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EDEXCEL A LEVEL MATHEMATICS: QUESTIONS BY TOPIC

Integration (8)

A Level Mathematics Topic 8

21 questions · 163 marks · Mark schemes included

Pearson Edexcel A Level Mathematics (9MA0), June 2018 - June 2025

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HOW TO USE THIS BOOKLET

Questions are grouped by specification sub-topic and run from the lowest mark tariff to the highest within each one.

Click any row below to jump straight to that sub-topic or question.

On every question page, click ANSWER in the top-right corner to jump to its mark scheme.

Diagrams, graphs, tables and data are printed exactly as they appeared in the original papers.

QUESTION	SOURCE	MARKS	PAGE
8.2 INTEGRATING STANDARD FUNCTIONS		2 Qs · 11 marks	3
1. Q7	Paper 1: Pure Mathematics 1 June2018	7	4
2. Q2	Paper 2: Pure Mathematics 2 June2025	4	8
8.3 DEFINITE INTEGRALS AND AREAS		4 Qs · 38 marks	10
3. Q10 <i>also Topic 7</i>	Paper 1: Pure Mathematics 1 June2024	9	11
4. Q16 <i>also Topic 3 & 5</i>	Paper 1: Pure Mathematics 1 June2022	9	15
5. Q7 <i>also Topic 7</i>	Paper 2: Pure Mathematics 2 November2021	9	18
6. Q12 <i>also Topic 3 & 5</i>	Paper 2: Pure Mathematics 2 October2020	11	22
8.4 INTEGRATION AS THE LIMIT OF A SUM		1 Qs · 3 marks	26
7. Q5	Paper 1: Pure Mathematics 1 June2025	3	27
8.5 INTEGRATION BY SUBSTITUTION AND BY PARTS		9 Qs · 78 marks	29
8. Q13	Paper 1: Pure Mathematics 1 June2018	7	30
9. Q12 <i>also Topic 6</i>	Paper 2: Pure Mathematics 2 November2021	7	34
10. Q11 <i>also Topic 6 & 9</i>	Paper 1: Pure Mathematics 1 November2021	8	38
11. Q13 <i>also Topic 5</i>	Paper 1: Pure Mathematics 1 June2024	8	42
12. Q10 <i>also Topic 6</i>	Paper 1: Pure Mathematics 1 October2020	10	46
13. Q13 <i>also Topic 2 & 6</i>	Paper 2: Pure Mathematics 2 June2023	13	50
14. Q13	Paper 1: Pure Mathematics 1 June2025	5	54
15. Q16	Paper 1: Pure Mathematics 1 June2025	7	56
16. Q15 <i>also Topic 7</i>	Paper 2: Pure Mathematics 2 June2025	13	59
8.6 INTEGRATION USING PARTIAL FRACTIONS		1 Qs · 7 marks	64
17. Q10 <i>also Topic 2</i>	Paper 2: Pure Mathematics 2 June2023	7	65
8.8 DIFFERENTIAL EQUATIONS IN CONTEXT		1 Qs · 9 marks	69
18. Q10 <i>also Topic 7</i>	Paper 2: Pure Mathematics 2 June2025	9	70
9.4 THE TRAPEZIUM RULE		3 Qs · 17 marks	74
19. Q1 <i>also Topic 9</i>	Paper 2: Pure Mathematics 2 October2020	5	75
20. Q5 <i>also Topic 9</i>	Paper 1: Pure Mathematics 1 June2023	6	77
21. Q5 <i>also Topic 9</i>	Paper 2: Pure Mathematics 2 June2022	6	79



QUESTIONS

21 questions · 163 marks · 7 sub-topics



SUB-TOPIC 8.2

INTEGRATING STANDARD FUNCTIONS

Sub-topic 8.2 · 2 questions · 11 marks · lowest tariff first



7. Given that $k \in \mathbb{Z}^+$

(a) show that $\int_k^{3k} \frac{2}{(3x-k)} dx$ is independent of k ,

(b) show that $\int_k^{2k} \frac{2}{(2x-k)^2} dx$ is inversely proportional to k . (3)



Question 7 continued

Lined area for writing the answer to Question 7.

(Total for Question 7 is 7 marks)



2. Find

$$\int \left(x^4 - 6x^{\frac{1}{2}} - 3 \right) dx$$

giving the answer in simplest form.

(4)



SUB-TOPIC 8.3

DEFINITE INTEGRALS AND AREAS

Sub-topic 8.3 · 4 questions · 38 marks · lowest tariff first



10.

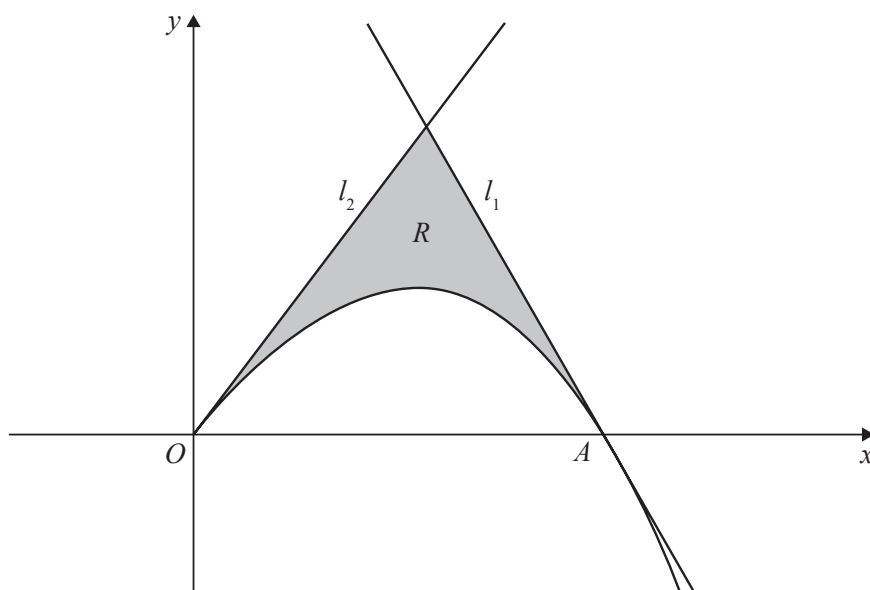


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve with equation

$$y = 8x - x^{\frac{5}{2}} \quad x \geq 0$$

The curve crosses the x -axis at the point A .

- (a) Verify that the x coordinate of A is 4

(1)

The line l_1 is the tangent to the curve at A .

- (b) Use calculus to show that an equation of line l_1 is

$$12x + y = 48$$

(3)

The line l_2 has equation $y = 8x$

The region R , shown shaded in Figure 3, is bounded by the curve, the line l_1 and the line l_2

- (c) Use algebraic integration to find the exact area of R .

(5)







(Total for Question 10 is 9 marks)







7.

In this question you should show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

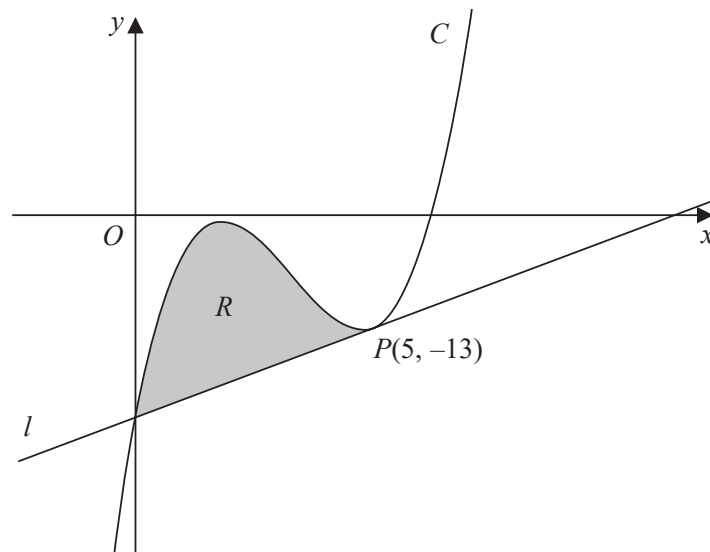


Figure 2

Figure 2 shows a sketch of part of the curve C with equation

$$y = x^3 - 10x^2 + 27x - 23$$

The point $P(5, -13)$ lies on C The line l is the tangent to C at P

- (a) Use differentiation to find the equation of l , giving your answer in the form $y = mx + c$ where m and c are integers to be found. (4)

- (b) Hence verify that l meets C again on the y -axis. (1)

The finite region R , shown shaded in Figure 2, is bounded by the curve C and the line l .

- (c) Use algebraic integration to find the exact area of R . (4)





(Total for Question 7 is 5 marks)



12.

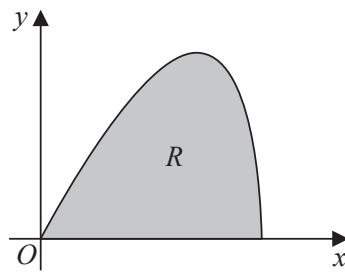


Figure 3

The curve shown in Figure 3 has parametric equations

$$x = 6 \sin t \quad y = 5 \sin 2t \quad 0 \leq t \leq \frac{\pi}{2}$$

The region R , shown shaded in Figure 3, is bounded by the curve and the x -axis.

- (a) (i) Show that the area of R is given by $\int_0^{\frac{\pi}{2}} 60 \sin t \cos^2 t \, dt$ (3)

- (ii) Hence show, by algebraic integration, that the area of R is exactly 20 (3)

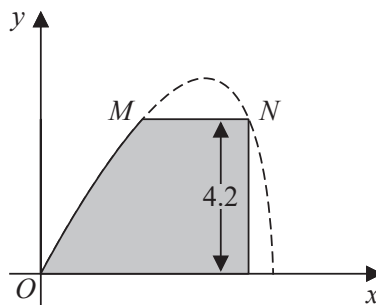


Figure 4

Part of the curve is used to model the profile of a small dam, shown shaded in Figure 4. Using the model and given that

- x and y are in metres
- the vertical wall of the dam is 4.2 metres high
- there is a horizontal walkway of width MN along the top of the dam

- (b) calculate the width of the walkway. (5)





(Total for Question 12 is 11 marks)



SUB-TOPIC 8.4

INTEGRATION AS THE LIMIT OF A SUM

Sub-topic 8.4 · 1 question · 3 marks · lowest tariff first



SUB-TOPIC 8.5

INTEGRATION BY SUBSTITUTION AND BY PARTS

Sub-topic 8.5 · 9 questions · 78 marks · lowest tariff first



13. Show that

$$\int_0^2 2x\sqrt{x+2} \, dx = \frac{32}{15}(2 + \sqrt{2})$$

(7)



Question 13 continued





Question 13 continued

(Total for Question 13 is 7 marks)



12. (a) Use the substitution $u = 1 + \sqrt{x}$ to show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}} \, dx = \int_p^q \frac{2(u-1)^3}{u} \, du$$

where p and q are constants to be found.

(3)

(b) Hence show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}} \, dx = A - B \ln 5$$

where A and B are constants to be found.

(4)

Question 12 continued



Question 12 continued



Question 12 continued

(Total for Question 12 is 7 marks)



11.

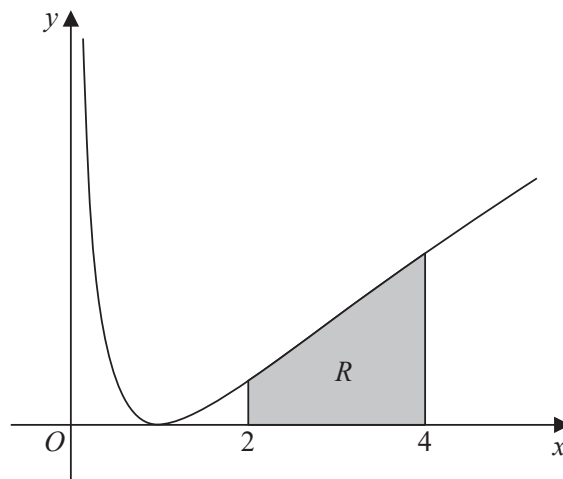


Figure 2

Figure 2 shows a sketch of part of the curve with equation

$$y = (\ln x)^2 \quad x > 0$$

The finite region R , shown shaded in Figure 2, is bounded by the curve, the line with equation $x = 2$, the x -axis and the line with equation $x = 4$

The table below shows corresponding values of x and y , with the values of y given to 4 decimal places.

x	2	2.5	3	3.5	4
y	0.4805	0.8396	1.2069	1.5694	1.9218

- (a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of R , giving your answer to 3 significant figures.

(3)

- (b) Use algebraic integration to find the exact area of R , giving your answer in the form

$$y = a(\ln 2)^2 + b \ln 2 + c$$

where a , b and c are integers to be found.

(5)

Question 11 continued



Question 11 continued



Question 11 continued

(Total for Question 11 is 8 marks)



13. (a) Given that a is a positive constant, use the substitution $x = a \sin^2 \theta$ to show that

$$\int_0^a x^{\frac{1}{2}} \sqrt{a-x} \, dx = \frac{1}{2} a^2 \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta \quad (4)$$

(b) Hence use algebraic integration to show that

$$\int_0^a x^{\frac{1}{2}} \sqrt{a-x} \, dx = k\pi a^2$$

where k is a constant to be found.

(4)



Question 13 continued





Question 10 continued

(Total for Question 10 is 10 marks)



Question 13 continued





Question 13 continued

(Total for Question 13 is 13 marks)



16.

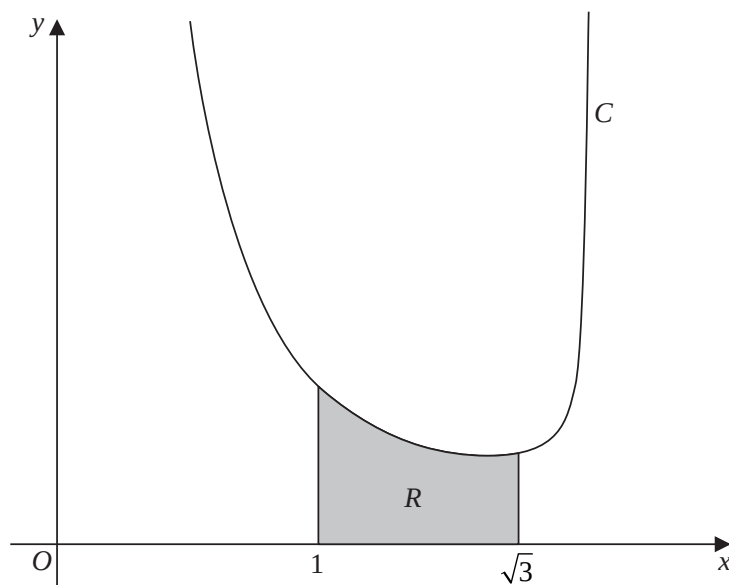


Figure 5

Figure 5 shows a sketch of the curve C with equation

$$y = \frac{1}{x^2 \sqrt{4 - x^2}} \quad 0 < x < 2$$

The region R , shown shaded in Figure 5, is bounded by C , the line with equation $x = 1$, the x -axis and the line with equation $x = \sqrt{3}$.

(a) Use the substitution $x = 2 \sin u$ to show that the area of R is given by

$$\int_a^b k \operatorname{cosec}^2 u \, du$$

where a , b and k are constants to be found.

(4)

(b) Hence, using algebraic integration, find the exact area of R .
Give your answer in simplest form.

(3)

15.

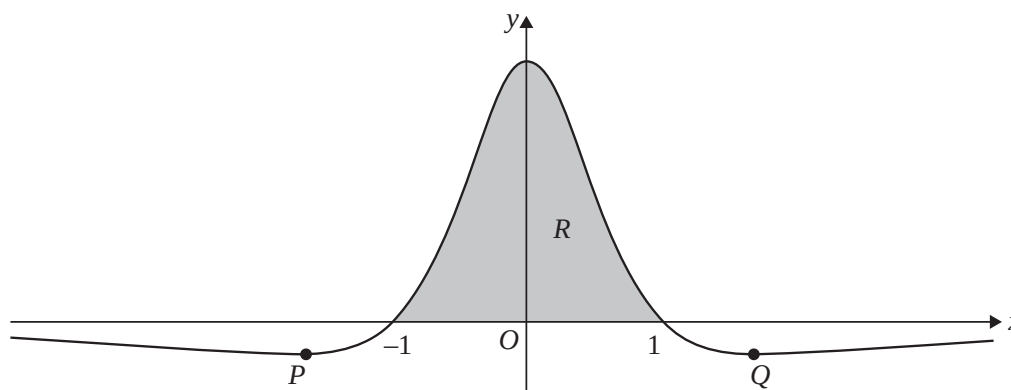


Figure 5

**In this question you must show all stages of your working.
Solutions relying on calculator technology are not acceptable.**

Figure 5 shows a sketch of part of the curve with equation $y = f(x)$, where

$$f(x) = \frac{1 - x^2}{(1 + x^2)^2}$$

The curve

- intersects the x -axis at -1 and 1
- has minimum turning points at P and Q

as shown in Figure 5.

