left(1 + \tan^2\left(\tan^{-1}\left(\frac{x-1}{2}\right)\right)\right) + 3$	M1	3.1a
	$\Rightarrow y = 2\left(1 + \left(\frac{x-1}{2}\right)^2\right) + 3 = \frac{1}{2}(x-1)^2 + 5$ *	A1	2.1
(b) Alternative 3:			
	$\frac{dy}{dx} = 2 \tan t = x - 1 \Rightarrow y = \int (x-1) dx = \frac{x^2}{2} - x + c$ $(3, 7) \rightarrow 7 = \frac{3^2}{2} - 3 + c \Rightarrow c = \frac{11}{2}$	M1	3.1a
	$\frac{x^2}{2} - x + \frac{11}{2} = \frac{1}{2}(x^2 - 2x) + \frac{11}{2} = \frac{1}{2}(x-1)^2 - \frac{1}{2} + \frac{11}{2} = \frac{1}{2}(x-1)^2 + 5$ *	A1	2.1

(c)	Attempts the lower limit for k: $\frac{1}{2}(x-1)^2 + 5 = -\frac{1}{2}x + k \Rightarrow x^2 - x + (11 - 2k) = 0$ $b^2 - 4ac = 1 - 4(11 - 2k) = 0 \Rightarrow k = \dots$	M1	2.1
	$(k =) \frac{43}{8}$	A1	1.1b
	Attempts the upper limit for k: $(x, y)_{t=-\frac{\pi}{4}}: t = -\frac{\pi}{4} \Rightarrow x = 2 \tan\left(-\frac{\pi}{4}\right) + 1 = -1, y = 2 \sec^2\left(-\frac{\pi}{4}\right) + 3 = 7$ $(-1, 7), y = -\frac{1}{2}x + k \Rightarrow 7 = \frac{1}{2} + k \Rightarrow k = \dots$	M1	2.1
	$(k =) \frac{13}{2}$	A1	1.1b
	$\frac{43}{8} < k \leq \frac{13}{2}$	A1	2.2a
		(5)	
(12 marks)			
Notes:			

(a) **Must use parametric differentiation to score any marks in this part and not e.g. Cartesian form**

M1: For the key step of attempting $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$. There must be some attempt to differentiate both

parameters however poor and divide the right way round so using $\frac{dy}{dx} = \frac{y}{x}$ scores M0.

This may be implied by e.g. $\frac{dx}{dt} = 2 \sec^2 t$, $\frac{dy}{dt} = 4 \sec^2 t \tan t$, $t = \frac{\pi}{4} \Rightarrow \frac{dx}{dt} = 4$, $\frac{dy}{dt} = 8 \Rightarrow \frac{dy}{dx} = 2$

A1: $\frac{dy}{dx} = \frac{4 \sec^2 t \tan t}{2 \sec^2 t}$. Correct expression in any form. May be implied as above.

Condone the confusion with variables as long as the intention is clear e.g.

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4 \sec^2 x \tan x}{2 \sec^2 x} (= 2 \tan x)$ and allow subsequent marks if this is interpreted correctly

M1: For attempting to find the values of x, y and the gradient at $t = \frac{\pi}{4}$ **AND** getting at least two correct.

Follow through on their $\frac{dy}{dx}$ so allow for any two of $x = 3$, $y = 7$, $\frac{dy}{dx} = 2$ (or their $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$)

Note that the $x = 3$, $y = 7$ may be seen as e.g. (3, 7) on the diagram. There must be a non-trivial $\frac{dy}{dx}$ for this mark e.g. they must have a $\frac{dy}{dx}$ to substitute into.

M1: For a correct attempt at the normal equation using their x and y at $t = \frac{\pi}{4}$ with the negative

reciprocal of their $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ having made some attempt at $\frac{dy}{dx}$ and all correctly placed.

For attempts using $y = mx + c$ they must reach as far as a value for c using their x and y at $t = \frac{\pi}{4}$

with the negative reciprocal of their $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ all correctly placed.

(b)

M1: Attempts to use $\sec^2 t = 1 + \tan^2 t$ oe to obtain an equation involving y and $(x-1)^2$ E.g. as above or e.g. $y = 2\sec^2 t + 3 = 2(1 + \tan^2 t) + 3 = 2\left(1 + \left(\frac{x-1}{2}\right)^2\right) + 3$ for M1 and then

$$y = \frac{1}{2}(x-1)^2 + 5^* \text{ for A1}$$

A1*: Proceeds with a clear argument to the given answer with no errors**Alternative 1:****M1:** Uses the given result, substitutes for x and attempts to use $\sec^2 t = 1 + \tan^2 t$ oe**A1:** Proceeds with a clear argument to the y parameter and makes a (minimal) conclusion e.g. “ $= y$ ”
QED, hence proven etc.**Alternative 2:****M1:** Uses the x parameter to obtain t in terms of arctan, substitutes into y and attempts to use
 $\sec^2 t = 1 + \tan^2 t$ oe**A1:** Proceeds with a clear argument to the given answer with no errors**Alternative 3:****M1:** Uses $\frac{dy}{dx}$ from part (a) to express $\frac{dy}{dx}$ in terms of x , integrates and uses (3, 7) to find “ c ” to reach
a Cartesian equation.**A1:** Proceeds with a clear argument to the given answer with no errors**Allow the marks for (b) to score anywhere in their solution e.g. if they find the Cartesian equation in part (a)**