(a) Use calculus to find the exact coordinates of P .

(5)

(b) Using the substitution $x = \tan \theta$ show that

$$\int_{-1}^1 f(x) dx = \int_{\alpha}^{\beta} \cos 2\theta d\theta$$

where α and β are constants to be found.

(5)

The finite region R , shown shaded in Figure 5, is bounded by the x -axis and the curve.

(c) Use algebraic integration to find the area of R .

(3)

Question 15 continued



Question 15 continued

(Total for Question 15 is 13 marks)

TOTAL FOR PAPER IS 100 MARKS



SUB-TOPIC 8.6

INTEGRATION USING PARTIAL FRACTIONS

Sub-topic 8.6 · 1 question · 7 marks · lowest tariff first









SUB-TOPIC 8.8

DIFFERENTIAL EQUATIONS IN CONTEXT

Sub-topic 8.8 · 1 question · 9 marks · lowest tariff first





Question 10 continued

(Total for Question 10 is 9 marks)



SUB-TOPIC 9.4

THE TRAPEZIUM RULE

Sub-topic 9.4 · 3 questions · 17 marks · lowest tariff first



MARK SCHEMES

official mark schemes



Question	Scheme	Marks	AOs
7 (a)	$\int \frac{2}{(3x-k)} dx = \frac{2}{3} \ln(3x-k)$	M1 A1	1.1a 1.1b
	$\int_k^{3k} \frac{2}{(3x-k)} dx = \frac{2}{3} \ln(9k-k) - \frac{2}{3} \ln(3k-k)$	dM1	1.1b
	$= \frac{2}{3} \ln\left(\frac{8k}{2k}\right) = \frac{2}{3} \ln 4$ oe	A1	2.1
		(4)	
(b)	$\int \frac{2}{(2x-k)^2} dx = -\frac{1}{(2x-k)}$	M1	1.1b
	$\int_k^{2k} \frac{2}{(2x-k)^2} dx = -\frac{1}{(4k-k)} + \frac{1}{(2k-k)}$	dM1	1.1b
	$= \frac{2}{3k} \left(\propto \frac{1}{k} \right)$	A1	2.1
		(3)	

(7 marks)

(a)

M1: $\int \frac{2}{(3x-k)} dx = A \ln(3x-k)$ Condone a missing bracket

A1: $\int \frac{2}{(3x-k)} dx = \frac{2}{3} \ln(3x-k)$

Allow recovery from a missing bracket if in subsequent work $A \ln 9k - k \rightarrow A \ln 8k$

dM1: For substituting k and $3k$ into their $A \ln(3x-k)$ and subtracting either way around

A1: Uses correct $\ln$ work and notation to show that $I = \frac{2}{3} \ln\left(\frac{8}{2}\right)$ or $\frac{2}{3} \ln 4$ oe (ie independent of k)

(b)

M1: $\int \frac{2}{(2x-k)^2} dx = \frac{C}{(2x-k)}$

dM1: For substituting k and $2k$ into their $\frac{C}{(2x-k)}$ and subtracting

A1: Shows that it is inversely proportional to k Eg proceeds to the answer is of the form $\frac{A}{k}$ with $A = \frac{2}{3}$

There is no need to perform the whole calculation. Accept from $-\frac{1}{(3k)} + \frac{1}{(k)} = \left(-\frac{1}{3} + 1\right) \times \frac{1}{k} \propto \frac{1}{k}$

If the calculation is performed it must be correct.

Do not isw here. They should know when they have an expression that is inversely proportional to k .

You may see substitution used but the mark is scored for the same result. See below

$u = 2x - k \rightarrow \left[\frac{C}{u} \right]$ for M1 with limits $3k$ and k used for dM1



Question	Scheme	Marks	AOs
2	$x^4 \rightarrow \dots x^5$ or $x^{\frac{1}{2}} \rightarrow \dots x^{\frac{3}{2}}$ or $-3 \rightarrow \dots x$	M1	1.1a
	Any of $\frac{1}{5}x^5$ or $-\frac{6}{\left(\frac{3}{2}\right)}x^{\frac{3}{2}}$ or $-3x$	A1	1.1b
	Any two of $\frac{1}{5}x^5$ or $-\frac{6}{\left(\frac{3}{2}\right)}x^{\frac{3}{2}}$ or $-3x$	A1	1.1b
	$\frac{1}{5}x^5 - 4x^{\frac{3}{2}} - 3x + c$	A1	1.1b
		(4)	
(4 marks)			
Notes			
M1: For increasing any power by one. Score for $x^n \rightarrow x^{n+1}$ in any term, including, $-3 \rightarrow \dots x$ where ... is a constant, but not for $+c$. Allow the indices to be unprocessed, e.g., x^{4+1} A1: One correct term which may be unsimplified and indices may be unprocessed. Condone e.g. $-3x^1$ or $-\frac{6}{\left(\frac{1}{2}+1\right)}x^{\frac{1}{2}+1}$ for this mark. Not scored for $+c$ A1: Two correct terms which may be unsimplified but indices must be processed. Condone $-3x^1$ for this mark. Not scored for $+c$ A1: cao Requires all terms simplified and $+c$ Ignore the LHS i.e. ignore what they call their integral. Allow $0.2x^5$ for $\frac{1}{5}x^5$ and $-4\sqrt{x^3}$ or $-4\sqrt{x^3}$ or $-4x\sqrt{x}$ or $-4x^{1.5}$ for $-4x^{\frac{3}{2}}$ Do not allow $-3x^1$ for this mark. Condone spurious integral signs e.g. $\int \frac{1}{5}x^5 - 4x^{\frac{3}{2}} - 3x + c$ or dx left in their answer ISW after a correct expression seen e.g. if they multiply through by 5 or e.g. try to solve $= 0$ Do not allow e.g. $\frac{1}{5}x^5 + -4x^{\frac{3}{2}} + -3x + c$			

Question	Scheme	Marks	AOs
10(a)	$8(4) - 4^{\frac{5}{2}} = 32 - 32 = 0$	B1	1.1b
		(1)	
(b)	$8 - \frac{5}{2}x^{\frac{3}{2}}$	B1	1.1b
	$x = 4 \Rightarrow \left\{ \frac{dy}{dx} \right\} 8 - \frac{5}{2} \times 8 = -12$ $\Rightarrow y\{-0\} = "-12"(x-4)$	M1	1.1b
	$12x + y = 48$ *	A1*	1.1b
		(3)	
(c)	Attempts to find one of the coordinates of the point of intersection $y = 8x, 12x + y = 48 \Rightarrow y = 19.2$ (or $x = 2.4$)	M1	1.1b
	Triangle area is $\frac{1}{2} \times 4 \times "19.2" \left(= 38.4 \text{ or } \frac{192}{5} \right)$ or $\int_0^{2.4} 8x \, dx + \int_{2.4}^4 "(48 - 12x)" \, dx$	dM1	3.1a
	$\int \left(8x - x^{\frac{5}{2}} \right) dx = 4x^2 - \frac{2}{7}x^{\frac{7}{2}}$	B1	1.1b
	$A = 38.4 - \left[4x^2 - \frac{2}{7}x^{\frac{7}{2}} \right]_0^4 = 38.4 - 64 + \frac{256}{7}$	ddM1	3.1a
	$= \frac{384}{35}$	A1	1.1b
		(5)	

(9 marks)**Notes****(a)****B1:** Substitutes $x = 4$ into the equation of the curve and verifies that $y = 0$. Accept " $8(4) - 4^{\frac{5}{2}} = 0$ "Alternatively, sets $8x - x^{\frac{5}{2}} = 0$ and solves with correct processing to achieve $x = 4$.As a minimum accept e.g. $8x - x^{\frac{5}{2}} = 0 \Rightarrow x^{\frac{3}{2}} = 8 \Rightarrow \{x = \} 4$ which may follow factorisation.**(b)****B1:** Correct differentiation. The $\frac{dy}{dx} =$ need not be present.**M1:** Correct method for finding the equation of the tangent at $A(4, 0)$.Requires substitution of $x = 4$ into their $\frac{dy}{dx}$ and an attempt at the equation of the line using this gradient. If using $y - y_1 = m(x - x_1)$ then condone the omission of the $- 0$.If $y = mx + c$ is used they must proceed as far as $c = \dots$ Accept $\frac{dy}{dx} = -12$ or $m = -12$ without explicit substitution of $x = 4$ provided $8 - \frac{5}{2}x^{\frac{3}{2}}$ is seen.

A1*: Correct work leading to the given equation having scored B1M1.

Condone $y + 12x = 48$ and apply isw once seen.

Do not condone $12x + y - 48 = 0$ (unless a correct equation $= 48$ is seen).

(c) **Note:** Condone poor notation such as missing dx or spurious $\int$ symbols throughout.

M1: Attempts to find either the x or y coordinate of the intersection of line l_1 and line l_2

You may need to check the diagram or limits to their integrals.

dM1: Correct method for the area of the triangle. e.g. Triangle area is $\frac{1}{2} \times 4 \times "19.2" \left(= 38.4 \text{ or } \frac{192}{5} \right)$

If integration is attempted then condone slips in their rearrangement of $12x + y = 48$ to $y = 48 - 12x$ and note that their integrals do need not to be evaluated, so for example

$$\text{look for } \int_0^{2.4} 8x \, dx + \int_{2.4}^4 "(48 - 12x)" \, dx \quad \left\{ = \frac{576}{25} + \frac{384}{25} = 23.04 + 15.36 \right\}$$

B1: Correct integration of **curve** ignoring limits, i.e. $4x^2 - \frac{2}{7}x^{\frac{7}{2}}$ but condone e.g. $\frac{8x^{1+1}}{2} - \frac{x^{\frac{5}{2}+1}}{\frac{7}{2}}$

ddM1: Fully correct strategy including substitution which would lead to an **exact** area.

Does not need to reach a value. Dependent on both previous M marks.

Implied by $38.4 - \frac{192}{7}$ or a correct final answer $\frac{384}{35}$

Note that the decimal approximation that might be seen is 10.97142857 and implies ddM0 unless there is evidence of substitution (which need not be evaluated).

A1: Correct exact value. Either $\frac{384}{35}$ or $10\frac{34}{35}$

Alternative using lines – curve:

M1: Attempts to find either the x or y coordinate of the intersection of line l_1 and line l_2

You may need to check the diagram or limits to their integrals.

dM1: Correct method for at least one part (0 to "2.4" or "2.4" to 4) of the area of R including limits.

Condone slips in their rearrangement of $12x + y = 48$ to $y = 48 - 12x$ and note that their integrals do need not to be evaluated, so for example

$$\text{look for } \int_0^{2.4} 8x - \left(8x - x^{\frac{5}{2}} \right) dx \text{ or } \int_{2.4}^4 "(48 - 12x) - \left(8x - x^{\frac{5}{2}} \right) dx \text{ (or a sum of both)}$$

B1: Correct integration of **both** regions ignoring limits. May be completed as a sum or separately.

Condone e.g. $\frac{x^{\frac{5}{2}+1}}{\frac{7}{2}}$ in place of $\frac{2}{7}x^{\frac{7}{2}}$ Note that each integral may have been simplified.

$$\int_{\dots}^{\dots} x^{\frac{5}{2}} dx \text{ and } \{+\} \int_{\dots}^{\dots} 48 - 20x + x^{\frac{5}{2}} dx \rightarrow \left[\frac{2}{7}x^{\frac{7}{2}} \right]_{\dots}^{\dots} \text{ and } \{+\} \left[48x - 10x^2 + \frac{2}{7}x^{\frac{7}{2}} \right]_{\dots}^{\dots}$$

ddM1: Fully correct strategy including substitution which would lead to an **exact** area.

Does not need to reach a value. Dependent on both previous M marks.

This approach requires:

- substitution of 0, their 2.4 and 4 in the correct places

- the $\frac{2}{7}(2.4)^{\frac{7}{2}} - \frac{2}{7}(2.4)^{\frac{7}{2}}$ to be cancelled (may be implied by a correct final answer $\frac{384}{35}$)

Note that the decimal approximation that might be seen is 10.97142857 and implies ddM0

unless there is evidence that the $\frac{2}{7}(2.4)^{\frac{7}{2}} - \frac{2}{7}(2.4)^{\frac{7}{2}}$ has been cancelled e.g. ~~6.118...~~ - ~~6.118...~~



Question	Scheme	Marks	AOs
16 (a)	Attempts $y \cdot \frac{dx}{dt} = (2 \sin 2t + 3 \sin t) \times 16 \sin t \cos t$ and uses $\sin 2t = 2 \sin t \cos t$	M1	2.1
	Correct expanded integrand. Usually for one of $(R) = \int \underline{48 \sin^2 t \cos t + 16 \sin^2 2t} dt$ $(R) = \int \underline{48 \sin^2 t \cos t + 64 \sin^2 t \cos^2 t} dt$ $(R) = \int \underline{24 \sin 2t \sin t + 16 \sin^2 2t} dt$	A1	1.1b
	Attempts to use $\cos 4t = 1 - 2 \sin^2 2t = (1 - 8 \sin^2 t \cos^2 t)$	M1	1.1b
	$R = \int_0^a 8 - 8 \cos 4t + 48 \sin^2 t \cos t dt$ *	A1*	2.1
	Deduces $a = \frac{\pi}{4}$	B1	2.2a
		(5)	
(b)	$\int 8 - 8 \cos 4t + 48 \sin^2 t \cos t dt = 8t - 2 \sin 4t + 16 \sin^3 t$	M1 A1	2.1 1.1b
	$\left[8t - 2 \sin 4t + 16 \sin^3 t \right]_0^{\frac{\pi}{4}} = 2\pi + 4\sqrt{2}$	M1 A1	2.1 1.1b
		(4)	
(9 marks)			
Notes:			

(a) Condone work in another variable, say $\theta \leftrightarrow t$ if used consistently for the first 3 marks

M1: For the key step in attempting $y \cdot \frac{dx}{dt} = (2 \sin 2t + 3 \sin t) \times 16 \sin t \cos t$ with an attempt to use

$\sin 2t = 2 \sin t \cos t$ Condone slips in finding $\frac{dx}{dt}$ but it must be of the form $k \sin t \cos t$

E.g. I $y \cdot \frac{dx}{dt} = (2 \sin 2t + 3 \sin t) \times k \sin t \cos t = (4 \sin t \cos t + 3 \sin t) \times k \sin t \cos t$

E.g. II $y \cdot \frac{dx}{dt} = (2 \sin 2t + 3 \sin t) \times k \sin t \cos t = (2 \sin 2t + 3 \sin t) \times \frac{k}{2} \sin 2t$

A1: A correct (expanded) integrand in t . Don't be concerned by the absence of $\int$ or dt or limits

$$(R) = \int \underline{48 \sin^2 t \cos t + 16 \sin^2 2t} dt \quad \text{or} \quad (R) = \int \underline{48 \sin^2 t \cos t + 64 \sin^2 t \cos^2 t} dt$$

but watch for other correct versions such as $(R) = \int \underline{24 \sin 2t \sin t + 16 \sin^2 2t} dt$

Question	Scheme	Marks	AOs
7(a)	$y = x^3 - 10x^2 + 27x - 23 \Rightarrow \frac{dy}{dx} = 3x^2 - 20x + 27$	B1	1.1b
	$\left(\frac{dy}{dx}\right)_{x=5} = 3 \times 5^2 - 20 \times 5 + 27 (= 2)$	M1	1.1b
	$y + 13 = 2(x - 5)$	M1	2.1
	$y = 2x - 23$	A1	1.1b
		(4)	
(b)	Both C and l pass through $(0, -23)$ and so C meets l again on the y -axis	B1	2.2a
		(1)	
(c)	$\pm \int (x^3 - 10x^2 + 27x - 23 - (2x - 23)) dx$ $= \pm \left(\frac{x^4}{4} - \frac{10}{3}x^3 + \frac{25}{2}x^2 \right)$	M1 A1ft	1.1b 1.1b
	$\left[\frac{x^4}{4} - \frac{10}{3}x^3 + \frac{25}{2}x^2 \right]_0^5$ $= \left(\frac{625}{4} - \frac{1250}{3} + \frac{625}{2} \right) (-0)$	dM1	2.1
	$= \frac{625}{12}$	A1	1.1b
		(4)	
(c) Alternative 1:			
	$\pm \int (x^3 - 10x^2 + 27x - 23) dx$ $= \pm \left(\frac{x^4}{4} - \frac{10}{3}x^3 + \frac{27}{2}x^2 - 23x \right)$	M1 A1	1.1b 1.1b
	$\left[\frac{x^4}{4} - \frac{10}{3}x^3 + \frac{27}{2}x^2 - 23x \right]_0^5 + \frac{1}{2} \times 5(23 + 13)$ $= -\frac{455}{12} + 90$	dM1	2.1
	$= \frac{625}{12}$	A1	1.1b
(c) Alternative 2:			
	$\int (x^3 - 10x^2 + 27x) dx = \left(\frac{x^4}{4} - \frac{10}{3}x^3 + \frac{27}{2}x^2 \right)$	M1 A1	1.1b 1.1b
	$\left[\frac{x^4}{4} - \frac{10}{3}x^3 + \frac{27}{2}x^2 \right]_0^5 - \frac{1}{2} \times 5 \times 10$	dM1	2.1
	$= \frac{625}{12}$	A1	1.1b
(9 marks)			

Notes

(a)

B1: Correct derivative

M1: Substitutes $x = 5$ into their derivative. This may be implied by their value for $\frac{dy}{dx}$ M1: Fully correct straight line method using $(5, -13)$ and their $\frac{dy}{dx}$ at $x = 5$

A1: cao. Must see the full equation in the required form.