(c)

M1: A full attempt to find the **lower** limit for k .

$$\frac{1}{2}(x-1)^2 + 5 = -\frac{1}{2}x + k \Rightarrow x^2 - x + (11 - 2k) = 0 \Rightarrow b^2 - 4ac = 1 - 4(11 - 2k) = 0 \Rightarrow k = \dots$$

Score **M1** for setting $\frac{1}{2}(x-1)^2 + 5 = -\frac{1}{2}x + k$, rearranging to 3TQ form and attempts $b^2 - 4ac \dots 0$ e.g. $b^2 - 4ac > 0$ or e.g. $b^2 - 4ac < 0$ correctly to find a value for k .**A1:** $k = \frac{43}{8}$ oe. Look for this **value** e.g. may appear in an inequality e.g. $k > \frac{43}{8}$, $k < \frac{43}{8}$ **An alternative method using calculus for lower limit:**

$$y = \frac{1}{2}(x-1)^2 + 5 \Rightarrow \frac{dy}{dx} = x-1, x-1 = -\frac{1}{2} \Rightarrow x = \frac{1}{2}$$

$$x = \frac{1}{2} \Rightarrow y = \frac{1}{2}\left(\frac{1}{2}-1\right)^2 + 5 = \frac{41}{8}$$

$$y = -\frac{1}{2}x + k \Rightarrow \frac{41}{8} = -\frac{1}{4} + k \Rightarrow k = \dots$$

Score **M1** for $\frac{dy}{dx} =$ “a linear expression in x ”, sets $= -\frac{1}{2}$, solves a linear equation to find x andthen substitutes into the given result in (b) to find y and then uses $y = -\frac{1}{2}x + k$ to find a valuefor k . **A1:** $k = \frac{43}{8}$ oe. Look for this **value** e.g. may appear in an inequality e.g. $k > \frac{43}{8}$, $k < \frac{43}{8}$

An alternative method using parameters for lower limit:

$$\begin{aligned}
 y &= -\frac{1}{2}x + k \Rightarrow 2\sec^2 t + 3 = -\frac{1}{2}(2\tan t + 1) + k \\
 \Rightarrow 2(1 + \tan^2 t) + 3 &= -\frac{1}{2}(2\tan t + 1) + k \Rightarrow 2\tan^2 t + \tan t + 5.5 - k = 0 \\
 b^2 - 4ac &= 0 \Rightarrow 1 - 4 \times 2(5.5 - k) = 0 \Rightarrow k = \frac{43}{8}
 \end{aligned}$$

Score **M1** for substituting parametric form of x and y into $y = -\frac{1}{2}x + k$, uses $\sec^2 t = 1 + \tan^2 t$ rearranges to 3TQ form and attempts $b^2 - 4ac \dots 0$ or e.g. $b^2 - 4ac > 0$ or $b^2 - 4ac < 0$ correctly to find a value for k .

A1: $k = \frac{43}{8}$ oe. Look for this **value** e.g. may appear in an inequality e.g. $k > \frac{43}{8}$, $k < \frac{43}{8}$

M1: A full attempt to find the **upper** limit for k . This requires an attempt to find the value of x and the value of y using $t = -\frac{\pi}{4}$, the substitution of these values into $y = -\frac{1}{2}x + k$ and solves for k .

A1: $k = \frac{13}{2}$. Look for this value e.g. may appear in an inequality.

A1: Deduces the correct range for k : $\frac{43}{8} < k \leq \frac{13}{2}$

Allow equivalent notation e.g. $\left(k \leq \frac{13}{2} \text{ and } k > \frac{43}{8}\right)$, $\left(k \leq \frac{13}{2} \cap k > \frac{43}{8}\right)$, $\left(\frac{43}{8}, \frac{13}{2}\right]$

But not e.g. $\left(k \leq \frac{13}{2}, k > \frac{43}{8}\right)$, $\left(k \leq \frac{13}{2} \cup k > \frac{43}{8}\right)$, $\left(k \leq \frac{13}{2} \text{ or } k > \frac{43}{8}\right)$ and do not allow if in terms of x .

Allow equivalent exact values for $\frac{43}{8}$, $\frac{13}{2}$

There may be other methods for finding the upper limit which are valid. If you are in any doubt if a method deserves credit then use Review.