(b)

B1: Makes a suitable deduction.

Alternative via equating l and C and factorising e.g.

$$x^3 - 10x^2 + 27x - 23 = 2x - 23$$

$$x^3 - 10x^2 + 25x = 0$$

$$x(x^2 - 10x + 25) = 0 \Rightarrow x = 0$$

So they meet on the y -axis

(c)

M1: For an attempt to integrate $x^n \rightarrow x^{n+1}$ for $\pm "C - l"$

A1ft: Correct integration in any form which may be simplified or unsimplified. (follow through their equation from (a))

If they attempt as 2 separate integrals e.g. $\int (x^3 - 10x^2 + 27x - 23) dx - \int (2x - 23) dx$ then

award this mark for the correct integration of the curve as in the alternative.

If they combine the curve with the line first then the subsequent integration must be correct or a correct ft for their line and allow for $\pm "C - l"$

dM1: Fully correct strategy for the area. Award for use of 5 as the limit and condone the omission of the -0 . **Depends on the first method mark.**

A1: Correct exact value

Alternative 1:M1: For an attempt to integrate $x^n \rightarrow x^{n+1}$ for $\pm C$ A1: Correct integration for $\pm C$

dM1: Fully correct strategy for the area e.g. correctly attempts the area of the trapezium and subtracts the area enclosed between the curve and the x -axis. Need to see the use of 5 as the limit condoning the omission of the -0 **and** a correct attempt at the trapezium **and** the subtraction.

May see the trapezium area attempted as $\int (2x - 23) dx$ in which case the integration and

use of the limits needs to be correct or correct follow through for their straight line equation.

Depends on the first method mark.

A1: Correct exact value

Note if they do $l - C$ rather than $C - l$ and the working is otherwise correct allow full marks if their final answer is given as a positive value. E.g. correct work with $l - C$ leading to $-\frac{625}{12}$ and

then e.g. hence area is $\frac{625}{12}$ is acceptable for full marks.

If the answer is left as $-\frac{625}{12}$ then score A0

Alternative 2:M1: For an attempt to integrate $x^n \rightarrow x^{n+1}$ for $(C + 23)$ A1: Correct integration for $(C + 23)$ 

subtracts from the area under the curve

Need to see the use of 5 as the limit condoning the omission of the “– 0” **and** a correct attempt at the triangle **and** the subtraction.

Depends on the first method mark.



Question	Scheme	Marks	AOs
12(a)(i)	$y \times \frac{dx}{dt} = 5 \sin 2t \times 6 \cos t \quad \text{or} \quad 5 \times 2 \sin t \cos t \times 6 \cos t$	M1	1.2



	$\begin{aligned} \text{(Area =)} \int 5 \sin 2t \times 6 \cos t \, dt &= \int 5 \times 2 \sin t \cos t \times 6 \cos t \, dt \\ &\text{or} \\ \int 5 \sin 2t \times 6 \cos t \, dt &= \int 60 \sin t \cos^2 t \, dt \end{aligned}$	dM1	1.1b
	$\text{(Area =)} \int_0^{\frac{\pi}{2}} 60 \sin t \cos^2 t \, dt *$	A1*	2.1*
		(3)	
(a)(ii)	$\int 60 \sin t \cos^2 t \, dt = -20 \cos^3 t$	M1 A1	1.1b 1.1b
	$\text{Area} = \left[-20 \cos^3 t \right]_0^{\frac{\pi}{2}} = 0 - (-20) = 20 *$	A1*	2.1
		(3)	
(b)	$5 \sin 2t = 4.2 \Rightarrow \sin 2t = \frac{4.2}{5}$	M1	3.4
	$t = 0.4986..., 1.072...$	A1	1.1b
	Attempts to find the x values at both t values	dM1	3.4
	$t = 0.4986... \Rightarrow x = 2.869...$ $t = 1.072 \Rightarrow x = 5.269...$	A1	1.1b
	Width of path = 2.40 metres	A1	3.2a
		(5)	
(11 marks)			

Notes:

(a)(i)

M1: Attempts to multiply y by $\frac{dx}{dt}$ to obtain $A \sin 2t \cos t$ but may apply $\sin 2t = 2 \sin t \cos t$ here

dM1: Attempts to use $\sin 2t = 2 \sin t \cos t$ within an integral which may be implied by

e.g. $A \int \sin 2t \times \cos t \, dt = \int k \sin t \cos^2 t \, dt$

A1*: Fully correct work leading to the given answer.

This must include $\sin 2t = 2 \sin t \cos t$ or e.g. $5 \sin 2t = 10 \sin t \cos t$ seen explicitly in their proof and a correct intermediate line that includes an integral sign and the “dt”

Allow the limits to just “appear” in the final answer e.g. working need not be shown for the limits.

(a)(ii)

M1: Obtains $\int 60 \sin t \cos^2 t \, dt = k \cos^3 t$. This may be attempted via a substitution of $u = \cos t$ to obtain

$$\int 60 \sin t \cos^2 t \, dt = k u^3$$

A1: Correct integration $-20 \cos^3 t$ or equivalent e.g. $-20u^3$

A1*: Rigorous proof with all aspects correct including the correct limits and the $0 - (-20)$ and

not just: $-20 \cos^3 \frac{\pi}{2} - (-20 \cos^3 0) = 20$

(b)

M1: Uses the given model and attempts to find value(s) of t when $\sin 2t = \frac{4.2}{5}$. Look for $2t = \sin^{-1} \frac{4.2}{5} \Rightarrow t = \dots$

A1: At least one correct value for t , correct to 2 dp. FYI $t = 0.4986..., 1.072...$ or in degrees $t = 28.57..., 61.42...$



Question	Scheme	Marks	AOs
5 (a)	$\lim_{\delta x \rightarrow 0} \sum_{x=1.44}^{2.89} \frac{2}{\sqrt{x}} \delta x = \int_{1.44}^{2.89} \frac{2}{\sqrt{x}} dx$	B1	1.2
		(1)	
(b)	$= [4\sqrt{x}]_{1.44}^{2.89} = 4 \times 1.7 - 4 \times 1.2$	M1	1.1b
	$= 2$	A1cso	1.1b
		(2)	
(3 marks)			

Notes:
Mark (a) and (b) together
(a)

B1: States that $\int_{1.44}^{2.89} \frac{2}{\sqrt{x}} dx$ or equivalent such as $2 \int_{1.44}^{2.89} x^{-\frac{1}{2}} dx$ or $2 \int_{1.44}^{2.89} x^{-0.5} dx$ but must include the limits and the dx. Condone $dx \leftrightarrow \delta x$ as it is very difficult to tell one from another sometimes.

(b)

M1: Uses $\int \frac{1}{\sqrt{x}} dx = a\sqrt{x}$ or $ax^{\frac{1}{2}}$ (allow a to be 1) **and** applies the given limits to their $ax^{\frac{1}{2}}$ subtracting either way round. (Condone with the constant of integration included) You do not need to be concerned by fractions within fractions as this is still of the required form e.g. $\frac{2x^{\frac{1}{2}}}{\frac{1}{2}}$. Only condone transcription errors of 2.89 or 1.44 when substituting the limits into the expression. This mark can be scored for

$$\text{e.g. } [4\sqrt{x}]_{1.44}^{2.89} = 4 \times \sqrt{2.89} - 4 \times \sqrt{1.44} \text{ or e.g. } \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} \rightarrow \frac{2(2.89)^{\frac{1}{2}} - 2(1.44)^{\frac{1}{2}}}{\frac{1}{2}}$$

May already be partially evaluated so allow e.g. $\frac{34}{5} - \frac{24}{5}$ o.e. provided it is not just 2.

A1: 2 cso

The method mark must have been awarded. Do not withhold this mark for poor notation or e.g. a missing dx in their solution.



Question	Scheme for Substitution		Marks	AOs
13	Chooses a suitable method for $\int_0^2 2x\sqrt{x+2} \, dx$ Award for - Using a valid substitution $u = \dots$, changing the terms to u's - integrating and using appropriate limits .		M1	3.1a
	Substitution $u = \sqrt{x+2} \Rightarrow \frac{dx}{du} = 2u$ oe	Substitution $u = x+2 \Rightarrow \frac{dx}{du} = 1$ oe	B1	1.1b
	$\int 2x\sqrt{x+2} \, dx$ $= \int A(u^2 \pm 2)u^2 du$	$\int 2x\sqrt{x+2} \, dx$ $= \int A(u \pm 2)\sqrt{u} du$	M1	1.1b
	$= Pu^5 \pm Qu^3$	$= Su^{\frac{5}{2}} \pm Tu^{\frac{3}{2}}$	dM1	2.1
	$= \frac{4}{5}u^5 - \frac{8}{3}u^3$	$= \frac{4}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}}$	A1	1.1b
	Uses limits 2 and $\sqrt{2}$ the correct way around	Uses limits 4 and 2 the correct way around	ddM1	1.1b
	$= \frac{32}{15}(2 + \sqrt{2}) *$		A1*	2.1
			(7)	

(7 marks)

M1: For attempting to integrate using substitution. Look for

- terms and limits changed to u 's. Condone slips and errors/omissions on changing $dx \rightarrow du$
- attempted multiplication of terms and raising of at least one power of u by one. Condone slips
- Use of at least the top correct limit. For instance if they go back to x 's the limit is 2

B1: For substitution it is for giving the substitution and stating a correct $\frac{dx}{du}$ Eg, $u = \sqrt{x+2} \Rightarrow \frac{dx}{du} = 2u$ or equivalent such as $\frac{du}{dx} = \frac{1}{2\sqrt{x+2}}$ **M1:** It is for attempting to get all aspects of the integral in terms of ' u '.All terms must be attempted including the dx . You are only condoning slips on signs and coefficients**dM1:** It is for using a correct method of expanding and integrating each term (seen at least once) . It is dependent upon the previous M**A1:** Correct answer in x or u See scheme**ddM1:** Dependent upon the previous M, it is for using the correct limits for their integral, **the correct way around****A1*:** Proceeds correctly to $= \frac{32}{15}(2 + \sqrt{2})$. **Note that this is a given answer**There must be at one least correct intermediate line between $\left[\frac{4}{5}u^5 - \frac{8}{3}u^3 \right]_{\sqrt{2}}^2$ and $\frac{32}{15}(2 + \sqrt{2})$

Question Alt	Scheme for by parts	Marks	AOs
13	Chooses a suitable method for $\int_0^2 2x\sqrt{x+2} \, dx$ Award for - using by parts the correct way around - and using limits	M1	3.1a
	$\int (\sqrt{x+2}) \, dx = \frac{2}{3}(x+2)^{\frac{3}{2}}$	B1	1.1b
	$\int 2x\sqrt{x+2} \, dx = Ax(x+2)^{\frac{3}{2}} - \int B(x+2)^{\frac{3}{2}}(dx)$	M1	1.1b
	$= Ax(x+2)^{\frac{3}{2}} - C(x+2)^{\frac{5}{2}}$	dM1	2.1
	$= \frac{4}{3}x(x+2)^{\frac{3}{2}} - \frac{8}{15}(x+2)^{\frac{5}{2}}$	A1	1.1b
	Uses limits 2 and 0 the correct way around	ddM1	1.1b
	$= \frac{32}{15}(2+\sqrt{2})$	A1*	2.1
		(7)	

M1: For attempting using by parts to solve It is a problem- solving mark and all elements do not have to be correct.

- the formula applied the correct way around. You may condone incorrect attempts at integrating $\sqrt{x+2}$ for this problem solving mark
- further integration, again, this may not be correct, and the use of at least the top limit of 2

B1: For $\int (\sqrt{x+2}) \, dx = \frac{2}{3}(x+2)^{\frac{3}{2}}$ oe May be awarded $\int_0^2 2x\sqrt{x+2} \, dx \rightarrow x^2 \times \frac{2(x+2)^{\frac{3}{2}}}{3}$

M1: For integration by parts the right way around. Award for $Ax(x+2)^{\frac{3}{2}} - \int B(x+2)^{\frac{3}{2}}(dx)$

dM1: For integrating a second time. Award for $Ax(x+2)^{\frac{3}{2}} - C(x+2)^{\frac{5}{2}}$

A1: $\frac{4}{3}x(x+2)^{\frac{3}{2}} - \frac{8}{15}(x+2)^{\frac{5}{2}}$ which may be un simplified

ddM1: Dependent upon the previous M, it is for using the limits 2 and 0 the **correct way around**

A1*: Proceeds to $= \frac{32}{15}(2+\sqrt{2})$. **Note that this is a given answer.**

At least one correct intermediate line must be seen. (See substitution). You would condone missing dx's



Question	Scheme	Marks	AOs
12(a)	$u = 1 + \sqrt{x} \Rightarrow x = (u-1)^2 \Rightarrow \frac{dx}{du} = 2(u-1)$ or $u = 1 + \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$	B1	1.1b
	$\int \frac{x}{1+\sqrt{x}} dx = \int \frac{(u-1)^2}{u} 2(u-1) du$ or $\int \frac{x}{1+\sqrt{x}} dx = \int \frac{x}{u} \times 2x^{\frac{1}{2}} du = \int \frac{2x^{\frac{3}{2}}}{u} du = \int \frac{2(u-1)^3}{u} du$	M1	2.1
	$\int_0^{16} \frac{x}{1+\sqrt{x}} dx = \int_1^5 \frac{2(u-1)^3}{u} du$	A1	1.1b
		(3)	
(b)	$2 \int \frac{u^3 - 3u^2 + 3u - 1}{u} du = 2 \int \left(u^2 - 3u + 3 - \frac{1}{u} \right) du = \dots$	M1	3.1a
	$= (2) \left[\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln u \right]$	A1	1.1b
	$= 2 \left[\frac{5^3}{3} - \frac{3(5)^2}{2} + 3(5) - \ln 5 - \left(\frac{1}{3} - \frac{3}{2} + 3 - \ln 1 \right) \right]$	dM1	2.1
	$= \frac{104}{3} - 2 \ln 5$	A1	1.1b
		(4)	
(7 marks)			
Notes			
(a) B1: Correct expression for $\frac{dx}{du}$ or $\frac{du}{dx}$ (or u') or dx in terms of du or du in terms of dx M1: Complete method using the given substitution. This needs to be a correct method for their $\frac{dx}{du}$ or $\frac{du}{dx}$ leading to an integral in terms of u only (ignore any limits if present) so for each case you need to see: $\frac{dx}{du} = f(u) \rightarrow \int \frac{x}{1+\sqrt{x}} dx = \int \frac{(u-1)^2}{u} f(u) du$ $\frac{du}{dx} = g(x) \rightarrow \int \frac{x}{1+\sqrt{x}} dx = \int \frac{x}{u} \times \frac{du}{g(x)} = \int h(u) du.$ In this case you can condone slips with coefficients e.g. allow $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \rightarrow \int \frac{x}{1+\sqrt{x}} dx = \int \frac{x}{u} \times \frac{x^{\frac{1}{2}}}{2} du = \int h(u) du$			

$$\text{but not } \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \rightarrow \int \frac{x}{1+\sqrt{x}} dx = \int \frac{x}{u} \times \frac{x^{-\frac{1}{2}}}{2} du = \int h(u) du$$

A1: All correct with correct limits and no errors. The “du” must be present but may have been omitted along the way but it must have been seen at least once before the final answer. The limits can be seen as part of the integral or stated separately.