Question	Scheme	Marks	AOs
5(a)	$-3 = \frac{t-1}{2} \Rightarrow t = -5 \Rightarrow y = 5((-5+2)^4$	M1	1.1b
	$(y =) 405$	A1	1.1b
		(2)	
(b)	$x = \frac{t-1}{2} \Rightarrow t = 2x+1 \Rightarrow y = 5((2x+1)+2)^4$	M1	1.1b
	$y = 5(2x+3)^4$	A1	1.1b
		(2)	
(c)	$((2x+3))^n \rightarrow \dots ((2x+3))^{n-1}$	M1	1.1b
	$\left(\frac{dy}{dx}\right) 40(2(-3)+3)^3$	dM1	1.1b
	$\left(\frac{dy}{dx}\right) -1080$	A1	1.1b
		(3)	
(7 marks)			
Notes			
(a) Mark parts (a) and (b) together.			
M1: Substitutes $x = -3$ into $x = \frac{t-1}{2}$, attempts to find t , and substitutes their t into $y = 5(t+2)^4$			
Condone slips. May be implied by substitution of $t = -5$ into y or by 405 or by $y = 5(-3)^4$			
Alternatively, they may substitute $x = -3$ into their $y = f(x)$			
A1: cao Answer only scores full marks and ISW after seeing 405 e.g. $(-5, 405)$			
Accept $(-3, 405)$ or e.g. $P = 405$			
(b) Mark parts (a) and (b) together.			
M1: Attempts to make t the subject of $x = \frac{t-1}{2}$ using the correct order of operations and			
substitutes into $y = 5(t+2)^4$ Condone slips e.g. dividing by 2 first: $t = \frac{x}{2} + 1$ or missing the			
$+2$ in $y = 5(t+2)^4$			
Alt 1: Writes $t = \left(\frac{y}{5}\right)^{\frac{1}{4}} - 2$, substitutes into $x = \frac{t-1}{2}$ and attempts to make y the subject using			
the correct order of operations, i.e., $2x = \left(\frac{y}{5}\right)^{\frac{1}{4}} - 3 \Rightarrow (2x+3)^4 = \frac{y}{5} \Rightarrow y = 5(2x+3)^4$			
Alt 2: Makes t the subject and then substitutes into an expanded $5(t+2)^4$			
Do not be too concerned about their expansion, but it must contain t^4 and a constant term.			
A1: cao and ISW after a correct answer seen. May be scored for $y = 5(2x+1+2)^4$			
Allow e.g. $y = 80x^4 + 480x^3 + 1080x^2 + 1080x + 405$ or e.g.			
$y = 5(2x+1)^4 + 40(2x+1)^3 + 120(2x+1)^2 + 160(2x+1) + 80$			
Their RHS may be unsimplified but do not allow if e.g. binomial coefficients are still present.			
Do not accept $f(x) = \dots$ It must be $y = \dots$			

(c) Note: Differentiation seen in (a) or (b) can score marks if used in (c)

M1: Reduces the power of their $(2x+3)^n$ by one to $Q(2x+3)^{n-1}$ where Q is a constant and could be 1. There should be no other terms using this method.

Alternatively, attempts to expand their $(2x+3)^n$ (may have been expanded in (b)) and reduces the power of x by one in at least one term.

They may use parametric differentiation, i.e., $\frac{dy}{dt} = a(t+2)^3$ and $\frac{dx}{dt} = b$ where a and b are

constants, leading to $\frac{dy}{dx} = \frac{a(t+2)^3}{b}$ i.e. they must divide the correct way round.

Condone attempts at the chain rule that reach e.g. $y' = \dots u^3$ but make a slip when substituting

back in for x or t , e.g., $\frac{dy}{dx} = (5x+3)^3$ or $\frac{dy}{dt} = (t+2)^2$, provided the intention is clear.

dM1: Substitutes $x = -3$ into their $\frac{dy}{dx}$ in terms of x which may be implied by their answer.

Using parametric differentiation, substitutes their t (found from an attempt at substituting $x = -3$ into $x = \frac{t-1}{2}$ which may have been seen in (a)) into their $\frac{dy}{dx}$ in terms of t .

Note for reference, if correct, the parametric differentiation is $\frac{dy}{dt} = 20(t+2)^3$ and $\frac{dx}{dt} = \frac{1}{2}$

leading to $\frac{dy}{dx} = 40(t+2)^3$

They may substitute their value of t into $\frac{dy}{dt}$ first before using the chain rule to reach $\frac{dy}{dx}$ which is acceptable and implies the first M mark.

A1: $\left(\frac{dy}{dx} = \right) -1080$

Correct answer only scores full marks.

May be seen labelled as m or something else.

Question	Scheme	Marks	AOs
11(a)	$\frac{dV}{dh} = 200$ oe e.g. $\frac{dh}{dV} = \frac{1}{200}$	B1	1.1b
	$\left(\frac{dh}{dt}\right) \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{200} \times \frac{k}{\sqrt{h}}$	M1	3.1a
	$\frac{dh}{dt} = \frac{\lambda}{\sqrt{h}}$ *	A1*	2.1
		(3)	
(b)	$\frac{dh}{dt} = \frac{\lambda}{\sqrt{h}} \Rightarrow \int h^{\frac{1}{2}} dh = \int \lambda dt \Rightarrow \dots h^{\frac{3}{2}} = \lambda t \{+c\}$	M1	1.1b
	$\frac{2}{3} h^{\frac{3}{2}} = \lambda t \{+c\}$ oe e.g. $h^{\frac{3}{2}} = \lambda t \{+c\}$	A1	1.1b
	$\frac{2}{3} (1.44)^{\frac{3}{2}} = \lambda \times 0 + c \Rightarrow c = 1.152 \left(= \frac{144}{125} \right)$	dM1	3.4
	$\frac{2}{3} (3.24)^{\frac{3}{2}} = \lambda \times 8 + "1.152" \Rightarrow \lambda = 0.342 \left(= \frac{171}{500} \right)$	ddM1	3.1b
	$h^{\frac{3}{2}} = 0.513t + 1.728$ oe e.g. $h^{\frac{3}{2}} = \frac{513}{1000}t + \frac{216}{125}$	A1	3.3
		(5)	
(b) Alternative:			
	$\frac{dh}{dt} = \frac{\lambda}{\sqrt{h}} \Rightarrow \frac{dt}{dh} = \frac{\sqrt{h}}{\lambda} \Rightarrow t = \dots h^{\frac{3}{2}} (+c)$	M1	1.1b
	$t = \frac{2h^{\frac{3}{2}}}{3\lambda} (+c)$ oe	A1	1.1b
	$0 = \frac{2(1.44)^{\frac{3}{2}}}{3\lambda} + c$ and $8 = \frac{2(3.24)^{\frac{3}{2}}}{3\lambda} + c$ $\Rightarrow \lambda = \dots \left(\frac{171}{500} \right)$ or $c = \dots \left(-\frac{64}{19} \right)$	dM1	3.4
	$\Rightarrow \lambda = \dots \left(\frac{171}{500} \right)$ and $c = \dots \left(-\frac{64}{19} \right)$	ddM1	3.1b
	$h^{\frac{3}{2}} = 0.513t + 1.728$ oe e.g. $h^{\frac{3}{2}} = \frac{513}{1000}t + \frac{216}{125}$	A1	3.3
		(5)	
(c)	$5^{\frac{3}{2}} = 0.513t + 1.728 \Rightarrow t = \dots$	M1	3.4
	$(t =)$ awrt 18.4 min	A1	3.2a
		(2)	
(10 marks)			
Notes			
(a)			