(b)

M1: Realises the requirement to cube the bracket and divide through by u and makes progress with the integration to obtain at least 3 terms of the required form e.g. 3 from ku^3 , ku^2 , ku , $k \ln u$

A1: Correct integration. This mark can be scored with the “2” still outside the integral or even if it has been omitted. But if the “2” has been combined with the integrand, the integration must be correct.

dM1: Completes the process by applying their “changed” limits and subtracts the right way round

Depends on the first method mark.

A1: Cao (Allow equivalents for $\frac{104}{3}$ e.g. $\frac{208}{6}$)



Question	Scheme	Marks	AOs
11(a)	$h = 0.5$	B1	1.1b
	$A \approx \frac{1}{2} \times \frac{1}{2} \{0.4805 + 1.9218 + 2(0.8396 + 1.2069 + 1.5694)\}$	M1	1.1b
	$= 2.41$	A1	1.1b
		(3)	
(b)	$\int (\ln x)^2 \, dx = x(\ln x)^2 - \int x \times \frac{2 \ln x}{x} \, dx$	M1 A1	3.1a 1.1b
	$= x(\ln x)^2 - 2 \int \ln x \, dx = x(\ln x)^2 - 2 \left(x \ln x - \int dx \right)$ $= x(\ln x)^2 - 2 \int \ln x \, dx = x(\ln x)^2 - 2x \ln x + 2x$	dM1	2.1
	$\int_2^4 (\ln x)^2 \, dx = \left[x(\ln x)^2 - 2x \ln x + 2x \right]_2^4$ $= 4(\ln 4)^2 - 2 \times 4 \ln 4 + 2 \times 4 - \left(2(\ln 2)^2 - 2 \times 2 \ln 2 + 2 \times 2 \right)$ $= 4(2 \ln 2)^2 - 16 \ln 2 + 8 - 2(\ln 2)^2 + 4 \ln 2 - 4$	ddM1	2.1
	$= 14(\ln 2)^2 - 12 \ln 2 + 4$	A1	1.1b
		(5)	
(8 marks)			
Notes			

(a)

 B1: Correct strip width. May be implied by $\frac{1}{2} \times \frac{1}{2} \{....\}$ or $\frac{1}{4} \times \{....\}$

M1: Correct application of the trapezium rule.

 Look for $\frac{1}{2} \times "h" \{0.4805 + 1.9218 + 2(0.8396 + 1.2069 + 1.5694)\}$ condoning slips in the digits.

The bracketing must be correct but it is implied by awrt 2.41

A1: 2.41 only. This is not awrt

(b)

 M1: Attempts parts the correct way round to achieve $\alpha x(\ln x)^2 - \beta \int \ln x dx$ o.e.

 May be unsimplified (see scheme). Watch for candidates who know or learn $\int \ln x dx = x \ln x - x$

 who may write $\int (\ln x)^2 dx = \int (\ln x)(\ln x) dx = \ln x(x \ln x - x) - \int \frac{x \ln x - x}{x} dx$

A1: Correct expression which may be unsimplified

 dM1: Attempts parts again to (only condone coefficient errors) to achieve $\alpha x(\ln x)^2 - \beta x \ln x \pm \gamma x$ o.e.

 ddM1: Applies the limits 4 and 2 to an expression of the form $\pm \alpha x(\ln x)^2 \pm \beta x \ln x \pm \gamma x$, subtracts and applies $\ln 4 = 2 \ln 2$ at least once. Both M's must have been awarded

A1: Correct answer

 It is possible to do $\int (\ln x)^2 dx$ via a substitution $u = \ln x$ but it is very similar.

 M1 A1, dM1: $\int u^2 e^u du = u^2 e^u - \int 2u e^u du = u^2 e^u - 2ue^u \pm 2e^u$

 ddM1: Applies appropriate limits and uses $\ln 4 = 2 \ln 2$ at least once to an expression of the form $u^2 e^u - \beta u e^u \pm \gamma e^u$ Both M's must have been awarded

Question	Scheme	Marks	AOs
13(a)	$x = a \sin^2 \theta \Rightarrow \frac{dx}{d\theta} = 2a \sin \theta \cos \theta$	B1	1.1b
	$\int x^{\frac{1}{2}} \sqrt{a-x} dx = \int \sqrt{a} \sin \theta \sqrt{a-a \sin^2 \theta} \times 2a \sin \theta \cos \theta \{d\theta\}$	M1	2.1
	$= \int \sqrt{a} \sin \theta \sqrt{a} \cos \theta \times 2a \sin \theta \cos \theta d\theta = 2a^2 \int \sin^2 \theta \cos^2 \theta d\theta$ $= 2a^2 \int \left(\frac{1}{2} \sin 2\theta\right)^2 d\theta$	dM1	3.1a
	Replaces or considers limits $\{x=0 \Rightarrow\} \theta=0, \{x=a \Rightarrow\} \theta=\frac{\pi}{2}$ $= \frac{1}{2} a^2 \int_0^{\frac{\pi}{2}} \sin^2 2\theta d\theta \quad *$	A1*	1.1b
		(4)	
(b)	$\dots \int \sin^2 2\theta d\theta \rightarrow \dots \int \frac{1-\cos 4\theta}{2} d\theta$	M1	1.1b
	$\rightarrow \dots \left(\frac{\theta}{2} - \frac{1}{8} \sin 4\theta\right)$	dM1	2.1
	$\mu \int \sin^2 2\theta d\theta \rightarrow \frac{\mu}{2} \left(\theta - \frac{1}{4} \sin 4\theta\right)$	A1	1.1b
	$\left\{ \int_0^a x^{\frac{1}{2}} \sqrt{a-x} dx \right\} = \frac{a^2}{4} \left(\frac{\pi}{2} - 0 - 0\right) = \frac{1}{8} \pi a^2$	A1	1.1b
		(4)	

(8 marks)

Notes

(a)

B1: $\frac{dx}{d\theta} = 2a \sin \theta \cos \theta$ or $dx = 2a \sin \theta \cos \theta d\theta$ o.e. seen or implied by their substitution.

Note that writing $x = a \sin^2 \theta = \frac{a}{2} (1 - \cos 2\theta) \rightarrow \frac{dx}{d\theta} = a \sin 2\theta$ is correct. Condone use of $\frac{dx}{da}$

M1: Attempts to substitute, fully replacing $x^{\frac{1}{2}}$ and $\sqrt{a-x}$ with θ 's and dx with their $dx = \dots$

Look for $x^{\frac{1}{2}} \sqrt{a-x} dx \rightarrow f(\theta)g(\theta)h(\theta)$ where

- $f(\theta)$ is an attempt at $\sqrt{a \sin^2 \theta}$ e.g. allow $a \sin \theta$ but just $a \sin^2 \theta$ is not condoned
- $g(\theta)$ is an attempt at $\sqrt{a - a \sin^2 \theta}$ but not $\sqrt{a} - \sqrt{a \sin^2 \theta}$ unless $\sqrt{a - a \sin^2 \theta}$ is attempted first
- $h(\theta) =$ their dx or their $\frac{dx}{d\theta}$ or $\frac{1}{\text{their } \frac{dx}{d\theta}}$ (in terms of θ only but condone da seen)

Condone slips provided the intention is clear, e.g. $x^{\frac{1}{2}} \rightarrow \sqrt{a} \sin^2 \theta$ but x must be eliminated. There is no need to replace the limits and condone the absence of $d\theta$ and/or the integral sign for this mark.



dM1: Attempts to use $\sin 2\theta = 2\sin \theta \cos \theta$ to convert an integral of the form $\int \sin^2 \theta \cos^2 \theta \, d\theta$ or
e.g. the form $\int \sin \theta \cos \theta \sin 2\theta \, d\theta$ to $\int \dots \sin^2 2\theta \, d\theta$

There is no need to replace the limits and condone the absence of $d\theta$ and/or the integral sign for this mark.

A1*: Replaces or considers limits $\{x=0 \Rightarrow\} \theta=0$, $\{x=a \Rightarrow\} \theta=\frac{\pi}{2}$ at some stage before the given answer and proceeds with no errors to the given answer. The replaced limits may appear with their integral symbol and do not have to be justified and do not have to appear on every line. Condone infrequent slips in notation, e.g. $\sin \theta^2$ in a line as long as it is not consistently poor. You must see the integral sign with the correct limits and the $d\theta$ together in the given answer.

(b)

M1: Adopts an appropriate strategy by using the double angle identity to obtain an integrable form

$$\dots \int \sin^2 2\theta \, d\theta \rightarrow \dots \int \frac{\pm 1 \pm \cos 4\theta}{2} \, d\theta \text{ which may be seen as}$$

$$\lambda \int \sin^2 2\theta \, d\theta \rightarrow \frac{\lambda}{2} \int \pm 1 \pm \cos 4\theta \, d\theta \text{ with the } \frac{1}{2} \text{ absorbed into their coefficient of the integral.}$$

dM1: Integrates into the form $\pm p\theta \pm q \sin 4\theta$

A1: Correct integration of $\mu \int \sin^2 2\theta \, d\theta$ to $\frac{\mu}{2} \left(\theta - \frac{1}{4} \sin 4\theta \right)$. Here μ may be 1.

Condone lack of limits here.

A1: Applies limits to the correct integral and proceeds to $\frac{1}{8} \pi a^2$ following correct work.

There is no need to see 0 substituted in and condone any omission of integral signs and/or $d\theta$

Note that $\frac{a^2}{4} \left[\theta + \frac{1}{4} \sin 4\theta \right]_0^{\frac{\pi}{2}} = \frac{1}{8} \pi a^2$ is incorrect and scores M1dM1A0A0

Use of $\sin^2 2\theta = \frac{\pm 1 \pm \cos k\theta}{2}$ with $k \neq 4$ scores M0dM0A0A0 but may lead to $\frac{1}{8} \pi a^2$

Condone use of x in place of θ e.g. $\frac{a^2}{4} \left[x - \frac{1}{4} \sin 4x \right]_0^{\frac{\pi}{2}} = \frac{1}{8} \pi a^2$

See overleaf for some alternative approaches.

Alternative 13(a) working backwards:

$$\frac{1}{2}a^2 \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta \qquad \frac{dx}{d\theta} = 2a \sin \theta \cos \theta \text{ score B1 (as in main scheme)}$$

$$= \frac{1}{2}a^2 \int_0^{\frac{\pi}{2}} (2 \sin \theta \cos \theta)(2 \sin \theta \cos \theta) \, d\theta$$

$$= a \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta \, dx$$

Score M1 here for using the double angle identity and replacing ... $\sin \theta \cos \theta \, d\theta$ with dx

$$= a \int_0^a \sqrt{\frac{x}{a}} \sqrt{1 - \frac{x}{a}} \, dx$$

Score dM1 here for a full attempt to replace all trig leading to everything in terms of x only

Must come from the form $\int \dots \sin \theta \cos \theta \, dx$

A1 fully correct with limits replaced / considered before the final line and the final line fully correct with limits, integral sign and dx as per the main scheme.

Alternative 13(b) via IBP Way 1:

$$\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta \qquad \begin{cases} u = \sin^2 2\theta & v' = 1 \\ u' = 4 \sin 2\theta \cos 2\theta & v = \theta \end{cases}$$

$$= \left[\theta \sin^2 2\theta \right]_0^{\frac{\pi}{2}} - 4 \int_0^{\frac{\pi}{2}} \theta \sin 2\theta \cos 2\theta \, d\theta$$

$$= \left[\theta \sin^2 2\theta \right]_0^{\frac{\pi}{2}} - 2 \int_0^{\frac{\pi}{2}} \theta \sin 4\theta \, d\theta \qquad \begin{cases} u = \theta & v' = \sin 4\theta \\ u' = 1 & v = -\frac{\cos 4\theta}{4} \end{cases}$$

$$= \left[\theta \sin^2 2\theta \right]_0^{\frac{\pi}{2}} - 2 \left(\left[-\frac{\theta \cos 4\theta}{4} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\frac{\cos 4\theta}{4} \, d\theta \right) \qquad \text{Score M1 here.}$$

$$= \left[\theta \sin^2 2\theta \right]_0^{\frac{\pi}{2}} - 2 \left[-\frac{\theta \cos 4\theta}{4} + \frac{\sin 4\theta}{16} \right]_0^{\frac{\pi}{2}} \quad \text{Score dM1 here, A1 if correct (ignoring limits).}$$

$$= \left[\theta \sin^2 2\theta + \frac{\theta \cos 4\theta}{2} - \frac{\sin 4\theta}{8} \right]_0^{\frac{\pi}{2}}$$

$$= \left(0 + \frac{\pi}{4} - 0 \right) - (0 + 0 - 0) = \frac{\pi}{4} \rightarrow \left\{ \int_0^a x^{\frac{1}{2}} \sqrt{a-x} \, dx \right\} = \frac{1}{8} \pi a^2 \text{ score A1* (with no errors).}$$



Alternative 13(b) via IBP Way 2:

$$\int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta \qquad \left\{ \begin{array}{ll} u = \sin 2\theta & v' = \sin 2\theta \\ u' = 2 \cos 2\theta & v = -\frac{\cos 2\theta}{2} \end{array} \right\}$$

$$= \left[-\frac{\sin 2\theta \cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos^2 2\theta \, d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta + \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta = \left[-\frac{\sin 2\theta \cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos^2 2\theta \, d\theta + \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta$$

$$\Rightarrow 2 \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta = \left[-\frac{\sin 2\theta \cos 2\theta}{2} \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 1 \, d\theta \quad \text{Score M1 here.}$$

$$\Rightarrow 2 \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta = \left[-\frac{\sin 2\theta \cos 2\theta}{2} + \theta \right]_0^{\frac{\pi}{2}} \quad \text{Score dM1 here, A1 if correct including the 2 (ignoring limits).}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta = \frac{1}{2} \left(0 + \frac{\pi}{2} \right) - \frac{1}{2} (0 + 0) = \frac{\pi}{4}$$

$$\rightarrow \left\{ \int_0^a x^{\frac{1}{2}} \sqrt{a-x} \, dx = \right\} = \frac{1}{8} \pi a^2 \quad \text{score A1* (with no errors).}$$



Question	Scheme	Marks	AOs
10 (a)	$x = u^2 + 1 \Rightarrow dx = 2u du$ oe	B1	1.1b
	Full substitution $\int \frac{3dx}{(x-1)(3+2\sqrt{x-1})} = \int \frac{3 \times 2u du}{(u^2+1-1)(3+2u)}$	M1	1.1b
	Finds correct limits e.g. $p = 2, q = 3$	B1	1.1b
	$= \int \frac{3 \times 2 \cancel{u} du}{u^2(3+2u)} = \int \frac{6 du}{u(3+2u)} *$	A1*	2.1
		(4)	
(b)	$\frac{6}{u(3+2u)} = \frac{A}{u} + \frac{B}{3+2u} \Rightarrow A = \dots, B = \dots$	M1	1.1b
	Correct PF. $\frac{6}{u(3+2u)} = \frac{2}{u} - \frac{4}{3+2u}$	A1	1.1b
	$\int \frac{6 du}{u(3+2u)} = 2 \ln u - 2 \ln(3+2u) \quad (+c)$	dM1 A1ft	3.1a 1.1b
	Uses limits $u = "3", u = "2"$ with some correct $\ln$ work leading to $k \ln b$ E.g. $(2 \ln 3 - 2 \ln 9) - (2 \ln 2 - 2 \ln 7) = 2 \ln \frac{7}{6}$	M1	1.1b
	$\ln \frac{49}{36}$	A1	2.1
		(6)	
(10 marks)			
Notes: Mark (a) and (b) together as one complete question			

(a)

B1: $dx = 2u du$ or exact equivalent. E.g. $\frac{dx}{du} = 2u, \frac{du}{dx} = \frac{1}{2}(x-1)^{-\frac{1}{2}}$

M1: Attempts a full substitution of $x = u^2 + 1$, including $dx \rightarrow \dots u du$ to form an integrand in terms of u .
Condone slips but there should be an attempt to use the correct substitution on the denominator.