B1: For $\frac{dV}{dh} = 200$ stated or used – may be implied by their chain rule attempt

M1: Requires:

- $\frac{dV}{dh} = p, p > 1$
- $\frac{dV}{dt} = \pm \frac{k}{\sqrt{h}}$ or e.g. $\frac{dV}{dt} = \pm \frac{1}{k\sqrt{h}}$ (or a suitable letter for k , which may be λ , but must **not** be a number)
- **application** of the correct chain rule $\left(\frac{dh}{dt}\right) \frac{dh}{dV} \times \frac{dV}{dt}$ or any equivalent with $\frac{dV}{dt} = \pm \frac{k}{\sqrt{h}}$ or $\pm \frac{1}{k\sqrt{h}}$ and their $\frac{dV}{dh}$ correctly placed. So $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{200k}{\sqrt{h}} = \frac{\lambda}{\sqrt{h}}$ scores M0 as $\frac{dh}{dV}$ is incorrectly placed.

A1*: A rigorous argument with all steps shown and simplifies to achieve $\frac{dh}{dt} = \frac{\lambda}{\sqrt{h}}$ with no errors.

Do not allow the use of λ for both constants. Allow use of e.g. $\frac{dV}{dt} = \pm \frac{1}{k\sqrt{h}}$ for full marks.

e.g. $\frac{dV}{dh} = 200, \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{200} \times \frac{\lambda}{\sqrt{h}} = \frac{\lambda}{200\sqrt{h}}$ scores B1M1A0* *unless* e.g. “let $\lambda = \frac{\lambda}{200}$ ” seen.

Allow correct work leading to e.g. $\frac{dh}{dt} = \frac{1}{200} \times \frac{k}{\sqrt{h}} \rightarrow \frac{\lambda}{\sqrt{h}}$ or $\frac{dh}{dt} = \frac{1}{200} \times \frac{k}{\sqrt{h}}$ so $\lambda = \frac{k}{200}$

There must be an attempt to link the $\frac{dh}{dt}$ with the $\frac{\lambda}{\sqrt{h}}$ which may be missing an = sign.

Allow an argument with $\frac{dV}{dt} = -\frac{k}{\sqrt{h}}$ e.g. $\frac{dV}{dh} = 200, \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{200} \times -\frac{k}{\sqrt{h}} = -\frac{\lambda}{\sqrt{h}}$

Withhold this A mark if there are notational errors e.g. $\frac{dV}{dt} = 200, \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{200} \times \frac{k}{\sqrt{h}} = \frac{\lambda}{\sqrt{h}}$

scores B1(implied)M1A0*

(b) Note that some candidates may work with e.g. $\lambda = \frac{k}{200}$ or e.g. $\lambda = 200k$ which is acceptable.

Candidates who do not have a λ e.g. assume $\frac{dh}{dt} = \frac{1}{200\sqrt{h}}$ or e.g. $\frac{dh}{dt} = \frac{200}{\sqrt{h}}$ then only the first 2 method marks are available (see note below). Condone use of other variables if the intention is clear but the final answer must be in terms of h and t .

M1: Separates the variables and integrates to obtain an equation of the form $...h^{\frac{3}{2}} = \lambda t \{+c\}$ oe

The constant of integration is not needed for this mark.

A1: $\frac{2}{3}h^{\frac{3}{2}} = \lambda t (+c)$ oe. The constant of integration is not needed for this mark.

Condone spurious notation for this intermediate mark e.g. integral signs left in after integrating.

dm1: Substitutes $t=0$ and $h=1.44$ and attempts to find c .

It is dependent on the previous method mark.

Do not be concerned with the “processing” to find “ c ” as long as they are using $t=0$ and $h=1.44$

May be implied by their value of c .

ddm1: Substitutes $t=8$ and $h=3.24$ and their c and attempts to find λ . Do not be concerned with the “processing” to find λ as long as they are using $t=8$ and $h=3.24$.

It is dependent on both previous method marks.