B1: Finds correct limits either states $p = 2, q = 3$ or sight of embedded values as limits to the integral

A1*: Clear reasoning including one fully correct intermediate line, including the integral signs, leading to the given expression ignoring limits. So B1, M1, B0, A1 is possible if the limits are incorrect, omitted or left as 5 and 10.

(b)

M1: Uses correct form of PF leading to values of A and B .

A1: Correct PF $\frac{6}{u(3+2u)} = \frac{2}{u} - \frac{4}{3+2u}$ (Not scored for just the correct values of A and B)

dM1: This is an overall problem solving mark. It is for using the correct PF form and integrating using $\ln$ s.
Look for $P \ln u + Q \ln(3+2u)$

A1ft: Correct integration for their $\frac{A}{u} + \frac{B}{3+2u} \rightarrow A \ln u + \frac{B}{2} \ln(3+2u)$ with or without modulus signs

M1: Uses their 2 and 3 as limits, with at least one correct application of the addition law or subtraction law leading to the form $k \ln b$ or $\ln a$. PF's must have been attempted. Condone bracketing slips. Alternatively changing the u 's back to x 's and use limits of 5 and 10.



Question	Scheme	Marks	AOs
13(a)	$3^{-2} \left(1 + \frac{x}{3}\right)^{-2} = 3^{-2}(1 + \dots x + \dots x^2)$	M1	1.1b
	$(-2)\left(\frac{x}{3}\right)$ or $\frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2$	M1	1.1b
	$\left(1 + \frac{x}{3}\right)^{-2} = 1 + (-2)\left(\frac{x}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2$	A1	1.1b
	$3^{-2} \left(1 + \frac{x}{3}\right)^{-2} = \frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}$	A1	2.1
		(4)	

(a)

M1: Attempts a binomial expansion by taking out a factor of 3^{-2} or $\frac{1}{3^2}$ or $\frac{1}{9}$ and achieves at least the first 3 terms in their expansion. May be seen separately e.g. $\frac{1}{9}$ and $(1 + \dots x + \dots x^2)$

M1: A correct method to find either the x or the x^2 term unsimplified.

Award for $(-2)(kx)$ or $\frac{(-2)(-2-1)}{2!}(kx)^2$ where $k \neq 1$. Condone invisible brackets.

A1: For a correct unsimplified or simplified expansion of $\left(1 + \frac{x}{3}\right)^{-2}$ e.g. $= 1 + (-2)\left(\frac{x}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2 - \dots$ or $1 - \frac{2x}{3} + \frac{x^2}{3} - \dots$ Do not condone missing brackets unless they are implied by subsequent work.

Condone $\left(-\frac{x}{3}\right)^2$ for $\left(\frac{x}{3}\right)^2$

Also allow this mark for 2 correct simplified terms from $\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}$ with both method marks scored.

A1: $\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}$ cao Allow terms to be listed. Ignore any extra terms. Isw once a correct **simplified** answer is seen.

Direct expansion, if seen, should be marked as follows:

$$\left((3+x)^{-2} = 3^{-2} - 2 \times 3^{-3} \times x + \frac{-2(-2-1)}{2!} \times 3^{-4} \times x^2\right)$$

$$\mathbf{M1:} \text{ For } (3+x)^{-2} = 3^{-2} + 3^{-3} \times \alpha x + 3^{-4} \times \beta x^2$$

M1: A correct method to find either the x or the x^2 term unsimplified.

Award for $(-2) \times 3^{-3} x$ or $\frac{(-2)(-2-1)}{2!} \times 3^{-4} x^2$. Condone invisible brackets.

A1: For a correct unsimplified or simplified expansion of $(3+x)^{-2}$ e.g. $3^{-2} - 2 \times 3^{-3} \times x + \frac{-2(-2-1)}{2!} \times 3^{-4} \times x^2$

Also award for at least 2 correct simplified terms from $\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}$ with both method marks scored.

A1: $\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}$ cao Allow terms to be listed. Ignore any extra terms. Isw once a correct **simplified** answer is seen.

Note that M0M1A1A0 is a possible mark trait in either method



Note regarding a possible misread in parts (b) and (c)

Some candidates are misreading $\int \frac{6x}{(3+x)^2} dx$ in parts (b) and (c) as $\int \frac{6}{(3+x)^2} dx$

If parts (b) and (c) are consistently attempted with $\int \frac{6}{(3+x)^2} dx$ then we will allow the M

marks in (b) **only**. **M1** for $x^n \rightarrow x^{n+1}$ applied to their expansion in part (a) or $6 \times$ (their expansion in part (a)) and **dM1** for substituting in 0.4 and 0.2 and subtracting either way round (may be implied).

No marks are available in part (c)

MARK PARTS (b) and (c) TOGETHER

(b)	$\int 6x \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27} \right) dx = \int \left(\frac{2x}{3} - \frac{4x^2}{9} + \frac{2x^3}{9} \right) dx = \dots$	M1	1.1b
	$\int \left(\frac{2x}{3} - \frac{4x^2}{9} + \frac{2x^3}{9} \right) dx = \frac{x^2}{3} - \frac{4x^3}{27} + \frac{x^4}{18} \text{ oe}$	A1	1.1b
	$\left[\frac{x^2}{3} - \frac{4x^3}{27} + \frac{x^4}{18} \right]_{0.2}^{0.4} = \left(\frac{(0.4)^2}{3} - \frac{4(0.4)^3}{27} + \frac{(0.4)^4}{18} \right) - \left(\frac{(0.2)^2}{3} - \frac{4(0.2)^3}{27} + \frac{(0.2)^4}{18} \right)$	dM1	3.1a
	$= \text{awrt } 0.03304 \text{ or } \frac{223}{6750}$	A1	1.1b
		(4)	

(b)

M1: Attempts to multiply their expansion from part (a) by $6x$ or just x and attempts to integrate. Condone copying slips and slips in expanding. Look for $x^n \rightarrow x^{n+1}$ at least once having multiplied by $6x$ or x . Ignore e.g. spurious integral signs.

A1: Correct integration, simplified or unsimplified.

$$\frac{x^2}{3} - \frac{4x^3}{27} + \frac{x^4}{18} \text{ oe e.g. } \frac{1}{9} \left(3x^2 - \frac{4x^3}{3} + \frac{x^4}{2} \right), 6 \left(\frac{x^2}{18} - \frac{2x^3}{81} + \frac{x^4}{108} \right)$$

If they have extra terms they can be ignored.

Ignore e.g. spurious integral signs.

dM1: An overall problem-solving mark for

- using part (a) by integrating $6x \times$ their binomial expansion and
- substituting in 0.4 and 0.2 and subtracting either way round (may be implied)

For evidence of using the correct limits we do not expect examiners to check so allow this mark if they obtain a value with (minimal) evidence of the use of the limits 0.4 and 0.2.

This could be e.g. $\left[f(x) \right]_{0.2}^{0.4} = \dots$ provided the first M was scored. If the integration was correct, evidence can be taken from answer of awrt 0.0330 if limits are not seen elsewhere. **Depends on the first M mark.**

A1: awrt 0.03304 (NB allow the exact value which is $\frac{223}{6750} = 0.033037037\dots$).

Is w following a correct answer.

Note answers which use additional terms in the expansion to give a different approximation score A0

Also note that the actual value is 0.032865...



Some may use integration by parts in (b) and the following scheme should be applied.

Integration by parts in (b):

Either by taking $u = 6x$ and $\frac{dv}{dx} = \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)$

$$\begin{aligned} \int 6x \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) dx &= 6x \times \left(\frac{1}{9}x - \frac{x^2}{27} + \frac{x^3}{81}\right) - 6 \int \left(\frac{1}{9}x - \frac{x^2}{27} + \frac{x^3}{81}\right) dx \\ &= 6x \left(\frac{1}{9}x - \frac{x^2}{27} + \frac{x^3}{81}\right) - \left(\frac{1}{3}x^2 - \frac{2x^3}{27} + \frac{6x^4}{324}\right) \end{aligned}$$

M1: A full attempt at integration by parts. This requires:

$$\int 6x \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) dx = kx \times f(x) - k \int f(x) dx = kx \times f(x) - kg(x)$$

Where $f(x)$ is an attempt to integrate their expansion from (a) with $x^n \rightarrow x^{n+1}$ at least once

and $g(x)$ is an attempt to integrate their $f(x)$ with $x^n \rightarrow x^{n+1}$ at least once

A1: Fully correct integration. Then **dM1A1** as in the main scheme

Or by taking $u = \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)$ and $\frac{dv}{dx} = 6x$

$$\begin{aligned} \int 6x \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) dx &= 3x^2 \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) - \int 3x^2 \times \left(-\frac{2}{27} + \frac{2x}{27}\right) dx \\ &= 3x^2 \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) - \int \left(\frac{6x^3}{27} - \frac{6x^2}{27}\right) dx = 3x^2 \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) - \left(\frac{6x^4}{108} - \frac{6x^3}{81}\right) \end{aligned}$$

M1: A full attempt at integration by parts. This requires:

$$\int 6x \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) dx = kx^2 \times f(x) - k \int x^2 g(x) dx = kx^2 \times f(x) - kh(x)$$

Where $f(x)$ is their expansion from (a) and $g(x)$ is an attempt to differentiate their $f(x)$ with

$x^n \rightarrow x^{n-1}$ at least once **and** $h(x)$ is an attempt to integrate their $x^2 g(x)$ with $x^n \rightarrow x^{n+1}$ at

least once

A1: Fully correct integration. Then **dM1A1** as in the main scheme



(c)	Overall problem-solving mark (see notes)	M1	3.1a
	$u = 3 + x \Rightarrow \int_{3.2}^{3.4} f(u) \, du \Rightarrow \int_{3.2}^{3.4} \frac{6(u-3)}{u^2} \, du = \int_{3.2}^{3.4} \frac{6}{u} - \frac{18}{u^2} \, du \Rightarrow \dots \ln u + \dots u^{-1}$	M1	1.1b
	$\int_{3.2}^{3.4} \frac{6(u-3)}{u^2} \, du = \int_{3.2}^{3.4} \frac{6}{u} - \frac{18}{u^2} \, du \Rightarrow 6 \ln u + 18u^{-1}$	A1	1.1b
	$\left[6 \ln u + 18u^{-1} \right]_{3.2}^{3.4} = \left(6 \ln 3.4 + \frac{18}{3.4} \right) - \left(6 \ln 3.2 + \frac{18}{3.2} \right) = \dots$	ddM1	1.1b
	$6 \ln \left(\frac{17}{16} \right) - \frac{45}{136} \text{ oe}$	A1	2.1
		(5)	
(c) Alt 1	Overall problem-solving mark (see notes)	M1	3.1a
	$\int 6x(3+x)^{-2} \, dx = \frac{\dots x}{3+x} \pm \dots \int (3+x)^{-1} \, dx = \frac{\dots x}{3+x} \pm \dots \ln(3+x) \text{ oe}$	M1	1.1b
	$= 6 \ln(3+x) - \frac{6x}{3+x} \text{ oe}$	A1	1.1b
	$\left(6 \ln(3+0.4) - \frac{6(0.4)}{3+0.4} \right) - \left(6 \ln(3+0.2) - \frac{6(0.2)}{3+0.2} \right) = \dots$	ddM1	1.1b
	$6 \ln \left(\frac{17}{16} \right) - \frac{45}{136} \text{ oe}$	A1	2.1

(c) Alt 2	Overall problem-solving mark (see notes)	M1	3.1a
	$\int 6x(3+x)^{-2} \, dx = \int \left(\frac{\dots}{(3+x)} + \frac{\dots}{(3+x)^2} \right) \, dx = \dots \ln(3+x) + \frac{\dots}{3+x} \text{ oe}$	M1	1.1b
	$= 6 \ln(3+x) + \frac{18}{3+x} \text{ oe}$	A1	1.1b
	$\left(6 \ln(3+0.4) + \frac{18}{3+0.4} \right) - \left(6 \ln(3+0.2) + \frac{18}{3+0.2} \right) = \dots$	ddM1	1.1b
	$6 \ln \left(\frac{17}{16} \right) - \frac{45}{136} \text{ oe}$	A1	2.1

(13 marks)**Notes****(c) There are various methods which can be used****M1:** An overall problem-solving mark for **all of**

- using an appropriate integration technique e.g. substitution, by parts or partial fractions – note that this may not be correct but mark positively if they have tried one of these approaches
- integrates one of their terms to a natural logarithm, e.g., $\frac{a}{3+x} \rightarrow b \ln(3+x)$ or $\frac{a}{u} \rightarrow b \ln u$
- substitutes in correct limits and subtracts either way round

M1: Integrates to achieve an expression of the required form for their chosen method

- substitution: $u = x + 3 \rightarrow \pm \frac{a}{u} \pm b \ln u$ or e.g. $u = (x+3)^2 \rightarrow \pm \frac{a}{\sqrt{u}} \pm b \ln u$
- parts: $\pm a \ln(3+x) \pm \frac{bx}{3+x}$ condone missing brackets e.g. $\dots \ln x + 3$ for $\dots \ln(3+x)$
- partial fractions: $\pm a \ln(3+x) \pm \frac{b}{3+x}$ condone missing brackets e.g. $\dots \ln 3 + x$ for $\dots \ln(3+x)$



A1: Correct integration for their method e.g.

- substitution: $u = x + 3 \rightarrow 6 \ln u + 18u^{-1}$ or e.g. $u = (x + 3)^2 \rightarrow 3 \ln u + \frac{18}{\sqrt{u}}$
- parts: $6 \ln(3 + x) - \frac{6x}{3 + x}$
- partial fractions: $6 \ln(3 + x) + \frac{18}{3 + x}$ or e.g. $3 \ln(9 + 6x + x^2) + \frac{18}{3 + x}$

Note that the above terms may appear “separated” but must be correct with the correct signs.
(ignore any reference to a constant of integration)

Do not condone missing brackets e.g. $6 \ln x + 3$ for $6 \ln(3 + x)$ unless they are implied by later work.

ddM1: Substitutes in the correct limits for their integral and subtracts either way round to find a value

Depends on both previous method marks.

For evidence of using the correct limits we do not expect examiners to check so allow this mark if they obtain a value with (minimal) evidence of the use of the appropriate limits.

This could be e.g. $[f(x)]_{0.2}^{0.4} = \dots$ provided both previous M marks were scored.

Note that for substitution they may revert back to $3 + x$ and so should be using 0.4 and 0.2

A1: A full and rigorous argument leading to $6 \ln\left(\frac{17}{16}\right) - \frac{45}{136}$ or exact equivalent e.g. $3 \ln\left(\frac{289}{256}\right) - \frac{45}{136}$ or

e.g. $-6 \ln\left(\frac{16}{17}\right) - \frac{45}{136}$

The brackets are not required around the $\frac{17}{16}$ and allow exact equivalents e.g. allow 1.0625 or $1\frac{1}{16}$

but not e.g. $\frac{3.4}{3.2}$. The $\frac{45}{136}$ must be exact or an exact equivalent. Also allow e.g. $6 \ln\left|\frac{17}{16}\right| - \frac{45}{136}$

Ignore spurious integral signs that may appear as part of their solution.

Question	Scheme	Marks	AOs												
13	$\int \frac{x}{(2x+1)^3} dx = \frac{x(2x+1)^{-2}}{-4} + \int \frac{(2x+1)^{-2}}{4} (dx)$	M1	3.1a												
	$= \dots + \frac{(2x+1)^{-1}}{-8}$	dM1	1.1b												
	$-\frac{x}{4(2x+1)^2} - \frac{1}{8(2x+1)}$	A1	1.1b												
	e.g. $= \left(-\frac{2}{4(2 \times 2+1)^2} - \frac{1}{8(2 \times 2+1)} \right) - \left(-\frac{0}{4(2 \times 0+1)^2} - \frac{1}{8(2 \times 0+1)} \right)$	ddM1	1.1b												
	e.g. $-\frac{2}{100} - \frac{1}{40} + \frac{1}{8} = \frac{2}{25}$ *	A1*	2.1												
		(5)													
13 Alt I	$\int \frac{x}{(2x+1)^3} dx = \int \frac{u-1}{4u^3} (du) \text{ where } u = 2x+1$	M1	3.1a												
	$= \int \frac{1}{4} u^{-2} - \frac{1}{4} u^{-3} du = \dots$	dM1	1.1b												
	$= -\frac{1}{4} u^{-1} + \frac{1}{8} u^{-2}$	A1	1.1b												
	$\int_0^2 \frac{x}{(2x+1)^3} dx = \left[-\frac{1}{4} u^{-1} + \frac{1}{8} u^{-2} \right]_1^5 = \left(-\frac{1}{20} + \frac{1}{200} \right) - \left(-\frac{1}{4} + \frac{1}{8} \right)$	ddM1	1.1b												
	e.g. $-\frac{1}{20} + \frac{1}{200} + \frac{1}{8} = \frac{2}{25}$ *	A1*	2.1												
		(5)													
(5 marks)															
Notes:															
M1: Obtains $\alpha x(2x+1)^{-2} \pm \beta \int (2x+1)^{-2} (dx)$ o.e. where $\alpha, \beta \neq 0$ but may be equal to each other (you do not need to be concerned about how they arrive at this) dM1: Uses a correct method to integrate an expression of the form $\pm \beta \int (2x+1)^{-2} (dx) \rightarrow \pm \gamma (2x+1)^{-1}$, $\beta, \gamma \neq 0$ It is dependent on the previous method mark. A1: $-\frac{x}{4(2x+1)^2} - \frac{1}{8(2x+1)}$ o.e. Allow this to be unsimplified Watch out for the DI method															
	<table><tr><td></td><td>D</td><td>I</td></tr><tr><td>+</td><td>x</td><td>$(2x+1)^{-3}$</td></tr><tr><td>-</td><td>1</td><td>$-\frac{(2x+1)^{-2}}{4}$</td></tr><tr><td>+</td><td>0</td><td>$\frac{(2x+1)^{-1}}{8}$</td></tr></table>		D	I	+	x	$(2x+1)^{-3}$	-	1	$-\frac{(2x+1)^{-2}}{4}$	+	0	$\frac{(2x+1)^{-1}}{8}$		
	D	I													
+	x	$(2x+1)^{-3}$													
-	1	$-\frac{(2x+1)^{-2}}{4}$													
+	0	$\frac{(2x+1)^{-1}}{8}$													

Giving correct integration e.g. $\int \frac{x}{(2x+1)^3} dx = -\frac{x(2x+1)^{-2}}{4} - \frac{(2x+1)^{-1}}{8}$

Score M1dM1 for obtaining $\pm px(2x+1)^{-2} \pm q(2x+1)^{-1}$, $p, q \neq 0$ and A1 for both correct.

ddM1: Substitutes 0 and 2 into an expression of the form $\pm px(2x+1)^{-2} \pm q(2x+1)^{-1}$, $p, q \neq 0$ or equivalent and subtracts either way round. Evidence of limits used cannot be the given answer. Condone slips but not directly evaluating the expression as 0 when x is 0. It is dependent on both of the previous method marks.

A1*: Shows **evidence of evaluating** after or when the limits are substituted in before proceeding to the given answer with **no errors seen including invisible brackets** and all previous marks scored. Condone missing/poor notation e.g. missing the dx throughout

Alternative method I – substitution using $u = 2x+1$

M1: Uses a suitable substitution e.g. $u = 2x+1$ and proceeds to $A \int \frac{u-1}{u^3} (du)$ o.e.

dM1: Splits into separate fractions and attempts to integrate $A \int u^{-2} - u^{-3} (du)$ Look for at least one correct index for one of the two terms i.e. $u^{-2} \rightarrow u^{-1}$ or $u^{-3} \rightarrow u^{-2}$. It is dependent on the previous method mark.

A1: $-\frac{1}{4}u^{-1} + \frac{1}{8}u^{-2}$ o.e.

ddM1: Substitutes correct limits (1 and 5 if in terms of u) into an expression of the correct form $\dots u^{-1} + \dots u^{-2}$ (or may have substituted back in terms of x and substitutes in 0 and 2 into an expression of the correct form $\pm p(2x+1)^{-2} \pm q(2x+1)^{-1}$, $p, q \neq 0$ o.e and subtracts either way round – see above for ddM1). Evidence of limits used cannot be the given answer. Condone slips. It is dependent on both of the previous method marks.

A1*: Shows **evidence of evaluating** after or when the limits are substituted in before proceeding to the given answer with **no errors seen including invisible brackets** and all previous marks scored. Condone missing/poor notation e.g. missing the du throughout

Alternative method II – partial fractions

M1: Writes $\frac{x}{(2x+1)^3}$ as $\frac{0.5}{(2x+1)^2} - \frac{0.5}{(2x+1)^3}$ o.e. Allow $\pm \frac{M}{(2x+1)^2} \pm \frac{N}{(2x+1)^3}$ (where M and N are constants)

dM1: $\int \frac{x}{(2x+1)^3} (dx) = \int \frac{"0.5"}{(2x+1)^2} (dx) - \int \frac{"0.5"}{(2x+1)^3} (dx) = \pm \dots (2x+1)^{-1} \pm \dots (2x+1)^{-2}$

A1: $-\frac{1}{4(2x+1)^1} + \frac{1}{8(2x+1)^2}$ o.e.

ddM1: Substitutes 0 and 2 into an expression of the correct form $\pm \dots (2x+1)^{-1} \pm \dots (2x+1)^{-2}$ and subtracts either way round. Evidence of limits used cannot be the given answer. Condone slips but not directly evaluating the expression as 0 when x is 0. It is dependent on both of the previous method marks.

A1*: Shows **evidence of evaluating** after or when the limits are substituted in before proceeding to the given answer with **no errors seen including invisible brackets** and all previous marks scored. Condone missing/poor notation e.g. missing the dx throughout.



Question	Scheme	Marks	AOs
16 (a)	$R = \int_1^{\sqrt{3}} \frac{1}{x^2 \sqrt{4-x^2}} dx$		
	$\int \frac{1}{x^2 \sqrt{4-x^2}} dx = \int \frac{1}{4 \sin^2 u \sqrt{4-4 \sin^2 u}} 2 \cos u (du)$	M1	3.1a
	Uses $1 - \sin^2 u = \cos^2 u \Rightarrow \sqrt{4-4 \sin^2 u} = 2 \cos u$	dM1	1.1b
	$= \int \frac{1}{4 \sin^2 u \times 2 \cos u} 2 \cos u (du) = \int \frac{1}{4} \operatorname{cosec}^2 u du$	A1	2.1
	Deduces $a = \frac{\pi}{6}$ and $b = \frac{\pi}{3}$	B1	2.2a
		(4)	
(b)	$\int \frac{1}{4} \operatorname{cosec}^2 u (du) = -\frac{1}{4} \cot u (+c)$	B1ft	1.2
	$= \left[-\frac{1}{4} \cot u \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = -\frac{1}{4} \cot \frac{\pi}{3} + \frac{1}{4} \cot \frac{\pi}{6}$	M1	2.1
	$= \frac{1}{2\sqrt{3}} \text{ or } \frac{\sqrt{3}}{6}$	A1	1.1b
		(3)	
(7 marks)			
Notes:			
(a)			
M1: Proceeds to $\frac{1}{A \sin^2 u \sqrt{\pm C \pm C \sin^2 u}} \times B \cos u$ o.e. or may be implied by $\frac{\cos u}{D \sin^2 u \cos u}$ o.e. in terms of u only $A, B, C, D \neq 0$ Requires $dx \rightarrow \dots \cos u (du)$ o.e. Condone the omission of the integral sign and du for this mark. Condone the $\pm B \cos u$ appearing after the du if present It cannot be implied by $\frac{1}{D \sin^2 u}$ or $\frac{\operatorname{cosec}^2 u}{D}$ where $D \neq 0$ dM1: Attempts to use $\pm 1 \pm \sin^2 u = \pm \cos^2 u$ to change $\sqrt{\pm C \pm C \sin^2 u}$ to $p \cos u$ o.e. (Must be seen somewhere in their solution which may be part of side-workings) May be seen as e.g. $\frac{1}{A \sin^2 u \sqrt{C \cos^2 u}} \times B \cos u$ or $\frac{1}{A \sin^2 u \times F \cos u} \times B \cos u$ It is dependent on the previous method mark. A1: $\int \frac{1}{4} \operatorname{cosec}^2 u du$ Ignore any limits or the absence of them for this mark. Requires the integral sign and du. Do not isw. Must be seen in (a) B1: Deduces $a = \frac{\pi}{6}$ and $b = \frac{\pi}{3}$ (in radians) May be with their final integral in the correct positions or stated separately which value is a and which value is b. Must be seen in (a).			

Alternative part (a) (Further Maths)

M1: $u = \arcsin\left(\frac{x}{2}\right) \Rightarrow \frac{du}{dx} = \frac{M}{\sqrt{1-Nx^2}}$ It cannot be implied by $\frac{1}{D\sin^2 u}$ or $\frac{\operatorname{cosec}^2 u}{D}$, $D \neq 0$

dM1: Uses $\frac{du}{dx} = \frac{1}{\sqrt{4-x^2}}$ o.e., substitutes $x = 2\sin u$ and proceeds to $\frac{1}{4\sin^2 u}$ It is dependent on the previous method mark.

A1: $\int \frac{1}{4} \operatorname{cosec}^2 u \, du$ Ignore any limits or the absence of them for this mark.

Requires the integral sign and du . Do not isw. Must be seen in (a)

B1: Deduces $a = \frac{\pi}{6}$ and $b = \frac{\pi}{3}$ (must be in radians)

May be with their final integral in the correct positions or stated separately which value is a and which value is b . Must be seen in (a).

(b) Condone x instead of u provided the appropriate limits are substituted into the function

B1ft: $k \operatorname{cosec}^2 u \rightarrow -k \cot u$ (may be left in terms of $k \neq 0$)

e.g. $-2 \operatorname{cosec}^2 u \rightarrow 2 \cot u$ is B1 (look for the sign to change as well)

Note some candidates may prefer to change

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{4} \operatorname{cosec}^2 u \, (du) = - \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{1}{4} \operatorname{cosec}^2 u \, (du) = \frac{1}{4} \cot u \, (+c) \text{ which scores B1ft}$$

M1: Uses the limits $\frac{\pi}{6}$ and $\frac{\pi}{3}$ either way round (or condone use of 30° and 60°) in an expression of the form $\pm q \cot u$ and subtracts (either way round).

Allow $q = 1$. May be implied by their final answer provided B1ft has been scored.

$$\text{May write as e.g. } -\frac{1}{4} \frac{\cos \frac{\pi}{3}}{\sin \frac{\pi}{3}} + \frac{1}{4} \frac{\cos \frac{\pi}{6}}{\sin \frac{\pi}{6}}$$

Do not condone a sign slip here e.g. $-\frac{1}{4} \cot \frac{\pi}{3} - \frac{1}{4} \cot \frac{\pi}{6}$ is M0.

The expression is sufficient but if just a value is stated following integration to the required form then you may need to check this on your calculator.

If no algebraic integration is seen then M0

If only decimal values are seen then M0

A1: $\frac{1}{2\sqrt{3}}$ or $\frac{\sqrt{3}}{6}$ provided the previous two marks have been scored.

Note that incorrect integration e.g. $\int k \operatorname{cosec}^2 u \, du = k \cot u \rightarrow \frac{1}{2\sqrt{3}}$ scores B0ftM1A0



Question	Scheme	Marks	AOs
15(a)	$(f'(x) =) \frac{\lambda x(1+x^2)^2 - \mu x(1-x^2)(1+x^2)}{(1+x^2)^4}$	M1	1.1b
	$= \frac{-2x(1+x^2)^2 - 4x(1-x^2)(1+x^2)}{(1+x^2)^4}$	A1	1.1b
	e.g. $\frac{-2x(1+x^2) - 4x(1-x^2)}{(1+x^2)^3} = 0$ $\Rightarrow (1+x^2) + 2(1-x^2) = 0$ $\Rightarrow 3 - x^2 = 0$ or e.g. $= \frac{-2x(1+x^2)[(1+x^2) + 2(1-x^2)]}{(1+x^2)^4}$ $= \frac{-2x(1+x^2)(3-x^2)}{(1+x^2)^4}$	dM1	3.1a
	e.g. $3 - x^2 = 0$ or $2x(1+x^2)(x^2 - 3) = 0$ $\Rightarrow x = \dots \Rightarrow y = \dots$	ddM1	2.1
	$\left(-\sqrt{3}, -\frac{1}{8}\right)$	A1	2.3
		(5)	

Notes

(a) There may be other valid ways to differentiate or to solve the resulting equation.

M1: Attempts the quotient rule to achieve $\frac{\pm \lambda x(1+x^2)^2 \pm \mu x(1-x^2)(1+x^2)}{(1+x^2)^4}$

but condone $\frac{\pm \lambda x(1+x^2)^2 \pm \mu x(1-x^2)(1+x^2)}{(1+x^2)^2}$ provided an incorrect quotient rule is not seen.

Alternatively, attempts the product rule on $f(x) = (1-x^2)(1+x^2)^{-2}$ to achieve

$$\pm \lambda x(1+x^2)^{-2} \pm \mu x(1-x^2)(1+x^2)^{-3}$$

There may be attempts at splitting the numerator to $f(x) = (1+x^2)^{-2} - x^2(1+x^2)^{-2}$ which should differentiate to $\pm \lambda x(1+x^2)^{-3} \pm \mu x(1+x^2)^{-2} \pm \gamma x^3(1+x^2)^{-3}$

In **all** cases, do not penalise signs of λ , μ and/or γ unless an incorrect quotient rule or an incorrect product rule is stated.

Any occurrences of $(1+x^2)^2$ may be replaced with $1+2x^2+x^4$

There is no need for a LHS e.g. $f'(x) =$ to be present so ignore an incorrect LHS.

Invisible brackets may be recovered/implied by later work.

A1: Correct differentiation. May be unsimplified. Ignore absence of (or incorrect) LHS.

The correct derivative using the quotient rule is $= \frac{-2x(1+x^2)^2 - 4x(1-x^2)(1+x^2)}{(1+x^2)^4}$ o.e.

using the product rule is $-2x(1+x^2)^{-2} - 4x(1-x^2)(1+x^2)^{-3}$ o.e.

and splitting the numerator is $-4x(1+x^2)^{-3} - 2x(1+x^2)^{-2} + 4x^3(1+x^2)^{-3}$ o.e.



You may need to check carefully for equivalent derivatives and ISW after a correct expression is seen. Note that some candidates may set $= 0$ at the start and may omit the denominator as a result – send to review if you are unsure in such cases.

dM1: Attempts to reduce the expression to a suitable form so that the roots can be found, either by cancelling a factor of $(1+x^2)$ and simplifying the other brackets

$$\text{e.g. } \frac{-2x(1+x^2)-4x(1-x^2)}{(1+x^2)^3} \rightarrow \text{e.g. } \frac{-6x+2x^3}{(1+x^2)^3} \text{ or } \frac{2x(x^2-3)}{(1+x^2)^3} \text{ or } 2x(x^2-3) (=0)$$

or by attempting to take x , $(1+x^2)$ or $\pm 2x(1+x^2)$ out as a factor and simplify the other brackets. If they only take out a factor of x then they must have $x(\pm Ax^4 \pm Bx^2 \pm C)$

$$\text{e.g. } \frac{-2x(1+x^2)[(1+x^2)+2(1-x^2)]}{(1+x^2)^4} \rightarrow \text{e.g. } \frac{-2x(1+x^2)(3-x^2)}{(1+x^2)^4} \text{ or } \frac{x(2x^4-4x^2-6)}{(1+x^2)^4}$$

Depends on the first M mark.

Note: they might attempt to expand the brackets first, or set $= 0$ and multiply through by $(1+x^2)^k$ and/or divide through by x or $2x$ at any stage.

Do not be concerned if any factors of $(1+x^2)$, x or 2 go “missing” during their work.

The $= 0$ does not explicitly need to be seen.

$$\text{Look for } \pm \frac{2x(1+x^2)(Ax^2+B)}{(1+x^2)^k} \text{ or } \pm \frac{(1+x^2)(Ax^3+Bx)}{(1+x^2)^k} \text{ or } \pm \frac{x(Ax^4+Bx^2+C)}{(1+x^2)^k} \text{ where}$$

- k may be 0, 1, 2, 3 or 4
- A and B (and C if present) are non-zero
- any of $2x$ or $(1+x^2)$ in the numerator and/or $(1+x^2)^k$ in the denominator may be absent

ddM1: Attempts to solve their numerator set $= 0$ using a valid non-calculator method **and** uses a solution for x to find a corresponding value for y .

The substitution may be implied by their value of y but, if not, the substitution must be seen.

- For either Ax^2+B or Ax^3+Bx , we require $A \neq B$ and $A \times B < 0$ From either expression

they can write down their $x = \pm \sqrt{\frac{-B}{A}}$ without working.