A1: Correct equation in the correct form from correct work. $h^{\frac{3}{2}} = 0.513t + 1.728$ or $h^{\frac{3}{2}} = \frac{513}{1000}t + \frac{216}{125}$

Must follow A1 earlier so do check if this has been obtained fortuitously. Allow 1.73 for 1.728

Note candidates who do not have a λ e.g. assume $\frac{dh}{dt} = \frac{1}{200\sqrt{h}}$ or e.g. $\frac{dh}{dt} = \frac{200}{\sqrt{h}}$ can use either $t=0$ and $h=1.44$ or $t=8$ and $h=3.24$ to find their constant of integration.

(b)Alternative:

M1: Finds the reciprocal of both sides and integrates to obtain an equation of the form $t = \dots h^{\frac{3}{2}} (+c)$

A1: $t = \frac{2h^{\frac{3}{2}}}{3\lambda} (+c)$ oe. The constant of integration is not needed for this mark.

dm1: Substitutes $t=0$ and $h=1.44$ and substitutes $t=8$ and $h=3.24$ and attempts to find λ or c .

It is dependent on the previous method mark.

Do not be concerned with the “processing” to find λ or c as long as they are using $t=0$ and $h=1.44$ and $t=8$ and $h=3.24$ and reach a value for λ or c . May be implied by their value(s).

ddm1: Complete attempt to find λ and c . **It is dependent on both previous method marks.**

Do not be concerned with the “processing” to find λ and c as long as they are using $t=0$ and $h=1.44$ and $t=8$ and $h=3.24$.

A1: Correct equation in the correct form. $h^{\frac{3}{2}} = 0.513t + 1.728$ or $h^{\frac{3}{2}} = \frac{513}{1000}t + \frac{216}{125}$

Must follow A1 earlier so do check if this has been obtained fortuitously. Allow 1.73 for 1.728

Special Case:

Some candidates are using the given equation in part (b) to find the value of A and the value of B using the given conditions. May score a maximum of 00110. This should be marked as follows:

M0A0: (No attempt to integrate)

M1: Substitutes $t=0$ and $h=1.44$ to find a value for B

dm1: Substitutes $t=8$ and $h=3.24$ with their value of B to find a value for A

A0: Since they have not used the given model.

(Allow full recovery in (c) if this equation is correct)

(c)

M1: Attempts to substitute $h=5$ into their equation which must be of the form $h^{\frac{3}{2}} = At + B$ or possibly a rearranged equation e.g. $h^{\frac{1}{2}} = \sqrt[3]{At + B}$ with values of A and B leading to a value for t .

Do not be concerned about the processing as long as they use $h=5$ and obtain a value for t even if t is negative.

A1: Awrt 18.4 minutes **following a correct equation in (b).**

The units are required but allow e.g. min, mins, but not just 18.4 and not m (which means metres)

Allow e.g. 18 minutes 25 seconds or 18 mins 26 secs

Note this may follow A0 in part (b) as they may have rearranged incorrectly in (b) but use a correct

equation in (c) e.g. $\frac{2}{3}h^{\frac{3}{2}} = \frac{171}{500}t + \frac{144}{125}$, $h^{\frac{1}{2}} = \sqrt[3]{\frac{513}{1000}t + \frac{216}{125}}$ or may come from the special case.

Apply isw following a correct time and units, e.g., 18.4 followed by 18 mins.



Question	Scheme	Marks	AOs
14 (a)	$\{u = 4 - \sqrt{h} \Rightarrow \} \frac{du}{dh} = -\frac{1}{2}h^{-\frac{1}{2}} \text{ or } \frac{dh}{du} = -2(4-u) \text{ or } \frac{dh}{du} = -2\sqrt{h}$	B1	1.1b
	$\left\{ \int \frac{dh}{4-\sqrt{h}} = \right\} \int \frac{-2(4-u)}{u} du$	M1	2.1
	$= \int \left(-\frac{8}{u} + 2 \right) du$	M1	1.1b
	$= -8\ln u + 2u \{+c\}$	M1	1.1b
		A1	1.1b
	$= -8\ln 4-\sqrt{h} + 2(4-\sqrt{h}) + c = -8\ln 4-\sqrt{h} - 2\sqrt{h} + k *$	A1*	2.1
		(6)	
(b)	$\left\{ \frac{dh}{dt} = \frac{t^{0.25}(4-\sqrt{h})}{20} = 0 \Rightarrow \right\} 4-\sqrt{h} = 0$	M1	3.4
	Deduces any of $0 < h < 16$, $0 \leq h < 16$, $0 < h \leq 16$, $0 \leq h \leq 16$, $h < 16$, $h \leq 16$ or all values up to 16	A1	2.2a
		(2)	
(c) Way 1	$\int \frac{1}{(4-\sqrt{h})} dh = \int \frac{1}{20} t^{0.25} dt$	B1	1.1b
	$-8\ln 4-\sqrt{h} - 2\sqrt{h} = \frac{1}{25} t^{1.25} \{+c\}$	M1	1.1b
		A1	1.1b
	$\{t = 0, h = 1 \Rightarrow \} -8\ln(4-1) - 2\sqrt{1} = \frac{1}{25} (0)^{1.25} + c$	M1	3.4
	$\Rightarrow c = -8\ln(3) - 2 \Rightarrow -8\ln 4-\sqrt{h} - 2\sqrt{h} = \frac{1}{25} t^{1.25} - 8\ln(3) - 2$ $\{h = 12 \Rightarrow \} -8\ln 4-\sqrt{12} - 2\sqrt{12} = \frac{1}{25} t^{1.25} - 8\ln(3) - 2$	dM1	3.1a
	$t^{1.25} = 221.2795202... \Rightarrow t = \sqrt[1.25]{221.2795...} \text{ or } t = (221.2795...)^{0.8}$	M1	1.1b
	$t = 75.154... \Rightarrow t = 75.2 \text{ (years) (3 sf) or awrt 75.2 (years)}$	A1	1.1b
	Note: You can recover work for part (c) in part (b)	(7)	
(c) Way 2	$\int_1^{12} \frac{20}{(4-\sqrt{h})} dh = \int_0^T t^{0.25} dt$	B1	1.1b
	$\left[20(-8\ln 4-\sqrt{h} - 2\sqrt{h}) \right]_1^{12} = \left[\frac{4}{5} t^{1.25} \right]_0^T$	M1	1.1b
		A1	1.1b
	$20(-8\ln(4-\sqrt{12}) - 2\sqrt{12}) - 20(-8\ln(4-1) - 2\sqrt{1}) = \frac{4}{5} T^{1.25} - 0$	M1	3.4
		dM1	3.1a
	$T^{1.25} = 221.2795202... \Rightarrow T = \sqrt[1.25]{221.2795...} \text{ or } T = (221.2795...)^{0.8}$	M1	1.1b
	$T = 75.154... \Rightarrow T = 75.2 \text{ (years) (3 sf) or awrt 75.2 (years)}$	A1	1.1b
	Note: You can recover work for part (c) in part (b)	(7)	
(15 marks)			