- For Ax^4+Bx^2+C they must show a valid non-calculator method for solving the quartic, treating it as a quadratic in x^2 and using the usual rules for solving a quadratic algebraically, e.g. $a = x^2 \rightarrow 2a^2 - 4a - 6 (=0) \rightarrow (a+1)(2a-6) (=0) \rightarrow x = \pm\sqrt{3}$. They must reach a value for x (and not just x^2) as well as finding a value for y .

Dependent on both previous method marks. They must use a value for x that is not -1 , 0 or 1 .

A1: Deduces the correct **exact** coordinates for $P \left(-\sqrt{3}, -\frac{1}{8} \right)$ o.e. $(-\sqrt{3}, -0.125)$

Requires all the previous marks to have been scored.

Condone $x = -\sqrt{3}$, $y = -0.125$ or e.g. $-\sqrt{3}$, $-\frac{1}{8}$

If there is more than one pair of coordinates given, then the correct coordinates must be clearly selected or any others clearly rejected.

Ignore any mistakes that occur if they multiply out the denominator $(1+x^2)^4$ after differentiation.

Question	Scheme	Marks	AOs
15(b)	$x = -1 \Rightarrow \alpha = -\frac{\pi}{4}$ and $x = 1 \Rightarrow \beta = \frac{\pi}{4}$	B1	2.2a
	$\frac{dx}{d\theta} = \sec^2 \theta$	B1	1.1b
	$\int \frac{1-x^2}{(1+x^2)^2} dx = \int \frac{1-\tan^2 \theta}{(1+\tan^2 \theta)^2} " \sec^2 \theta " d\theta$ $\text{or} = \int \frac{1-\tan^2 \theta}{\sec^4 \theta} " \sec^2 \theta " d\theta$	M1	1.1b
	$= \int (1 - \tan^2 \theta) \cos^2 \theta d\theta$ $= \int \left(1 - \frac{\sin^2 \theta}{\cos^2 \theta} \right) \cos^2 \theta d\theta$ $= \int (\cos^2 \theta - \sin^2 \theta) d\theta$ $= \int \frac{2 - 1 - \tan^2 \theta}{\sec^2 \theta} d\theta$ $= \int \left(\frac{2}{\sec^2 \theta} - \frac{1 + \tan^2 \theta}{\sec^2 \theta} \right) d\theta$ $= \int (2 \cos^2 \theta - 1) d\theta$	dM1	3.1a
	$= \int \cos 2\theta d\theta *$	A1*	2.1
		(5)	

Notes

(b) Mark parts (b) and (c) together.

B1: Deduces the correct limits for the integral in θ which may be seen separately as side working or within their integral work. This mark cannot be scored working in degrees.

Allow $\frac{3\pi}{4}$ instead of $-\frac{\pi}{4}$ but not decimal approximations.

B1: $\frac{dx}{d\theta} = \sec^2 \theta$ or equivalent e.g. $\frac{dx}{d\theta} = 1 + x^2$ or $\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta}$ (coming from $x = \frac{\sin \theta}{\cos \theta}$) but must be correct, so not e.g. $\frac{dx}{d\theta} = \sec^2 x$ unless recovered.

M1: Makes a complete attempt at using the substitution $x = \tan \theta$
Requires:

- An attempt at using $\frac{dx}{d\theta}$ to replace dx with $d\theta$ either way round, so if $\frac{dx}{d\theta} = g(\theta)$ allow either $dx = g(\theta)\{d\theta\}$ **or** $dx = \frac{1}{g(\theta)}\{d\theta\}$. $\frac{dx}{d\theta}$ must be a function of θ and not a constant.
- All terms in x replaced with $\tan \theta$ (or $1 + x^2$ with $\sec^2 \theta$)
Condone the absence of $d\theta$ but dx must no longer be present.
Condone if they fail to square the denominator or e.g. a slip in missing a θ
Use of notation such as $d(\tan \theta)$ is correct but does not score the M1 until replaced with $\sec^2 \theta \{d\theta\}$ or their derivative of $\tan \theta$ (which must not be a constant multiple of $\tan \theta$).

dM1: Uses trigonometric identities e.g. $\pm 1 \pm \tan^2 \theta = \pm \sec^2 \theta$, $\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\sec \theta = \frac{1}{\cos \theta}$,

$\pm \cos^2 \theta \pm \sin^2 \theta = \pm 1$ to simplify the integral to an expression of the form
 $\pm a \sin^2 \theta \pm b \cos^2 \theta \pm c$ where one of a , b or c may be 0.

The algebra should essentially be correct but condone e.g. sign slips or errors collecting terms.



Alternatively, e.g. $\int \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} d\theta = \int \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta} d\theta = \int \frac{\cos^2 \theta - \sin^2 \theta}{1} d\theta$

They must have replaced dx with $d\theta$ the correct way round for this mark, so if $\frac{dx}{d\theta} = g(\theta)$

then they must have $dx = g(\theta)\{d\theta\}$ and **not** $dx = \frac{1}{g(\theta)}\{d\theta\}$

It is acceptable to replace e.g. $\tan^2 \theta \cos^2 \theta$ with $\sin^2 \theta$

Dependent on the previous method mark.

A1*: cso Arrives at $\int \cos 2\theta d\theta$ with sufficient working shown and no incorrect work ignoring limits. The final line must be fully correct, including the integral sign and $d\theta$ (ignoring limits). Note that this may be seen in part (c) and may score the mark.

All trigonometric identities must be fully correct.

Condone one or two slips in a missing θ but not frequent omissions.

Condone missing integral signs in their intermediate work, but it must be present on the final line.

dx must be replaced with $d\theta$ at some stage before the final line but does not need to be present in every intermediate line of working.

Condone notational errors throughout e.g. $\sin \theta^2$ provided they are recovered.

This mark is independent from any work to do with limits, i.e., B0B1M1dM1A1* is possible.

Question	Scheme	Marks	AOs
15(c)	$\int \cos 2\theta \, d\theta = \frac{1}{2} \sin 2\theta$	M1	1.1b
	$\left[\frac{1}{2} \sin 2\theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{2} \sin \left(2 \times \frac{\pi}{4} \right) - \frac{1}{2} \sin \left(2 \times -\frac{\pi}{4} \right)$	dM1	1.1b
	$= 1$	A1	2.1
		(3)	
(13 marks)			
Notes			
(c)	Mark parts (b) and (c) together.		
M1:	$\int \cos 2\theta \, d\theta \rightarrow \pm \frac{1}{2} \sin 2\theta \{+c\}$		
dM1:	Substitutes their changed limits (not 1 and -1) into $\pm \frac{1}{2} \sin 2\theta$ and subtracts either way round. May be implied e.g. $\frac{1}{2} + \frac{1}{2}$ provided they have the correct limits. Dependent on the previous method mark.		
A1:	Area of 1 found following correct work. This mark requires clear substitution of the correct limits into $\frac{1}{2} \sin 2\theta$ the correct way round. If they have the limits the wrong way round and achieve an answer of -1 they cannot just make the answer positive for this mark. They may use limits from 0 to $\frac{\pi}{4}$ and multiply their result by 2. $\frac{3\pi}{4}$ may be used in place of $-\frac{\pi}{4}$ as the lower limit which is acceptable. Condone any spurious integral symbol accompanying $\frac{1}{2} \sin 2\theta$ The substitution may be implied by e.g. $\frac{1}{2} - -\frac{1}{2}$ or $\frac{1}{2} \sin \left(\frac{\pi}{2} \right) - \frac{1}{2} \sin \left(-\frac{\pi}{2} \right)$ but must follow sight of $\frac{1}{2} \sin 2\theta$ Condone use of limits -45 and 45 (degrees) the correct way round for full marks in part (c). Note that use of a calculator on the original integral will give the correct answer. An answer of 1 scores no marks without evidence of scoring both method marks above including the substitution.		

Question	Scheme	Marks	AOs
10(a)	e.g. $\frac{3kx-18}{(x+4)(x-2)} \equiv \frac{A}{x+4} + \frac{B}{x-2} \Rightarrow 3kx-18 \equiv A(x-2) + B(x+4)$ or $\frac{3kx-18}{(x+4)(x-2)} \equiv \frac{A}{x-2} + \frac{B}{x+4} \Rightarrow 3kx-18 \equiv A(x+4) + B(x-2)$	B1	1.1b
	$6k-18=6B \Rightarrow B=\dots$ or $-12k-18=-6A \Rightarrow A=\dots$ or $3kx-18 \equiv (A+B)x + 4B - 2A \Rightarrow A+B=3k, -18=4B-2A$ $\Rightarrow A=\dots$ or $B=\dots$	M1	1.1b
	$\frac{2k+3}{x+4} + \frac{k-3}{x-2}$	A1	1.1b
		(3)	
(b)	$\int \left(\frac{"2k+3"}{x+4} + \frac{"k-3"}{x-2} \right) dx = \dots \ln(x+4) + \dots \ln(x-2)$	M1	1.2
	$("2k+3") \ln(x+4) + ("k-3") \ln(x-2)$	A1ft	1.1b
	$("2k+3") \ln(5) - ("k-3") \ln(5) \Rightarrow ("k+6") \ln 5 = 21 \Rightarrow k = \dots$	dM1	3.1a
	$(k =) \frac{21}{\ln 5} - 6$	A1	2.2a
		(4)	
(7 marks)			
Notes			
(a) B1: Correct form for the partial fractions and sets up the correct corresponding identity which may be implied by two equations in A and B if they are comparing coefficients. M1: Either - substitutes $x=2$ or $x=-4$ in an attempt to find A or B in terms of k - expands the rhs, collects terms and compares coefficients in an attempt to find A or B in terms of k Or may be implied by one correct fraction (numerator and denominator) You may see candidates substituting two other values of x and then solving simultaneous equations. A1: Achieves $\frac{2k+3}{x+4} + \frac{k-3}{x-2}$ with no errors. Must be the correct partial fractions not just for correct numerators. May be seen in (b). Correct answer implies B1M1A1. One correct fraction only B0M1A0			



(b)

M1: Attempts to find $\int \left(\frac{"2k+3"}{x+4} + \frac{"k-3"}{x-2} \right) dx$. Score for either $\frac{\cdots}{x+4} \rightarrow \dots \ln(x+4)$ or $\frac{\cdots}{x-2} \rightarrow \dots \ln(x-2)$

Allow the ... to be in terms of k or just constants but there must be no x terms.

Condone invisible brackets for this mark.

A1ft: $("2k+3") \ln|x+4| + ("k-3") \ln|x-2|$

but condone round brackets e.g. $("2k+3") \ln(x+4) + ("k-3") \ln(x-2)$ or equivalent e.g.

$("2k+3") \ln(x+4) + ("k-3") \ln(2-x)$

Follow through their partial fractions with numerators which must both be in terms of k .

Condone missing brackets as long as they are recovered later e.g. when applying limits.

dM1: A full attempt to find the value of k . To score this mark they must have attempted to integrate their partial fractions, substituted in the correct limits, subtracted either way round, set = 21 and attempted to solve to find k . Condone omission of the terms containing $\ln(1)$ or $\ln(-1)$.

Note that e.g. $\ln(-5)$ or $\ln(5)$ must be seen but may be disregarded **after** substitution and subtraction.

Do not be concerned with the processing as long as they proceed to $k = \dots$

Condone if they use x instead of k after limits have been used as long as the intention is clear.

A1: Deduces $(k =) \frac{21}{\ln 5} - 6$ or exact equivalent e.g. $\frac{21 - 6 \ln 5}{\ln 5}$, $\frac{21 - 3 \ln 25}{\ln 5}$.

Allow recovery from expressions that contain e.g. $\ln(-5)$ as long as it is dealt with subsequently.

Also allow recovery from invisible brackets. Condone $x = \frac{21}{\ln 5} - 6$

Some candidates may use substitution in part (b) e.g.

$$\int \left(\frac{"2k+3"}{x+4} + \frac{"k-3"}{x-2} \right) dx = \int \left(\frac{"2k+3"}{x+4} \right) dx + \int \left(\frac{"k-3"}{x-2} \right) dx$$

$$u = x+4 \Rightarrow \int \left(\frac{"2k+3"}{x+4} \right) dx = \int \left(\frac{"2k+3"}{u} \right) du = \dots \ln u$$

$$u = x-2 \Rightarrow \int \left(\frac{"k-3"}{x-2} \right) dx = \int \left(\frac{"k-3"}{u} \right) du = \dots \ln u$$

Score **M1** for integrating at least once to an appropriate form as in the main scheme e.g. $\dots \ln u$

A1ft: For $("2k+3") \ln|u| + ("k-3") \ln|u|$

but condone $("2k+3") \ln u + ("k-3") \ln u$ which may be seen separately

Follow through their "A" and "B" in terms of k .

Condone missing brackets as long as they are recovered later e.g. when applying limits.

dM1: A full attempt to find the value of k . To score this mark they must have attempted to integrate their partial fractions using substitution, substituted in the correct changed limits and subtracts either way

round, set = 21 and attempted to solve to find k . Do not be concerned with processing as long as they proceed to $k = \dots$. Condone omission of terms which contain e.g. $\ln(1)$ or $\ln(-1)$.

Note that e.g. $\ln(-5)$ or $\ln(5)$ must be seen but may be disregarded **after** substitution and subtraction.

$$[(2k+3)\ln u]_1^5 + [(k-3)\ln u]_{-5}^{-1} = 21 \Rightarrow (2k+3)\ln 5 - (2k+3)\ln 1 + (k-3)\ln 1 - (k-3)\ln 5 = 21$$

$$\Rightarrow (2k+3)\ln 5 - (k-3)\ln 5 = 21 \Rightarrow (k+6)\ln 5 = 21 \Rightarrow k = \dots$$

A1: $k = \frac{21}{\ln 5} - 6$ or exact equivalent e.g. $\frac{21-6\ln 5}{\ln 5}$, $\frac{21-3\ln 25}{\ln 5}$, $21\log_5 e - 6$.

Allow recovery from expressions that contain e.g. $\ln(-5)$ as long as it is dealt with subsequently.

Also allow recovery from invisible brackets.

Question	Scheme	Marks	AOs
10(a)	$\frac{dV}{dt} = 0.45$ or $\frac{dV}{dt} = \pm 0.3V$	M1	3.1b
	$\frac{dV}{dt} = 0.45 - \frac{3}{10}V$ $20\frac{dV}{dt} = 9 - 6V$ *	A1*	2.1
		(2)	
(b)	e.g. $\frac{1}{9-6V} \frac{dV}{dt} = \frac{1}{20} \rightarrow \int \frac{1}{9-6V} dV = \int \frac{1}{20} dt$	B1	1.1b
	$\frac{1}{9-6V} \rightarrow \dots \ln 9-6V $	M1	1.1b
	$-\frac{1}{6} \ln 9-6V = \frac{t}{20} (+c)$	A1	1.1b
	$-\frac{1}{6} \ln 9-6V = \frac{t}{20} + c$ $9-6V = Ae^{-\frac{3t}{10}}$ $t=0, V=0.25 \Rightarrow A=(7.5)$	dM1	3.1a
	$9-6V = 7.5e^{-\frac{3t}{10}}$ $V = \frac{3}{2} - \frac{5}{4}e^{-\frac{3t}{10}}$	A1	2.1
		(5)	
Notes			
(a) Marks for part (a) may not be scored in part (b) M1: Either $\frac{dV}{dt} = 0.45$ or $\frac{dV}{dt} = \pm 0.3V$ o.e. e.g. $\frac{dV}{dt} = \frac{9}{20}$ or $\frac{dV}{dt} = \pm \frac{3}{10}V$ seen or implied by e.g. $\frac{dV}{dt} = 0.45 - \frac{3}{10}V$ (but not implied by just stating the given answer). Condone use of $\dot{V}$ It may be seen as part of their $\frac{dV}{dt}$ e.g. $\frac{dV}{dt} = 0.45 + V + 0.3V$ scores M1A0* Condone e.g. change in volume = (inflow – outflow =) $0.45 - 0.3V$ for this mark. A1*: Achieves $20\frac{dV}{dt} = 9 - 6V$ with no errors, following $\frac{dV}{dt} = 0.45 - \frac{3}{10}V$ o.e. (including the $\frac{dV}{dt}$ or $\dot{V}$ but note that it must be $\frac{dV}{dt}$ in the final line and not $\dot{V}$). change in volume = $0.45 - 0.3V \rightarrow 20\frac{dV}{dt} = 9 - 6V$ scores M1A0*. Ignore any units used in their working for both marks.			
(b) B1: Separates the variables correctly, e.g., $\int \frac{1}{9-6V} dV = \int \frac{1}{20} dt$ or $\int \frac{20}{9-6V} dV = \int \{1\} dt$ o.e. The integral symbol and/or dV and/or dt may be implied if they go on to integrate both sides to the correct form $\dots \ln \alpha(9-6V) = \dots t (+c)$ with or without the modulus brackets.			