Notes for Question 14	
(a)	
B1:	See scheme. Allow $du = -\frac{1}{2}h^{-\frac{1}{2}}dh$, $dh = -2(4-u)du$, $dh = -2\sqrt{h}du$ o.e.
M1:	Complete method for applying $u = 4 - \sqrt{h}$ to $\int \frac{dh}{4 - \sqrt{h}}$ to give an expression of the form $\int \frac{k(4-u)}{u} du$; $k \neq 0$
Note:	Condone the omission of an integral sign and/or du
M1:	Proceeds to obtain an integral of the form $\int \left(\frac{A}{u} + B \right) \{du\}$; $A, B \neq 0$
M1:	$\int \left(\frac{A}{u} + B \right) \{du\} \rightarrow D \ln u + Eu$; $A, B, D, E \neq 0$; with or without a constant of integration
A1:	$\int \left(-\frac{8}{u} + 2 \right) \{du\} \rightarrow -8 \ln u + 2u$; with or without a constant of integration
A1*:	dependent on all previous marks Substitutes $u = 4 - \sqrt{h}$ into their integrated result and completes the proof by obtaining the printed result $-8 \ln 4 - \sqrt{h} - 2\sqrt{h} + k$. Condone the use of brackets instead of the modulus sign.
Note:	They must combine $2(4)$ and their $+c$ correctly to give $+k$
Note:	Going from $-8 \ln 4 - \sqrt{h} + 2(4 - \sqrt{h}) + c$ to $-8 \ln 4 - \sqrt{h} - 2\sqrt{h} + k$, with no intermediate working or with no incorrect working is required for the final A1* mark.
Note:	Allow A1* for correctly reaching $-8 \ln 4 - \sqrt{h} - 2\sqrt{h} + c + 8$ and stating $k = c + 8$
Note:	Allow A1* for correctly reaching $-8 \ln 4 - \sqrt{h} + 2(4 - \sqrt{h}) + k = -8 \ln 4 - \sqrt{h} - 2\sqrt{h} + k$
Alternative (integration by parts) method for the 2nd M, 3rd M and 1st A mark	
$\left\{ \int \frac{-2(4-u)}{u} du = \int \frac{2u-8}{u} du \right\} = (2u-8) \ln u - \int 2 \ln u du = (2u-8) \ln u - 2(u \ln u - u) \{+ c\}$	
2 nd M1:	Proceeds to obtain an integral of the form $(Au + B) \ln u - \int A \ln u \{du\}$; $A, B \neq 0$
3 rd M1:	Integrates to give $D \ln u + Eu$; $D, E \neq 0$; which can be simplified or un-simplified with or without a constant of integration.
Note:	Give 3 rd M1 for $(2u-8) \ln u - 2(u \ln u - u)$ because it is an un-simplified form of $D \ln u + Eu$
1 st A1:	Integrates to give $(2u-8) \ln u - 2(u \ln u - u)$ or $-8 \ln u + 2u$ o.e. with or without a constant of integration.
(b)	
M1:	Uses the context of the model and has an understanding that the tree keeps growing until $\frac{dh}{dt} = 0 \Rightarrow 4 - \sqrt{h} = 0$. Alternatively, they can write $\frac{dh}{dt} > 0 \Rightarrow 4 - \sqrt{h} > 0$
Note:	Accept $h = 16$ or 16 used in their inequality statement for this mark.
A1:	See scheme
Note:	A correct answer can be given M1 A1 from any working.

Notes for Question 14	
(c)	Way 1
B1:	Separates the variables correctly. dh and dt should not be in the wrong positions, although this mark can be implied by later working. Condone absence of integral signs.
M1:	Integrates $t^{0.25}$ to give $\lambda t^{1.25}$; $\lambda \neq 0$
A1:	Correct integration. E.g. $-8\ln 4-\sqrt{h} - 2\sqrt{h} = \frac{1}{25}t^{1.25}$ or $20(-8\ln 4-\sqrt{h} - 2\sqrt{h}) = \frac{4}{5}t^{1.25}$ $-8\ln 4-\sqrt{h} + 2(4-\sqrt{h}) = \frac{1}{25}t^{1.25}$ or $20(-8\ln 4-\sqrt{h} + 2(4-\sqrt{h})) = \frac{4}{5}t^{1.25}$ with or without a constant of integration, e.g. k , c or A
Note:	There is no requirement for modulus signs.
M1:	Some evidence of applying both $t = 0$ and $h = 1$ to their model (which can be a changed equation) which contains a constant of integration, e.g. k , c or A
dM1:	dependent on the previous M mark Complete process of finding their constant of integration, followed by applying $h = 12$ and their constant of integration to their changed equation
M1:	Rearranges their equation to make $t^{\text{their } 1.25} = \dots$ followed by a correct method to give $t = \dots$; $t > 0$
Note:	$t^{\text{their } 1.25} = \dots$ can be negative, but their ' $t = \dots$ ' must be positive
Note:	"their 1.25" cannot be 0 or 1 for this mark
Note:	Do not give this mark if $t^{\text{their } 1.25} = \dots$ (usually $t^{0.25} = \dots$) is a result of substituting $t = 12$ (or $t = 11$) into the given $\frac{dh}{dt} = \frac{t^{0.25}(4-\sqrt{h})}{20}$. Note: They will usually write $\frac{dh}{dt}$ as either 12 or 11.
A1:	awrt 75.2
(c)	Way 2
B1:	Separates the variables correctly. dh and dt should not be in the wrong positions, although this mark can be implied by later working.
Note:	Integral signs and limits are not required for this mark.
M1:	Same as Way 1 (ignore limits)
A1:	Same as Way 1 (ignore limits)
M1:	Applies limits of 1 and 12 to their model (i.e. to their changed expression in h) and subtracts
dM1:	dependent on the previous M mark Complete process of applying limits of 1 and 12 and 0 and T (or ' t ') appropriately to their changed equation
M1:	Same as Way 1
A1:	Same as Way 1

Question	Scheme	Marks	AOs
10(a)	$\frac{dV}{dt} = 0.45$ or $\frac{dV}{dt} = \pm 0.3V$	M1	3.1b
	$\frac{dV}{dt} = 0.45 - \frac{3}{10}V$ $20\frac{dV}{dt} = 9 - 6V$ *	A1*	2.1
		(2)	
(b)	e.g. $\frac{1}{9-6V} \frac{dV}{dt} = \frac{1}{20} \rightarrow \int \frac{1}{9-6V} dV = \int \frac{1}{20} dt$	B1	1.1b
	$\frac{1}{9-6V} \rightarrow \dots \ln 9-6V $	M1	1.1b
	$-\frac{1}{6} \ln 9-6V = \frac{t}{20} (+c)$	A1	1.1b
	$-\frac{1}{6} \ln 9-6V = \frac{t}{20} + c$ $9-6V = Ae^{-\frac{3t}{10}}$ $t=0, V=0.25 \Rightarrow A=(7.5)$	dM1	3.1a
	$9-6V = 7.5e^{-\frac{3t}{10}}$ $V = \frac{3}{2} - \frac{5}{4}e^{-\frac{3t}{10}}$	A1	2.1
		(5)	
Notes			
(a) Marks for part (a) may not be scored in part (b) M1: Either $\frac{dV}{dt} = 0.45$ or $\frac{dV}{dt} = \pm 0.3V$ o.e. e.g. $\frac{dV}{dt} = \frac{9}{20}$ or $\frac{dV}{dt} = \pm \frac{3}{10}V$ seen or implied by e.g. $\frac{dV}{dt} = 0.45 - \frac{3}{10}V$ (but not implied by just stating the given answer). Condone use of $\dot{V}$ It may be seen as part of their $\frac{dV}{dt}$ e.g. $\frac{dV}{dt} = 0.45 + V + 0.3V$ scores M1A0* Condone e.g. change in volume = (inflow – outflow =) $0.45 - 0.3V$ for this mark. A1*: Achieves $20\frac{dV}{dt} = 9 - 6V$ with no errors, following $\frac{dV}{dt} = 0.45 - \frac{3}{10}V$ o.e. (including the $\frac{dV}{dt}$ or $\dot{V}$ but note that it must be $\frac{dV}{dt}$ in the final line and not $\dot{V}$). change in volume = $0.45 - 0.3V \rightarrow 20\frac{dV}{dt} = 9 - 6V$ scores M1A0*. Ignore any units used in their working for both marks.			
(b) B1: Separates the variables correctly, e.g., $\int \frac{1}{9-6V} dV = \int \frac{1}{20} dt$ or $\int \frac{20}{9-6V} dV = \int \{1\} dt$ o.e. The integral symbol and/or dV and/or dt may be implied if they go on to integrate both sides to the correct form $\dots \ln \alpha(9-6V) = \dots t (+c)$ with or without the modulus brackets.			