M1:	Attempts to integrate the reciprocal term $\frac{\beta}{9-6V} \rightarrow \dots \ln 9-6V $ or $\rightarrow \dots \ln \alpha(6V-9) $ for some constant β (and α if used). Condone e.g. $\frac{20}{9-6V} \rightarrow \dots \ln 9-6V$ or $\rightarrow \dots \ln 6V-9$
A1:	Correct integration for both sides. They do not need the $+c$ for this mark. Note scoring this mark implies the earlier B1 (unless it is a verification attempt – see SC). Note that e.g. $-\frac{1}{6} \ln 3-2V = \frac{t}{20} (+c)$ or $-\frac{10}{3} \ln 2V-3 = t (+c)$ are also correct. $-\frac{1}{6} \ln(9-6V) = \frac{t}{20} (+c)$ is also correct. Condone log being used in place of ln.
dM1:	Requires constant of integration now. Substitutes (or states) $t = 0$ and $V = 0.25$ and finds a value for c , which may be “A” = e^c if they rearrange first to eliminate ln terms. Dependent on the previous method mark. Do not be concerned about their processing to find c or “A” = e^c and does not need to be exact.
A1:	Achieves the required form e.g. $V = \frac{3}{2} - \frac{5}{4} e^{-\frac{3t}{10}}$ with no errors and clear working. Allow equivalent fractions or decimals e.g. $V = 1.5 - 1.25e^{-\frac{6}{20}t}$
SC:	Attempts by verification may score maximum B0M1A1dM1A0 – see below.
Alt:	Use of an integrating factor – see below.
(b)	Special Case: Attempts by verification may score maximum B0M1A1dM1A0
B0:	This mark may not be scored via this approach.
M1:	Differentiates $V = P - Qe^{-kt}$ to the form $\frac{dV}{dt} = \alpha e^{-kt}$ where α is a constant (note it should be $\frac{dV}{dt} = Qke^{-kt}$) and substitutes both this and $V = P - Qe^{-kt}$ into $20 \frac{dV}{dt} = 9 - 6V$ and deduces a value for P or k by comparing coefficients.
A1:	Correct values for both P and k .
dM1:	Substitutes (or states) $t = 0$ and $V = 0.25$ and finds a value for Q . Requires a value for P to have been found using the above approach.
A0:	This mark may not be scored via this approach.
	Alternative: Using Integrating Factor (Further Maths)
B1:	Deduces the correct integrating factor for the equation, $e^{0.3t}$ This should come from $\frac{dV}{dt} + 0.3V = 0.45 \Rightarrow \text{I.F.} = e^{\int 0.3dt} = e^{0.3t}$ May be implied by sight of $\frac{d(Ve^{0.3t})}{dt} = \dots$
M1:	Fully multiplies through by their integrating factor and integrates both sides. Score for $Ve^{kt} = \int \dots e^{kt} dt = \dots e^{kt}$ Condone missing dt
A1:	Correct integration $Ve^{0.3t} = \int 0.45e^{0.3t} dt = \frac{3}{2} e^{0.3t} (+c)$
dM1:	As main scheme.
A1:	As main scheme.

Question	Scheme	Marks	AOs
10(c)	Examples: (1) $\frac{dV}{dt} = 0 \Rightarrow V = (1.5)$ (or e.g. max V is 1.5) (2) As $t \rightarrow \infty$, $e^{-0.3t} \rightarrow 0$ (or $V \rightarrow "1.5"$) (3) Flow in = flow out at max V so $0.3V = 0.45 \Rightarrow V = 1.5$ (4) As $e^{-0.3t} > 0$, $V < "1.5"$ (5) When $V > 1.5$, $\frac{dV}{dt} < 0$ (6) $V = 2 \Rightarrow \frac{dV}{dt} = -0.15$ or compares $\frac{dV_{\text{out}}}{dt}$ ($= 0.6$) against $\frac{dV_{\text{in}}}{dt}$ ($= 0.45$) at $V = 2$ (7) $V = 2 \Rightarrow "1.5" - "1.25"e^{-0.3t} = 2 \Rightarrow e^{-0.3t} < 0$ (8) $"1.5" - "1.25"e^{-0.3t} = 2 \Rightarrow \ln(-0.4)$ is undefined (condone e.g. gives a maths error)	M1	3.2a
	- The (upper) limit for V is 1.5 (m^3) so no (the container will not become full) (first 4 bullets) - If $V = 2$ (or $V > 1.5$), it would be emptying so no (it can never be full) (bullets 5, 6) - No, as the equation cannot be solved (or is not true/doesn't work) when $V = 2$ (bullets 7, 8)	A1ft	2.4
		(2)	
(9 marks)			
Notes			
(c) M1: See main scheme. If using the answer to part (b) it must be of the form $V = P - Qe^{-kt}$ but there is no limitation on the values of their P , Q or k . Substitution of a large value for t may score this mark but it is unlikely to be recovered to score the A1 unless they reference e.g. V_{max} being "1.5". Reference to an (upper) limit of "1.5" or their P can imply the method mark. If setting $V = 2$ in their equation they must reach either $\ln(-\text{ve})$ or solve the equation to reach a value for t to score this mark. A1ft: Must conclude "no" or equivalent e.g. "the container will not become full". Makes a correct interpretation for their method (see bullets 1-8) with a clear conclusion e.g. "no". To score this mark through ft, their V must be of the form $V = P - Qe^{-kt}$ with $k > 0$, $Q > 0$ and $0 < P < 2$ if used (but note that they can still use the answer to part (a) to score both marks via bullets 1, 3, 5 or 6). Allow "it" in place of the "container"/"tank". Just stating "the equation cannot be solved when $V = 2$ " without any evidence is M0A0. There must be no incorrect working if solving their equation or contradictory statements such as " t cannot be negative" but condone notational errors provided the intention is clear.			

Question	Scheme	Marks	AOs
1(a)	$h = 0.5$	B1	1.1a
	$A \approx \frac{0.5}{2} \{0.5774 + 0.8452 + 2(0.7071 + 0.7746 + 0.8165)\}$	M1	1.1b
	= awrt 1.50	A1	1.1b
	For reference: The integration on a calculator gives 1.511549071 The full accuracy for y values gives 1.504726147 The accuracy from the table gives 1.50475		
		(3)	
(b)	$3 \times \text{their (a)}$ If (a) is correct, allow awrt 4.50 or awrt 4.51 even with no working. Only allow 4.5 if (a) is correct and working is shown e.g. 3×1.5 If (a) is incorrect allow $3 \times \text{their (a)}$ given to at least 3sf but do not be too concerned about the accuracy (as they may use rounded or rounded value from (a)) For reference the integration on a calculator gives 4.534647213	B1ft	2.2a
		(1)	
(c)	<u>This mark depends on the B1 having been awarded in part (b) with awrt 4.5</u> Look for a sensible comment. Some examples: • The answer is accurate to 2 sf or one decimal place • Answer to (b) is accurate as $4.535 \approx 4.50$ • Very accurate as 4.535 to 2 sf is 4.5 • $4.51425 < 4.535$ so my answer is underestimate but not too far off • It is an underestimate but quite close • It is a very good estimate • High accuracy • (Quite) accurate • It is less than 1% out • $4.535 - 4.5 = 0.035$ so not far out But not just “it is an underestimate” or Calculates percentage error correctly using awrt 4.50 or awrt 4.51 or 4.5 (No comment is necessary in these cases although one may be given) Examples: $\left \frac{4.535 - 4.50}{4.535} \right \times 100 = 0.77\% \quad \text{or} \quad \left \frac{4.535 - 4.51}{4.535} \right \times 100 = 0.55\% \quad \text{or}$ $\left \frac{4.535 - 4.51425}{4.535} \right \times 100 = 0.46\% \quad \text{or} \quad \left \frac{4.50}{4.535} \right \times 100 = 99\%$ In these cases don't be too concerned about accuracy e.g. allow 1sf. This mark should be withheld if there are any contradictory statements	B1	3.2b
		(1)	
(5 marks)			

Notes:

(a)



B1: States or uses $h = 0.5$. May be implied by $\frac{1}{4} \times \{ \dots \}$ below.

M1: Correct attempt at the trapezium rule.

Look for $\frac{1}{2}h \times (0.5774 + 0.8452 + 2(0.7071 + 0.7746 + 0.8165))$ condoning slips on the terms but must use all y values



Question	Scheme	Marks	AOs
5(a)	$h = 0.2$	B1	1.1b
	$\frac{1}{2} \times 0.2 \times \{a + 13.5 + 2(16.8 + b + 20.2 + 18.7)\} = 17.59$	M1	1.1b
	e.g. $\Rightarrow a + 13.5 + 2b + 111.4 = 175.9 \Rightarrow a + 2b = 51^*$	A1*	2.1
		(3)	
(b)	$a + 16.8 + b + 20.2 + 18.7 + 13.5 = 97.2 \Rightarrow a + b = 28 \Rightarrow a = \dots$ (or $b = \dots$)	M1	3.1a
	$a = 5$ or $b = 23$	A1	1.1b
	$a = 5$ and $b = 23$	A1	1.1b
		(3)	
(6 marks)			
Notes			
(a) B1: States or uses $h = 0.2$ o.e. M1: Forms the equation $\frac{1}{2} \times 0.2 \times \{a + 13.5 + 2(16.8 + b + 20.2 + 18.7)\} = 17.59$ o.e. but condone copying slips. They may have added some of the y values together so as a minimum accept e.g. $0.1 \times \{a + 13.5 + 2(55.7 + b)\} = 17.59$ Condone invisible brackets as long as they are recovered or implied in further work before achieving the given answer. Condone the use of $\approx$ for this mark. Allow this mark if they add the areas of individual trapezia e.g. $\frac{\text{their } h}{2} \{a + 16.8\} + \frac{\text{their } h}{2} \{16.8 + b\} + \frac{\text{their } h}{2} \{b + 20.2\} + \frac{\text{their } h}{2} \{20.2 + 18.7\} + \frac{\text{their } h}{2} \{18.7 + 13.5\}$ Condone copying slips but it must be a complete method using all the trapezia. h must be numerical but condone $h = 1$ A1*: A rigorous argument leading to $a + 2b = 51$ from correct working and no errors seen including brackets, although do not penalise a missing trailing bracket at the end e.g. $\frac{1}{2} \times 0.2 \times \{a + 13.5 + 2(16.8 + b + 20.2 + 18.7)\} = 17.59 \Rightarrow \dots \Rightarrow a + 2b = 51 \text{ could score B1M1A1but}$ $\frac{1}{2} \times 0.2 \times a + 13.5 + 2(16.8 + b + 20.2 + 18.7) = 17.59 \Rightarrow \dots \Rightarrow a + 2b = 51 \text{ could score max B1M1A0}$ provided later work implied correct brackets. Both sets of brackets must be dealt with correctly before proceeding to the final answer such that e.g. $\dots \Rightarrow a + 2b + 124.9 = 175.9 \Rightarrow a + 2b = 51 \text{ is M1A1}^*$ $\dots \Rightarrow a + 13.5 + 33.6 + 2b + 40.4 + 37.4 = 175.9 \Rightarrow a + 2b = 51 \text{ is M1A1}^*$ $\dots \Rightarrow 0.1a + 1.35 + 3.36 + 0.2b + 4.04 + 3.74 = 17.59 \Rightarrow a + 2b = 51 \text{ is M1A1}^*$ Note that $a + 2b \approx 51$ as the final answer is A0*			
(b) M1: Attempts to form the equation $a + 16.8 + b + 20.2 + 18.7 + 13.5 = 97.2$, condoning copying errors, (may just be stated as e.g. $a + b = 28$ o.e.) and attempts to solve their equation simultaneously with the given equation (or condone their equation from part (a)). Do not be too concerned with the process here as calculators may be used. Score if values for a or b are reached from a pair of simultaneous equations. A1: for $a = 5$ or $b = 23$ A1: for both $a = 5$ and $b = 23$			

Question	Scheme	Marks	AOs
5(a)	States or uses $h = 1.5$	B1	1.1a
	Full attempt at the trapezium rule $= \frac{h}{2} \{1.63 + 2.63 + 2 \times (2 + 2.26 + 2.46)\}$	M1	1.1b
	= awrt 13.3 or $\frac{531}{40}$	A1	1.1b
		(3)	
(b)(i)	$\int_3^9 \log_3(2x)^{10} dx = 10 \times "13.3" = \text{awrt } 133 \text{ or e.g. } \frac{531}{4}$	B1ft	2.2a
(ii)	$\int_3^9 \log_3 18x dx = \int_3^9 \log_3(9 \times 2x) dx = \int_3^9 2 + \log_3 2x dx$ $= [2x]_3^9 + \int_3^9 \log_3 2x dx = 18 - 6 + \int_3^9 \log_3 2x dx = \dots$	M1	3.1a
	Awrt 25.3 or $\frac{1011}{40}$	A1ft	1.1b
		(3)	
(6 marks)			
Notes:			

(a)

B1: States or uses $h = 1.5$ **M1:** A full attempt at the trapezium rule.Look for $\frac{\text{their } h}{2} \{1.63 + 2.63 + 2 \times (2 + 2.26 + 2.46)\}$ but condone copying slipsNote that $\frac{\text{their } h}{2} \{1.63 + 2.63 + 2 \times (2 + 2.26 + 2.46)\}$ scores M0 unless the missing brackets are recovered or implied by their answer. You may need to check.

Allow this mark if they add the areas of individual trapezia e.g.

$$\frac{\text{their } h}{2} \{1.63 + 2\} + \frac{\text{their } h}{2} \{2 + 2.26\} + \frac{\text{their } h}{2} \{2.26 + 2.46\} + \frac{\text{their } h}{2} \{2.46 + 2.63\}$$

Condone copying slips but must be a complete method using all the trapezia.

A1: awrt 13.3 (Note full accuracy is 13.275) or exact equivalent.**Note that the calculator answer is 13.324 so you must see correct working to award awrt 13.3**Use of $h = -1.5$ leading to a negative area can score B1M1A0 but allow full marks if then stated as positive.

(b)(i)

B1ft: Deduces that $\int_3^9 \log_3(2x)^{10} dx = 10 \times "13.3" = \text{awrt } 133$ FT on their 13.3 look for 3sf accuracy but follow through on e.g. their rounded answer to part (a) so if 13 was their answer to part (a) then allow 130 here **following a correct method**.A correct method must be seen here but a minimum is e.g. $10 \times "13.3" = "133"$ Note that $\int_3^9 \log_3(2x)^{10} dx = 133.2414316\dots$ so a correct method must be seen to award marks.**Attempts to apply the trapezium rule again in any way score M0 as the instruction in the question was to use the answer to part (a).**

(b)(ii)

M1: Shows correct log work to relate the given question to part (a)

Must reach as far as e.g. $[2x]_3^9 + \int_3^9 \log_3 2x \, dx = \dots$ with correct use of limits on $[2x]_3^9$ which

may be implied or equivalent work e.g. finds the area of the rectangle as 2×6

A1ft: **Correct working** followed by awrt 25.3 but ft on their 13.3 so allow for $12 +$ their answer to part (a) **following correct work** as shown.

Note that $\int_3^9 \log_3 18x \, dx = 25.32414\dots$ so a correct method must be seen to award marks.

Some examples of an acceptable method are:

$$\int_3^9 \log_3 18x \, dx = \int_3^9 \log_3 (9 \times 2x) \, dx = \int_3^9 2 + \log_3 2x \, dx = 6 \times 2 + "13.3" = 25.3$$

$$\int_3^9 \log_3 18x \, dx = \int_3^9 \log_3 (9 \times 2x) \, dx = \int_3^9 2 + \log_3 2x \, dx = 12 + "13.3" = 25.3$$

$$\int_3^9 \log_3 18x \, dx = \int_3^9 \log_3 (9 \times 2x) \, dx = \int_3^9 2 + \log_3 2x \, dx = [2x]_3^9 + \int_3^9 \log_3 2x \, dx = 25.3$$

BUT just $12 + "13.3" = 25.3$ scores M0

Attempts to apply the trapezium rule again in any way score M0 as the instruction in the question was to use the answer to part (a).