M1:	Attempts to integrate the reciprocal term $\frac{\beta}{9-6V} \rightarrow \dots \ln 9-6V $ or $\rightarrow \dots \ln \alpha(6V-9) $ for some constant β (and α if used). Condone e.g. $\frac{20}{9-6V} \rightarrow \dots \ln 9-6V$ or $\rightarrow \dots \ln 6V-9$
A1:	Correct integration for both sides. They do not need the $+c$ for this mark. Note scoring this mark implies the earlier B1 (unless it is a verification attempt – see SC). Note that e.g. $-\frac{1}{6} \ln 3-2V = \frac{t}{20} (+c)$ or $-\frac{10}{3} \ln 2V-3 = t (+c)$ are also correct. $-\frac{1}{6} \ln(9-6V) = \frac{t}{20} (+c)$ is also correct. Condone log being used in place of ln.
dM1:	Requires constant of integration now. Substitutes (or states) $t = 0$ and $V = 0.25$ and finds a value for c , which may be “A” = e^c if they rearrange first to eliminate ln terms. Dependent on the previous method mark. Do not be concerned about their processing to find c or “A” = e^c and does not need to be exact.
A1:	Achieves the required form e.g. $V = \frac{3}{2} - \frac{5}{4} e^{-\frac{3t}{10}}$ with no errors and clear working. Allow equivalent fractions or decimals e.g. $V = 1.5 - 1.25e^{-\frac{6}{20}t}$
SC:	Attempts by verification may score maximum B0M1A1dM1A0 – see below.
Alt:	Use of an integrating factor – see below.
(b)	Special Case: Attempts by verification may score maximum B0M1A1dM1A0
B0:	This mark may not be scored via this approach.
M1:	Differentiates $V = P - Qe^{-kt}$ to the form $\frac{dV}{dt} = \alpha e^{-kt}$ where α is a constant (note it should be $\frac{dV}{dt} = Qke^{-kt}$) and substitutes both this and $V = P - Qe^{-kt}$ into $20 \frac{dV}{dt} = 9 - 6V$ and deduces a value for P or k by comparing coefficients.
A1:	Correct values for both P and k .
dM1:	Substitutes (or states) $t = 0$ and $V = 0.25$ and finds a value for Q . Requires a value for P to have been found using the above approach.
A0:	This mark may not be scored via this approach.
	Alternative: Using Integrating Factor (Further Maths)
B1:	Deduces the correct integrating factor for the equation, $e^{0.3t}$ This should come from $\frac{dV}{dt} + 0.3V = 0.45 \Rightarrow \text{I.F.} = e^{\int 0.3dt} = e^{0.3t}$ May be implied by sight of $\frac{d(Ve^{0.3t})}{dt} = \dots$
M1:	Fully multiplies through by their integrating factor and integrates both sides. Score for $Ve^{kt} = \int \dots e^{kt} dt = \dots e^{kt}$ Condone missing dt
A1:	Correct integration $Ve^{0.3t} = \int 0.45e^{0.3t} dt = \frac{3}{2} e^{0.3t} (+c)$
dM1:	As main scheme.
A1:	As main scheme.

Question	Scheme	Marks	AOs
10(c)	Examples: (1) $\frac{dV}{dt} = 0 \Rightarrow V = (1.5)$ (or e.g. max V is 1.5) (2) As $t \rightarrow \infty$, $e^{-0.3t} \rightarrow 0$ (or $V \rightarrow "1.5"$) (3) Flow in = flow out at max V so $0.3V = 0.45 \Rightarrow V = 1.5$ (4) As $e^{-0.3t} > 0$, $V < "1.5"$ (5) When $V > 1.5$, $\frac{dV}{dt} < 0$ (6) $V = 2 \Rightarrow \frac{dV}{dt} = -0.15$ or compares $\frac{dV_{\text{out}}}{dt}$ ($= 0.6$) against $\frac{dV_{\text{in}}}{dt}$ ($= 0.45$) at $V = 2$ (7) $V = 2 \Rightarrow "1.5" - "1.25"e^{-0.3t} = 2 \Rightarrow e^{-0.3t} < 0$ (8) $"1.5" - "1.25"e^{-0.3t} = 2 \Rightarrow \ln(-0.4)$ is undefined (condone e.g. gives a maths error)	M1	3.2a
	- The (upper) limit for V is 1.5 (m^3) so no (the container will not become full) (first 4 bullets) - If $V = 2$ (or $V > 1.5$), it would be emptying so no (it can never be full) (bullets 5, 6) - No, as the equation cannot be solved (or is not true/doesn't work) when $V = 2$ (bullets 7, 8)	A1ft	2.4
		(2)	
(9 marks)			
Notes			
(c) M1: See main scheme. If using the answer to part (b) it must be of the form $V = P - Qe^{-kt}$ but there is no limitation on the values of their P , Q or k . Substitution of a large value for t may score this mark but it is unlikely to be recovered to score the A1 unless they reference e.g. V_{max} being "1.5". Reference to an (upper) limit of "1.5" or their P can imply the method mark. If setting $V = 2$ in their equation they must reach either $\ln(-\text{ve})$ or solve the equation to reach a value for t to score this mark. A1ft: Must conclude "no" or equivalent e.g. "the container will not become full". Makes a correct interpretation for their method (see bullets 1-8) with a clear conclusion e.g. "no". To score this mark through ft, their V must be of the form $V = P - Qe^{-kt}$ with $k > 0$, $Q > 0$ and $0 < P < 2$ if used (but note that they can still use the answer to part (a) to score both marks via bullets 1, 3, 5 or 6). Allow "it" in place of the "container"/"tank". Just stating "the equation cannot be solved when $V = 2$ " without any evidence is M0A0. There must be no incorrect working if solving their equation or contradictory statements such as " t cannot be negative" but condone notational errors provided the intention is clear.			